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Abstract. Cuts of real classes are studied in detail. Really representable pairs of cuts, i.e. pairs of form $\langle \underline{X}, \overline{X}\rangle$ where X is a real class, are fully described.

Keywords: Alternative set theory, lower cut, upper cut, real class

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This paper is a direct continuation of [**K–Z 1988**] contributing to the problematics of cuts in the AST. It is devoted to a detailed investigation of cuts of real classes. In Section 1 the existence of a tight approximation by σ– or π–classes both from in and out for each real class is proved. In Section 2 some connections between the cuts of a figure in an arbitrary indiscernibility equivalence and its monads are established. As a consequence, an estimation for the gap between the [illegible] and upper cut of a real class is derived. Some of these conclusions have alr[illegible]mplicitly appeared in [**S–Ve 1981**]. Finally, in Section 3 really representable[illegible]s of cuts are completely cut–theoretically characterized.

For the used notions, notations, conventions and results concerning cut arithmetic and cuts of classes the reader is referred to [**K–Z 1988**].

1. Approximation of real classes. In order to extend the amount of classes known to have a cut we state some technical results, some of them slightly generalizing some from [**Tz 1987**].

1.1.Lemma. *Let $\{X_n; n \in FN\}$ be a sequence of classes such that $X_n \subseteq X_{n+1}$ and the class $\{u; X_n \subseteq u\}$ is revealed for each n. Let $X = \bigcup\{X_n; n \in FN\}$. Then*

$$\overline{X} = \bigcup\{\overline{X_n}; n \in FN\}.$$

PROOF: Since $X_n \subseteq X$ for each $n, \bigcup\{\overline{X_n}; n \in FN\} \leq \overline{X}$. Conversely, let $a \notin \bigcup\{\overline{X_n}; n \in FN\}$. We put $Y_n = \{u; X_n \subseteq u \hat{\approx} a\}$. Then $\{Y_n; n \in FN\}$ is a sequence of revealed classes and $\emptyset \neq Y_{n+1} \subseteq Y_n$ for each n. Hence there is a $u \in \bigcap\{Y_n; n \in FN\}$. Then $X \subseteq u \hat{\approx} a$, therefore $a \notin \overline{X}$. ∎

1.2. Corollary. *Let $\{X_n; n \in FN\}$ be a sequence of classes such that $X_n \subseteq X_{n+1}, X_n$ has a cut and the class $\{u; X_n \subseteq u\}$ is revealed for each n. Let $X = \bigcup\{X_n; n \in FN\}$. Then X has a cut and $|X| = \bigcup\{|X_n|; n \in FN\}$.*

PROOF: $|X| \leq \overline{X} = \bigcup\{|X_n|; n \in FN\} \leq \underline{X}$. ∎

In a similar way the following dual statements can be derived.

1.3. Lemma. *Let $\{X_n; n \in FN\}$ be a sequence of classes such that $X_{n+1} \subseteq X_n$ and $\{u; u \subseteq X_n\}$ is a revealed class for each n. Let $X = \bigcap\{X_n; n \in FN\}$. Then*

$$\underline{X} = \bigcap\{\underline{X_n}; n \in FN\}.$$

1.4. Corollary. *Let $\{X_n; n \in FN\}$ be a sequence of classes such that $X_{n+1} \subseteq X_n, X_n$ has a cut and the class $\{u; u \subseteq X_n\}$ is revealed for each n. Let $X = \bigcap\{X_n; n \in FN\}$. Then X has a cut and*

$$|X| = \bigcap\{|X_n|; n \in FN\}.$$

A particular consequence of 1.2 and 1.4 is

1.5. Theorem. *If X is a σ-class or a π-class, then X has a cut which is a σ-class or a π-class, respectively.*

However, if $a \notin FN$, then the class

$$X = ((a - FN) \times \{0\}) \cup (FN \times \{1\}),$$

being a union of a π-class and of a σ-class which are even separated by the Sd-class $V \times \{0\}$, still has not a cut. Indeed, $\underline{X} = a - FN, \overline{X} = a + FN$.

For classes without cut one can still look for some approximating sub- and superclasses with cuts. Arbitrarily good loose approximations are possible.

