Some new results on accretive multivalued operators

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Abstract. Let A be a multivalued accretive operator on a separable Banach space. Then the set of all points in a domain $D(A)$ of A, at which A is not norm continuous, forms a first category set. If an accretive operator A on a general Banach space admits an extension which is norm–weak upper semicontinuous on $\operatorname{int} D(A)$, then A is norm continuous on a residual subset of $\operatorname{int} D(A)$. As a consequence we obtain generic continuity on $\operatorname{int} D(A)$ for any accretive operator on a reflexive Fréchet smooth Banach space.

Each maximal accretive operator on a Banach space X has convex values iff the norm on X is Gâteaux smooth. An analogous necessary and sufficient condition for weak closedness of values of any maximal accretive operator is given, too

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0. Introduction.

A lot of nonlinear problems of applied mathematics lead to monotone or accretive operators on Banach spaces which are defined in an analogous way ($T : X \to 2^{X^*}$ is monotone iff $\langle x-y, x^*-y^*\rangle \geq 0$ whenever $x^* \in T(x)$ and $y^* \in T(y)$; for the definition of an accretive operator see Definition 2). In this paper we deal with accretive multivalued operators and derive several theorems analogous to well–known results for monotone operators. However, the properties of accretive operators depend much more on geometrical properties of the space in question.

Using a method of Preiss and Zajíček [7] we prove that for any accretive operator A on a separable Banach space, the set M of all points x with $A(x)$ nonempty and such that A is not norm continuous at x, is a first category set. In uniformly Fréchet smooth separable Banach spaces this method gives σ–porosity of M. For a monotone operator, this set is σ–porous (and even something more) in any Banach space with a separable dual [7].

It is a well known fact that every monotone operator T on an Asplund space is norm continuous on a residual subset of $\operatorname{int} D(T)$ (interior of domain of T) [4]. Using the method of separable reduction [2], we prove generic norm continuity on $\operatorname{int} D(A)$ of an accretive operator A, having a norm–weak upper semicontinuous extension on $\operatorname{int} D(A)$, in a general Banach space. As a consequence we obtain generic norm continuity on $\operatorname{int} D(A)$ of any accretive operator on a reflexive Fréchet smooth Banach space. (For an analogous result, obtained by Kenderov's methods, see [5].)

In the last section we derive a necessary and sufficient condition which a Banach space ought to satisfy so that any maximal accretive operator on X has convex, respectively weak closed values. Note that maximal monotone operators have always convex and weak* closed values.

1. Preliminaries.

In this paper, X will always be a Banach space over the reals R and $B(x,r) = \{y \in X : \|x-y\| < r\}$ will be an open ball centered at x and having radius r.

For a continuous convex function f on X and $x, v \in X$, we shall denote $\partial f(x) = \{x^* \in X^* : f(z) \geq f(x) + \langle z - x, x^* \rangle$ for any $z \in X\}$ (a subdifferential of f at x) and

$$f'(x,v) = \lim_{t \downarrow 0}(f(x+tv) - f(x))/t = \sup\{\langle v, x^* \rangle : x^* \in \partial f(x)\}$$

(one–sided derivative of f at x in the direction v).

Let us denote $q(x) = \|x\|, Q(x) = \frac{1}{2}\|x\|^2$ and $J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} = \partial Q(x)$ for $x \in X$. It is easy to compute that

$$(1) \qquad Q'(x,v) = \|x\| q'(x,v) \quad \text{for any } x, v \in X.$$

The multivalued mapping $J : X \to 2^X$ is called a duality map and its properties are closely related to geometrical properties of X : X is Gâteaux (respectively Fréchet) smooth if and only if J is singlevalued (singlevalued and continuous, respectively) (cf. [**1**]).

Definition 1. X is said to be uniformly Fréchet smooth if it is Gâteaux smooth and the limit $\lim\limits_{t \to 0} \frac{\|x+tv\| - \|x\|}{t} = q'(x,v)$ is uniform on $\{(x,v) \in X \times X : \|x\| = 1, \|v\| = 1\}$.

It is evident that uniformly Fréchet smooth spaces are Fréchet smooth.

