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Anatoli P. Kombarov

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On rectangular covers of $X^2\backslash\Delta$

A.P.KOMBAROV

Abstract. A paracompact Σ–space X has a G_δ–diagonal iff there exists a locally finite (in $X^2\backslash\Delta$) rectangular open cover of $X^2\backslash\Delta$.

Keywords: paracompact Σ–space, G_δ–diagonal

Classification: 54F65

It is proved in [1] that every regular Σ–space X with $X^2\backslash\Delta$ paracompact has a G_δ–diagonal. The proof of this theorem in [1] uses an existence of a locally finite cover of $X^2\backslash\Delta$ by open sets whose closures miss $\Delta = \{(x,x) : x \in X\}$.

Theorem 1. *A paracompact Σ–space X has a G_δ–diagonal iff there exists a locally finite (in $X^2\backslash\Delta$) rectangular open cover of $X^2\backslash\Delta$.*

A family ϑ of subsets of X^2 is called rectangular if $\vartheta = \{U_\alpha \times W_\alpha : \alpha \in A\}$. All spaces are assumed to be regular and T_1.

PROOF: A paracompact space X with a G_δ–diagonal is submetrizable (see [**2**, Corollary 2.9]). So, if τ is a topology on X^2, then there exists a topology τ' such that $\tau' \subseteq \tau$ and (X^2,τ') is metrizable. There exists a locally finite (in $X^2\backslash\Delta$) rectangular open cover of $X^2\backslash\Delta$ in τ' ([**3**, Proposition 1]) and therefore in τ.

We prove the converse assertion. We need the following Lemma 1 which is similar to Lemma 2 of [**1**].

Lemma 1. *Suppose ϑ is a locally finite (in $X^2\backslash\Delta$) rectangular open cover of $X^2\backslash\Delta$ and $x \in X$. If $x \in \overline{M}$ for some countable $M \subseteq X\backslash\{x\}$, then x is a G_δ-point.*

PROOF: For each $m \in M$, let $U_m \times W_m \subset X^2\backslash\Delta$ be a basic open neighbourhood of (x,m) such that the number $n(U_m \times W_m) = |\{V \in \vartheta : V \cap (U_m \times W_m) \neq \emptyset\}|$ is minimal. We prove that $\{x\} = \cap\{U_m : m \in M\}$. If $y \in \cap\{U_m : m \in M\}$, $y \neq x$, then $(y,x) \in P \times Q \in \vartheta$. Note that $x \notin \overline{P}$. Since $x \in \overline{M}$, there exists $m \in M$ such that $(y,m) \in P \times Q$. Then $H = (U_m \setminus \overline{P}) \times W_m$ is a basic open neighbourhood of (x,m), $H \subseteq U_m \times W_m$, $H \cap (P \times Q) = \emptyset$ and $P \times Q \in \vartheta$, but $n(U_m \times W_m)$ is minimal. This is a contradiction. ∎

Proposition 1. *Suppose ϑ is a locally finite rectangular open cover of $X^2\backslash\Delta$ and X is a strong Σ–space. Then each point of X is G_δ.*

PROOF: Let $x \in X$. If x is not a G_δ–point and X is a strong Σ–space, then there exists a compact space $B \subseteq X$ such that $x \in B$ and x is not isolated in B, [**4**]. Let $\lambda = \{P : P \times Q \in \vartheta, P \cap B \neq \emptyset, x \in Q\}$. If we choose $z(P) \in P \cap B$ for each $P \in \lambda$, then $Z = \{z(P) : P \in \lambda\}$ is discrete because λ is locally finite in $X \setminus \{x\}$. From compactness of B it follows that $x \in \overline{M}$ for every infinite $M \subseteq Z$. Now Lemma 1 completes the proof of Proposition 1. ∎

Lemma 2. *Let $U, W \subseteq X$ and $x \notin \overline{U} \cap \overline{W}$. Then there exists an open neighbourhood G of x such that $G^2 \cap (U \times W) = \emptyset$.*

We omit the easy proof of Lemma 2.

Proposition 2. *Let X be a strong Σ-space and ϑ be a locally finite rectangular open cover of $X^2 \setminus \Delta$. Then X has a G_δ-diagonal.*

We confine ourselves to showing how Proposition 2 can be proved by following the proof of Theorem 4 in [1]. See [1] for the beginning of the proof up to the condition (iv). For our proof ϑ should be taken to be a locally finite rectangular open cover of $X^2 \setminus \Delta$.

(iv) If $i < j < n$, $x_i \neq x_j$, $U \times W = V \in \vartheta$, $x \notin \overline{U} \cap \overline{W}$, and $(\{x_i\} \times G(s \upharpoonright j+1)) \cap V \neq \emptyset$, then $G(s^\frown \langle x \rangle)^2 \cap V = \emptyset$. It follows from Lemma 2 that $G(s^\frown \langle x \rangle)$ exists. Now we can formally follow the proof given in [1] up to a cluster point $p \in C$.

Now suppose $\cap\{G(s \upharpoonright n) : n \in \omega\}$ contains a point $q \neq p$. Let $(p, q) \in U \times W = V \in \vartheta$. Let x_i, x_j, $x_n \in U$, $i < j < n$, $x_i \neq x_j$. Then $x_n \notin \overline{W}$ and hence $x_n \notin \overline{U} \cap \overline{W}$. We see that $(x_i, q) \in (\{x_i\} \times G(s \upharpoonright j+1)) \cap V$. By (iv), we have $G(s \upharpoonright n)^2 \cap V = \emptyset$, contradicting $(p, q) \in G(s \upharpoonright n)^2 \cap V$.

So a strong Σ-space X has a W_δ-diagonal. Then X has a G_δ-diagonal (see [**2**, Theorem 4.14, Theorem 6.6]).

Proposition 2 completes the proof of Theorem 1. ■

Corollary. *A paracompact p-space X is metrizable iff there exists a locally finite rectangular open cover of $X^2 \setminus \Delta$.*

Let αZ denote the one–point compactification of an uncountable discrete space Z. It is easy to see that there exists a point–finite rectangular open cover of $(\alpha Z)^2 \setminus \Delta$.

Theorem 2. *A Lindelöf β-space X has a G_δ-diagonal iff there exists a countable rectangular open cover of $X^2 \setminus \Delta$.*

We note that every Σ-space is a β-space (see [**2**, Definition 7.7]). Theorem 2 can be deduced from Corollary 2.9 and Theorem 7.9 of [**2**].

Remark. It is easy to see that if there exists a countable rectangular open cover of $X^2 \setminus \Delta$ and X is hereditarily Lindelöf then X has a G_δ-diagonal. The author does not know an example of a Lindelöf space X without a G_δ-diagonal such that there exists a countable rectangular open cover of $X^2 \setminus \Delta$.

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Mech.-Math.faculty, Moscow State University, Moscow 119899, USSR

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