1.6. Proposition. *Let X be an arbitrary class and A, B be cuts such that $A < \underline{X}, \overline{X} <$ (∀ Then there are classes Y, Z such that $Y \subseteq X \subseteq Z$ and $|Y| = A, |Z| = B$.*

PROOF: We will show that Y, Z can even be chosen of the form $Y = F''A, Z = F''B$ for some Sd-function F such that $B \leq \text{dom}(F) \in \mathcal{C}$. If $B = N$, then $Z = V$ will do. Without loss of generality we can assume that $B < N$. Then there are sets u, v, w such that $A \leq |u|, |v| \leq B \leq |w|, u \subseteq X \subseteq v \subseteq w$ and a map $f : |w| \approx w$ such that $f''A \subseteq u \subseteq f''|v| = v \subseteq f''B \subseteq w$. ∎

Let us concentrate on tight approximations, i.e. those achieving the values $\underline{X}$ resp. $\overline{X}$, now.

1.7. Theorem. *Let X be an arbitrary class.*

(a) *If X is a semiset and $\underline{X}$ is a σ-cut, then there is a σ-class $Y \subseteq X$ such that $|Y| = \underline{X}$.*
(b) *If $\overline{X}$ is a π -class, then there is a π-class $Z \supseteq X$ such that $|Z| = \overline{X}$.*

PROOF:

(a) Let $\underline{X} = \bigcup\{a_n; n \in FN\}$, where $a_n \leq a_{n+1}$ for each n. A sequence $\{u_n; n \in FN\}$ such that $a_n \stackrel{\frown}{\prec} u_n \subseteq u_{n+1} \subseteq X$ for each n can easily be constructed by induction. Then $Y = \bigcup\{u_n; n \in FN\}$ has the required properties.
(b) If $\overline{X} = N$, we put $Z = V$. Otherwise, there is a set $v \supseteq X$. Then by 3.1.7 (c) from **[K–Z 1988]** $\underline{v \setminus X} = |v| - \overline{X}$ is a σ-cut and by (a) there is a σ-class $Y \subseteq v \setminus X$ such that $|Y| = |v| - \overline{X}$. Then $Z = v \setminus Y$ is a π-class, $X \subseteq Z$ and $|Z| = |v| - |Y| = \overline{X}$. ∎

1.8. Remark. However, the condition "X is a semiset" cannot be omitted from (a). Namely, in [**Č–K 1982**] it is proved that for any w no monad in the indiscernibility equivalence E_w contains a subclass, which is a proper Sd–class. Hence it cannot even contain a subclass, which is a σ–class and not a semiset (each monad is a π–class!). On the other hand, as we shall see within short, there is at least one monad M in E_w, which is not a semiset, i.e. $|M| = N$.

For the idea of the proof of the following theorem we are indebted to K.Čuda.

1.9. Theorem. *Let X be a real class.*

(a) *If $\underline{X}$ is a π–class, then there is a π–class $Y \subseteq X$ such that $|Y| = \underline{X}$.*
(b) *If $\overline{X}$ is a σ–class, then there is a σ–class $Z \supseteq X$ such that $|Z| = \overline{X}$.*

PROOF:

(a) Let w be a set such that X is a figure in the indiscernibility equivalence E_w. Then, by 1.2.1 (b) from [**K–Z 1988**], $\underline{X}$ is a figure in E_w, too. Since $\underline{X}$ is a π–class, there is an $a \leq \underline{X}$ such that $\underline{X} = E''_w a$. Then there is also a $u \subseteq X$ such that $a \mathbin{\hat{\approx}} u$. We put $Y = E''_w u \subseteq X$. Then $Y$is a π–class and $|Y| \leq \underline{X}$. Conversely, Y, and therefore also $|Y|$ are figures in E_w, and $a \leq |Y|$. Hence $\underline{X} = E''_w a \leq |Y|$.
(b) follows from (a) by duality in exactly the same way as in the proof of 1.7.

■

Summarizing 1.7 and 1.9, we obtain

1.10. Theorem. *If X is a real class, then there are classes Y, Z such that $Y \subseteq X \subseteq Z$, each of them is either a σ–class or a π–class, and $|Y| = \underline{X}, |Z| = \overline{X}$.*

2. Cuts of figures and monads. We start the investigation of connections between the cuts of a figure and its monads in a given indiscernibility equivalence by clearing the semiset case.

2.1. Theorem. *Let R be an indiscernibility equivalence and X be a class such that $R''\{x\}$ is a semiset for each $x \in X$. Then $R''X$ is a semiset.*