Lemma 1. *The following assertions are equivalent:*

(i) *X is uniformly Fréchet smooth;*
(ii) *J is singlevalued and uniformly continuous on $\{x \in X : \|x\| = 1\}$;*
(iii) *J is singlevalued and uniformly continuous on*

$$\{x \in X : r_1 \leq \|x\| \leq r_2\} \textit{ whenever } 0 < r_1 < r_2.$$

PROOF: For the proof of the equivalence (i)⇔ (ii) see [**1**]. The equivalence (ii)⇔(iii) is an easy consequence of the fact that $J(tx) = tJ(x)$ for any $t \in R$. ■

For any $u \in X$ and $a \in R$ denote

$$(2) \qquad \begin{aligned} E_{u,a} &= \{x \in X : \langle u, x^* \rangle > a\|u\| \cdot \|x\| \text{ for each } x^* \in J(x)\}, \\ F_{u,a} &= \{x \in X : \langle x, u^* \rangle \geq a\|x\| \cdot \|u\| \text{ for some } u^* \in J(u)\}. \end{aligned}$$

Lemma 2.

(i) *For any $a \geq -1$*

$$E_{u,a} = \{x \in X : \|x - tu\| < \|x\| - at\|u\| \textit{ for some } t > 0\}.$$

(ii) *Let $u \neq 0$ or $a \leq 1$. Then*

$$F_{u,a} = \{x \in X : \|u + tx\| \geq \|u\| + at\|x\| \textit{ for any } t > 0\}.$$

PROOF:

(i) If $u = 0$ both the sets are empty. Let $u \neq 0$. For any $x \in X$ the set $J(x)$ is weak* compact in X^*. Hence by (1)

$$\begin{aligned} E_{u,a} &= \{x \in X : \min\{\langle u, x^*\rangle : x^* \in J(x)\} > a\|u\| \cdot \|x\|\} = \\ &= \{x \in X : -\max\{\langle -u, x^*\rangle : x^* \in J(x)\} > a\|u\| \cdot \|x\|\} = \\ &= \{x \in X : Q'(x, -u) < -a\|u\| \cdot \|x\|\} = \{x \in X \setminus \{0\} : q'(x, -u) < -a\|u\|\} = \\ &= \{x \in X \setminus \{0\} : \frac{\|x - tu\| - \|x\|}{t} < -a\|u\| \text{ for some } t > 0\} = \\ &= \{x \in X : \|x - tu\| < \|x\| - at\|u\| \text{ for some } t > 0\} \text{ for any } a \geq -1. \end{aligned}$$

(ii) If $u = 0$ and $a \leq 1$ then both the sets are equal to X. Let $u \neq 0$ and $a \in R$. Then, similarly as in (i), we get

$$\begin{aligned} F_{u,a} &= \{x \in X : \max\{\langle x, u^*\rangle : u^* \in J(u)\} \geq a\|x\| \cdot \|u\|\} = \\ &= \{x \in X : Q'(u, x) \geq a\|x\| \cdot \|u\|\} = \{x \in X : q'(u, x) \geq a\|x\|\} = \\ &= \{x \in X : \frac{\|u + tx\| - \|u\|}{\|t\|} \geq a\|x\| \text{ for any } t > 0\} = \\ &= \{x \in X : \|u + tx\| \geq \|u\| + at\|x\| \text{ for any } t > 0\}. \quad \blacksquare \end{aligned}$$

Definition 2. Let $A : X \to 2^X$ be a multivalued mapping, $D(A) = \{u \in X : A(u) \neq \emptyset\}$ be its domain and $G(A) = \{(u, x) \in X \times X : u \in D(A),\ x \in A(u)\}$ be its graph. A is said to be accretive if for any $(u, x) \in G(A)$, $(v, y) \in G(A)$ there exists $w^* \in J(u - v)$ such that $\langle x - y, w^*\rangle \geq 0$.

Note that Lemma 2, (ii) for $a = 0$ gives an equivalent definition of an accretive operator (cf. Kato [**3**]):

$$\text{(3)} \qquad \begin{aligned} &A : X \to 2^X \text{ is accretive iff } \|u - v + t(x - y)\| \geq \|u - v\| \\ &\text{whenever } (u, x) \in G(A), \quad (v, y) \in G(A) \text{ and } t > 0. \end{aligned}$$

Definition 3. $A : X \to 2^X$ is maximal accretive if A is accretive and $G(A)$ is a proper subset of graph of no accretive operator on X.

2. Continuity on separable Banach spaces.

Definition 4. Let P, S be topological spaces and $A : P \to 2^S$. We shall say that A is upper semicontinuous (u.s.c.) at a point $u_0 \in D(A)$ if for any open set $V \subset S$ containing $A(u_0)$ there exists an open set $U \subset P$ containing u_0 such that $A(u) \subset V$ for any $u \in U$. A is said to be continuous at $u_0 \in D(A)$ if A is u.s.c. at u_0 and $A(u_0)$ is a singleton.

We state the following well-known and easy lemma without a proof.

Lemma 3. *Let P, S be topological spaces and $A : P \to 2^S$ be a multivalued mapping with $D(A) = P$. Then the following two conditions are equivalent:*

(i) *A is u.s.c. at any point of P;*

(ii) *The set $A^{-1}(C) = \{u \in P : A(u) \cap C \neq \emptyset\}$ is closed in P for any closed subset C of S.*

Let us define a system $\mathcal{M}$ of certain small sets.

Definition 5. For any $a > 0$, let $\mathcal{M}_a$ be the system of all sets M with the following property:

for any $u \in M$ and any $\varepsilon > 0$ there exist $z \in B(u, \varepsilon)$ and $v \in X \setminus \{0\}$ such that $M \cap (z + E_{v,a}) = \emptyset$.

Now we define $\mathcal{M}$ as the system of all sets M such that for any $a > 0$ M is a countable union of sets from $\mathcal{M}_a$.

The sets from $\mathcal{M}_a$ and $\mathcal{M}$ are analogous respectively to a–angle porous sets and angle small sets from [7].

Lemma 4. *Each set from $\mathcal{M}$ is of the first Baire category.*

PROOF: Choose an arbitrary $a \in (0, 1)$. Then for any $v \in X \setminus \{0\}$ the set $E_{v,a}$ is open by Lemma 2, (i) and contains all vectors of the from tv with $t > 0$. Consequently all sets from $\mathcal{M}_a$ are nowhere dense and hence each set $M \in \mathcal{M}$, being a countable union of sets from $\mathcal{M}_a$, is a first category set. ■

Definition 6 (cf. [6], [8]). For $M \subset X$, $x \in X$ and $d > 0$ denote $\gamma(x, d, M) = \sup\{r > 0 : B(z, r) \subset B(x, d) \setminus M$ for some $z \in X\}$. A set M is said to be porous if $\limsup_{d \downarrow 0} \gamma(x, d, M)/d > 0$ for any $x \in M$. A set is termed σ-porous if it can be written as a union of countably many porous sets.

Lemma 5. *Let X be uniformly Fréchet smooth and $0 < a < 1$. Then there exists $r > 0$ such that $B(u, r\|u\|) \subset E_{u,a}$ for any $u \neq 0$.*

PROOF: Denote $c = (1-a)/2$ and $P = \{x \in X : 1 - c \leq \|x\| \leq 1 + c\}$. The duality map J is singlevalued and uniformly continuous on P by Lemma 1. Consequently there exists $\delta > 0$ such that $\|J(x_1) - J(x_2)\| < ac = c(1-2c)$ whenever $x_1, x_2 \in P$ and $\|x_1 - x_2\| < \delta$. Put $r = \min(c, \delta)$.

Let $\|u\| = 1$ and let $x \in B(u, r)$ be an arbitrary point. Then $\|x - u\| < c$, $\|x - u\| < \delta$ and $u, x \in P$. Consequently

$$
\begin{gathered}
2\langle u, J(x)\rangle = \langle u, J(u)\rangle + \langle x, J(x)\rangle - \langle x - u, J(x-u)\rangle + \langle u - x, J(x)\rangle + \\
\langle x, J(x-u)\rangle - \langle u, J(x-u)\rangle + \langle u, J(x) - J(u)\rangle \geq \|u\|^2 + \|x\|^2 - \\
- \|x-u\|^2 - 2\|x\| \cdot \|u - x\| - \|u\| \cdot \|x - u\| - \|u\| \cdot \|J(x) - J(u)\| > \\
> 1 + \|x\|^2 - c^2 - 2c\|x\| - c - c(1-2c) = \|x\|^2 + (1-c)^2 - 2c\|x\| \geq \\
\geq 2(1-c)\|x\| - 2c\|x\| = 2(1-2c)\|x\| = 2a\|x\| = 2a\|x\| \cdot \|u\|.
\end{gathered}
$$