PROOF: Let $\{R_n; n \in FN\}$ be a generating sequence of R and Y be a countable dense subclass of X, i.e. $Y \subseteq X \subseteq R''_n Y$ for each n. Let $n(x)$ denote for each $x \in X$ the least natural number n such that $R''_n\{x\}$ is a semiset, hence a set. Such an n exists for each $x \in X$, because in the opposite case $R''\{x\} = \bigcap\{R''_n\{x\}; n \in FN\}$ were not a semiset. We put

$$Z = \bigcup\{R''_{n(y)}\{y\}; y \in Y\}.$$

Obviously, Z is a semiset. We will prove the inclusion $R''X \subseteq Z$. Let $x \in X$. There is a $y \in Y$ such that $\langle x, y\rangle \in R_{n(x)+2}$. Then $R''\{x\} \subseteq R''_{n(x)+1}\{y\} \subseteq R''_{n(x)}\{x\}$, hence $R''_{n(x)+2}\{y\}$ is a semiset and $n(y) \leq n(x) + 1$. Consequently, $R''\{x\} \subseteq R''_{n(y)}\{y\}$. ■

2.2. Corollary. *If R is an indiscernibility equivalence and X is a figure in R which is not a semiset, then there is a monad $M \subseteq X$ in R which is not a semiset.*

In Particular, $\overline{X} = N$ implies $\underline{X} = N$ for each real class X.

2.3. Theorem. *Let R be an indiscernibility equivalence, A be a σ-cut and X be a class such that $|R''\{x\}| \leq A$ for each $x \in X$. Then $\overline{R''X} \leq A \cdot FN$.*

PROOF: The proof of 2.1 can be repeated almost literally. One has only to change the meaning of $n(x)$ to denote the least natural number n such that $|R''_n\{x\}| \leq A$ for $x \in X$. Then for the class Z, constructed as above, obviously $\overline{Z} \leq A \cdot FN$ holds. ■

2.4. Theorem. *Let X be a real class and $a \in N$. If $\underline{X} \leq a \leq \overline{X}$ then*

$$a/FN \leq \underline{X} \leq a \leq \overline{X} \leq a \cdot FN.$$

PROOF: If $\underline{X} < a/FN$, then $\underline{X} \leq b$ for some $b < a/FN$. From 2.3 it follows $\overline{X} \leq b \cdot FN < a/FN$ — a contradiction. Similarly, if $\underline{X} \leq a$ then $\overline{X} \leq a \cdot FN$ by the same theorem. ■

An immediate consequence is the following:

2.5. Theorem. *Let X be a real class.*

(a) *If $\underline{X}$ is an additive σ-cut, then $\underline{X} = \overline{X}$.*
(b) *If $\overline{X}$ is an additive π-cut, then $\underline{X} = \overline{X}$.*
(c) *If A is an additive σ-cut and $\underline{X} \leq A$, then $\overline{X} \leq A$.*
(d) *If B is an additive π-cut and $B \leq \overline{X}$, then $B \leq \underline{X}$.*

As we shall see within short, for each infinite b there is a real class X such that $\underline{X} = b/FN$ and $\overline{X} = b \cdot FN$, so that the maximal gap allowed by 2.4 always is achieved, hence 2.5 cannot be strengthened.

3. Really representable pairs of cuts. A pair of cuts $\langle A, B\rangle$ is called representable if there is a class X such that $A = \underline{X}, B = \overline{X}$.

The following result is due to A.Tzouvaras [**Tz 1987**]. We restate it here for the sake of completeness in a slightly modified form without proof.

3.1. Theorem. *A pair $\langle A, B\rangle$ of cuts is representable if and only if $A \leq B$ and $(A \in N \vee B \in N) \Rightarrow A = B$.*

A pair of cuts $\langle A, B\rangle$ is called really representable if there is a real class X such that $A = \underline{X}, B = \overline{X}$.

Obviously, each really representable pair of cuts is representable and consists of two real cuts (i.e. σ- or π-cuts).

In the proof of the characterization theorem of really representable pairs of cuts we will utilize the following lemma.

3.2. Lemma. *For each infinite b there is a real class Y such that $\underline{Y} = b/FN$, $\overline{Y} = \mathrm{int}(b)$.*

PROOF: In [**Č–V 1979**] for each infinite b two real classes Y, Z such that $Y \cap Z = \emptyset, Y \cup Z \subseteq b$ and $\mathrm{int}(b) \leq \overline{Y}, \mathrm{int}(b) \leq \overline{Z}$ are constructed. Then using the results from [**K–Z 1988**] one obtains

$$\underline{Y} \leq \underline{b \setminus Z} = b - \overline{Z} \leq b - \mathrm{int}(b) = b/FN.$$

Since $\lfloor b/2 \rfloor \in \overline{Y}$ and $\lfloor b/2 \rfloor / FN = b/FN$, 2.4 implies $\underline{Y} = b/FN$, and by a symmetry argument $\underline{Z} = b/FN$, as well. Consequently

$$\overline{Y} \leq \overline{b \setminus Z} = b - \underline{Z} = b - b/FN = \operatorname{int}(b),$$

hence $\overline{Y} = \operatorname{int}(b)$. ■

3.3. Theorem. *A pair of cuts is really representable if and only if it is exactly of one of the following ten types:*

$$\begin{array}{lll}
(1) & \langle A, A \rangle & (A \text{ - } \textit{real}) \\
(2) & \langle a - A, a + A \rangle & (A \text{ - } \textit{additive, real, } 0 < A < a) \\
(3) & \langle a + b/FN, a + \operatorname{int}(b) \rangle & \multirow{2}{*}{$\Big\} \; (FN < b)$} \\
(4) & \langle a + b/FN, a + \operatorname{cl}(b) \rangle & \\
(5) & \langle a - b/FN, a + \operatorname{int}(b) \rangle & \Big\} \; (FN < b < a \cdot FN) \\
(6) & \langle a - b/FN, a + \operatorname{cl}(b) \rangle & \\
(7) & \langle a + b/FN, a + b \cdot FN \rangle & (FN < b) \\
(8) & \langle a - b/FN, a + b \cdot FN \rangle & (FN < b < a \cdot FN) \\
(9) & \langle a - b \cdot FN, a - b/FN \rangle & \Big\} \; (FN < b < a/FN) \\
(10) & \langle a - b \cdot FN, a + b/FN \rangle &
\end{array}$$

Proof: will consist of three parts. The first one, left to the reader, requires to check that none two of the ten types do overlap. In the second part we will produce (for each particular choice of admitted parameters A, a, b) a sequence of real classes $X_i (i = 1, \ldots, 10)$ such that the cuts of the class X_i are exactly of the i–th type. For given $b \notin FN$ we denote by Y the class assured by 3.2. We put

$$\begin{aligned}
X_1 &= A, \\
X_2 &= (a - A) \cup (A \times \{0\}), \\
X_3 &= a \cup (Y \times \{0\}), \\
X_4 &= (a + b/FN) \cup (Y \times \{0\}), \\
X_5 &= (a - b/FN) \cup (Y \times \{0\}), \\
X_6 &= (a - b/FN) \cup (Y \times \{0\}) \cup (b/FN \times \{1\}), \\
X_7 &= a \cup (Y \times FN), \\
X_8 &= (a - b/FN) \cup (Y \times FN), \\
X_9 &= u \setminus (Y \times FN), \text{ where } Y \times FN \subseteq u \subseteq N^2, u \mathrel{\widehat{\approx}} a, \\
X_{10} &= b/FN \cup X_9.
\end{aligned}$$

The verification that the classes X_i have the desired cuts, using the results on cut arithmetic from [K–Z 1988], is just a matter of routine, now. Finally, we have to show that for each real class X the pair$\langle \underline{X}, \overline{X}\rangle$ falls under some of the listed types. This will take place in four lemmas. The conclusion that their presumptions exhaust all possibilities follows then from the Sochor's theorem stated in [**S 1988**] (see also 2.3.9 in [**K–Z 1988**]) and from results of Section 2.

In what follows, X denotes a real class such that $\underline{X} < \overline{X}$.

3.4. Lemma. *Let $\underline{X} = a + B$, where B is a nonzero additive cut. Then there exists a $b \notin FN$ such that $B = b/FN$ and $\overline{X}$ is one of the following three cuts: $a + \mathrm{int}(b), a + \mathrm{cl}(b), a + b \cdot FN$. In other words, the pair $\langle \underline{X}, \overline{X}\rangle$ is either of type (3), (4) or (7).*

PROOF: There is a set $u \subseteq X$ such that $u \hat{\approx} a$. Then for $Y = X \setminus u$ it holds $\underline{Y} = B < \overline{Y}$, since $\underline{X} < \overline{X}$. Then $B = c/FN$ for some $c \notin FN$ by 2.4, and $\overline{Y} \leq c \cdot FN$ by the same theorem. If $\overline{Y} < c \cdot FN$, then there are two possibilities:

(1) $\overline{Y} = b + A$ for some b such that $c/FN < b < c \cdot FN$ and some real additive cut $A \leq c/FN$. Then $b/FN = c/FN = B$. We will prove that $A = B$. If $A < B$, then $A < \lfloor b/d \rfloor$ for some $d \notin FN$, and there is a set $v \hat{\approx} \lfloor b(1 + 1/d) \rfloor$ such that $Y \subseteq v$. Then the following computation

$$\overline{v \setminus Y} = \lfloor b(1 + 1/d) \rfloor - b/FN = \mathrm{int}(b) > \lfloor b/d \rfloor \cdot FN,$$
$$\underline{v \setminus Y} = \lfloor b/d \rfloor - A \simeq \lfloor b/d \rfloor$$

contradicts 2.4.