Hence $x \in E_{u,a}$ and the needed inclusion is proved for $\|u\| = 1$. For an arbitrary $u \neq \emptyset$ we have

$$B(u, r\|u\|) = \|u\| \cdot B(\frac{u}{\|u\|}, r) \subset \|u\| \cdot E_{u/\|u\|,a} = E_{u,a}$$

and the proof is complete. ■

Theorem 1. *Let X be a separable Banach space and $A : X \to 2^X$ be an accretive operator. Then the set*

$$M = \{u \in D(A) : A \textit{ is not norm continuous at } u\}$$

is in $\mathcal{M}$.

PROOF: It is easy to see that $M = \{u \in D(A) : \lim_{\delta \downarrow 0} \operatorname{diam} A(B(u,\delta)) > 0\}$. Let C be a countable dense set in X and let $a > 0$. Then M is a countable union of sets $M_{n,d} = \{u \in D(A) : \lim_{\delta\downarrow 0} \operatorname{diam} A(B(u,\delta)) \geq \frac{1}{n}$ and $\operatorname{dist}(d, A(u)) \leq \frac{a}{2n}\}, d \in C, n = 1, 2, \dots$. Clearly, it suffices to prove that of the sets $M_{n,d}$ is in $\mathcal{M}_a$. Let $n, d, u \in M_{n,d}$ and $\varepsilon > 0$ be fixed. There exist $z \in B(u,\varepsilon)$ and $\tilde{z} \in A(z)$ such that $\|\tilde{z} - d\| \geq \frac{1}{2n}$, since $\operatorname{diam} A(B(u,\varepsilon)) \geq \frac{1}{n}$. Put $v = \tilde{z} - d$. Choose an arbitrary $y \in M_{n,d}$. There exists $\tilde{y} \in A(y)$ such that $\|\tilde{y} - d\| \leq \frac{a}{2n}$. Since A is accretive, there exists $w^* \in J(y - z)$ such that $\langle \tilde{y} - \tilde{z}, w^* \rangle \geq 0$. Then

$$\langle v, w^* \rangle = \langle \tilde{z} - d, w^* \rangle = \langle \tilde{y} - d, w^* \rangle - \langle \tilde{y} - \tilde{z}, w^* \rangle \leq \langle \tilde{y} - d, w^* \rangle \leq$$
$$\leq \|\tilde{y} - d\| \cdot \|w^*\| \leq \frac{a}{2n}\|w^*\| \leq a\|v\| \cdot \|w^*\|.$$

Consequently $y - z \notin E_{v,a}$ and thus $M_{n,d} \cap (z + E_{v,a}) = \emptyset$. The proof is complete. ■

As an immediate consequence of Theorem 1, Lemma 4 and Lemma 5 we state

Theorem 2. *Let X be a separable Banach space and $A : X \to 2^X$ be an accretive operator. Then the set*

$$M = \{u \in D(A) : A \textit{ is not norm continuous at } u\}$$

is a first category set. If in addition X is uniformly Fréchet smooth then M is σ-porous.

3. Non-separable case.

Theorem 3. *Let X be a Banach space and $U \subset X$ be a nonempty open set. Let $A : X \to 2^X$ be an accretive operator with $U \subset \operatorname{int} D(A)$ and such that there exists an accretive operator $\widetilde{A} : X \to 2^X$ with the following properties:*

(i) $G(A) \subset G(\widetilde{A})$,

(ii) *$\widetilde{A}$ is norm–weak u.s.c. at each point $u \in U$.*

Then the set $H = \{u \in U : A$ is norm continuous at $u\}$ is a dense G_δ subset of U.

PROOF: Clearly $H = \bigcap_{n=1}^{\infty} \{u \in U : \lim_{\delta \downarrow 0} \operatorname{diam} A(B(u,\delta)) < 1/n\}$. H is a G_δ set since each member of the intersection is open. It suffices to prove that $U \setminus H$ is of the first Baire category.