(2) $\overline{Y} = b - A$ for some b such that $c/FN < b < c \cdot FN$ and some real additive cut $A < b$. Then $A \leq b/FN = c/FN = B$, again. There is a $v \hat{\approx} b$ such that $Y \subseteq v$. Then $\underline{v \setminus Y} = A$ and $\overline{v \setminus Y} = b - b/FN = \mathrm{int}(b)$. Consequently $A = b/FN$. This completes the proof. ∎

3.5. Lemma. *$\underline{X} = a - A, \overline{X} = c - B$ where $a \leq c < a \cdot FN$ and $A < a, B < c$ are nonzero additive cuts. Then under putting $b = c - a$ it holds either*

$$a \not\simeq c, A = B = b/FN \text{ and}$$
$$\underline{X} = a - b/FN, \overline{X} = a + \mathrm{int}(b),$$

or

$$a \simeq c, A = b \cdot FN, B = b/FN \text{ and}$$
$$\underline{X} = a - b \cdot FN, \overline{X} = a - b/FN,$$

i.e. $\langle \underline{X}, \overline{X}\rangle$ is either of the type (5) or (9).

PROOF: There exists a set $u \hat{\approx} c$ such that $X \subseteq u$. Then $\underline{u \setminus X} = B, \overline{u \setminus X} = b + A$, and the conclusion follows from the previous lemma. ∎

3.6. Lemma. *Let* $\underline{X} = a - A, \overline{X} = c + B$, *where* $A \neq B$ *are nonzero additive cuts,* $A < a \leq c$. *Then there is a* $b > FN$ *such that either*

$$\underline{X} = a - b/FN, \overline{X} = a + b \cdot FN \text{ and } b < a \cdot FN,$$

or

$$\underline{X} = c - b \cdot FN, \overline{X} = c + b/FN \text{ and } b < c/FN,$$

i.e. $\langle \underline{X}, \overline{X} \rangle$ *is either of the type (8) or (10).*

PROOF: Assume that $A < B$. Then $A \leq b/FN < b \cdot FN \leq B$ for each $b \in B \setminus A$. We can find a $b \in B \setminus A$ such that $b \leq a$. Then there is a set $u \subseteq X, u \hat{\approx} a - b$. Then $\underline{X \setminus u} = b - A, \overline{X \setminus u} = c - a + b + B$. From 2.4 it follows $\overline{X \setminus u} \leq b \cdot FN$, hence $B = b \cdot FN = \overline{X \setminus u}$, and $A = b/FN$. Then $\underline{X} = a - b/FN$ and $\overline{X \setminus u} = B, \overline{X} = \overline{X \setminus u} + |u| = b \cdot FN + a - b = a + b \cdot FN$.

If $B < A$, then there is a $d \in A \setminus B, d \leq a$, and a set $v \hat{\approx} c + d, X \subseteq v$. Then $\underline{v \setminus X} = d - B, \overline{v \setminus X} = c + d - a + A$. By the previous part of the proof, there is a $b < d \cdot FN, b > FN$ such that $\underline{v \setminus X} = d - b/FN, \overline{v \setminus X} = d + b \cdot FN$. Since $b < d \cdot FN < A < a, b \cdot FN < A < a \leq c$ and $b < c/FN$. Finally, $\underline{X} = c - b \cdot FN, \overline{X} = c + b/FN$. ■

3.7. Lemma. *Let* $\underline{X} = a - A, \overline{X} = c + A$, *where* $A < a$ *is a nonzero additive cut and* $a \leq c$. *Then either*

$$\underline{X} = a - A, \overline{X} = a + A$$

or there exists a $b < a \cdot FN, b > FN$ *such that*

$$\underline{X} = a - b/FN, \overline{X} = a + \mathrm{cl}(b).$$

In other words, $\langle \underline{X}, \overline{X} \rangle$ *is either of the type (2) or (6).*

PROOF: Assume that $\overline{X} \neq a + A$ and denote $b = c - a$. Obviously $A < b$, i.e. $A \leq b/FN$. It suffices to show that $A = b/FN$. If $A < b/FN$, then there is a d such that $A < d/FN < d \cdot FN < b/FN$. Taking a set $u \subseteq X, u \hat{\approx} a - d$, one obtains

$$\underline{X \setminus u} = d - A,$$
$$\overline{X \setminus u} = c - a + d + A = b + d + A > b > d \cdot FN$$

– a contradiction to 2.4. ■

Finally, if $\underline{X} = \overline{X}$, then the pair $\langle \underline{X}, \overline{X} \rangle$ is of type (1). This completes the proof of Theorem 3.3. ■

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