Let on the contrary $U \setminus H$ be a first category set. Then there exist a positive integer m_0 such that the set

$$D_{m_0} = \{u \in U : \text{ there exist } x \in A(u) \text{ and a sequence } \{(v_k, y_k)\} \subset G(A) \\ \text{such that } \lim_{k\to\infty} v_k = u \text{ and } \|y_k - x\| \geq 1/m_0 \text{ for } k = 1,2,\dots\}$$

is not nowhere dense. Hence there exists an open nonempty subset G of U such that D_{m_0} is dense in G.

We shall construct a sequence $Y_0 \subset Y_1 \subset Y_2 \subset \dots$ of separable subspaces of X by induction.

Choose $u_0 \in D_{m_0} \cap G$ arbitrarily. There exist $x_0 \in A(u_0)$ and a sequence $\{(v_k, y_k)\} \subset G(A)$ such that $\lim_{k\to\infty} v_k = u_0$ and $\|y_k - x_0\| \geq 1/m_0$. Define $Y_0 = \operatorname{lin}(\{u_0\} \cup \{x_0\} \cup \{v_k\}_1^\infty \cup \{y_k\}_1^\infty)$. Clearly Y_0 is separable.

Let $Y_0, Y_1, \dots, Y_s$ be defined. There exists a sequence $\{c_i^{(s)}\}_{i=1}^\infty$ which is a countable dense subset of $Y_s \cap G$. (Note that $Y_s \cap G$ is nonempty since it contains u_0.) For any $i = 1,2,\dots$ there exists a sequence $\{u_{i,n}^{(s)}\}_{n=1}^\infty \subset D_{m_0} \cap G$ such that $\lim_{n\to\infty} u_{i,n}^{(s)} = c_i^{(s)}$. By the definition of D_{m_0}, for any $i,n = 1,2,\dots$ there exist $x_{i,n}^{(s)} \in A(u_{i,n}^{(s)})$ and a sequence $\{(v_{i,n,k}^{(s)}, y_{i,n,k}^{(s)})\}_{k=1}^\infty \subset G(A)$ such that

$$\text{(4)} \qquad \lim_{k\to\infty} v_{i,n,k}^{(s)} = u_{i,n}^{(s)} \quad \text{and} \quad \|y_{i,n,k}^{(s)} - x_{i,n}^{(s)}\| \geq 1/m_0 \qquad \text{for any } k.$$

Define

$$Y_{s+1} = \operatorname{lin}(Y_s \cup \{u_{i,n}^{(s)}\}_{i,n=1}^\infty \cup \{x_{i,n}^{(s)}\}_{i,n=1}^\infty \cup \{v_{i,n,k}^{(s)}\}_{i,n,k=1}^\infty \cup \{y_{i,n,k}^{(s)}\}_{i,n,k=1}^\infty).$$

Put $Y = \overline{\bigcup_{s=0}^{\infty} Y_s}$. It is evident that Y is a closed separable subspace of X and $G_Y = G \cap Y$ is a nonempty open set in Y.

For any $w \in Y$ put $A_Y(w) = \widetilde{A}(w) \cap Y$. The operator $A_Y : Y \to 2^Y$ is accretive on Y.

Let $w \in G_Y$ and $\delta > 0$ by fixed. It is easy to see that there exist positive integers s, i, n, k such that

$$\text{(5)} \qquad \|w - u_{i,n}^{(s)}\| < \delta \text{ and } \|w - v_{i,n,k}^{(s)}\| < \delta.$$

Hence $D(A_Y)$ is dense in G_Y. But $D(A_Y) \cap G_Y = \widetilde{A}^{-1}(Y) \cap G_Y$. Consequently $D(A_y) \cap G_Y$ is closed in G_Y since Y is weak–closed and $\widetilde{A}$ is norm–weak u.s.c. on G (Lemma 3). Hence $G_Y \subset D(A_Y)$.

Now $\lim_{\delta \downarrow 0} \operatorname{diam} A_Y(B(w,\delta)) \geq 1/m_0$ for any $w \in G_Y$, by (4) and (5) ($B(w,\delta)$ is a ball in Y). Consequently an accretive operator A_Y is not norm continuous at any $w \in G_Y$. But this is in contradiction with Theorem 2. ■

The idea of the following two proofs is due to L.Zajíček.

Lemma 6. *Let $u_0 \in X, \varepsilon > 0$ and let $A : B(u_0, \varepsilon) \to X$ be an accretive (singlevalued) mapping such that $\|A(u)\| \leq r$ for u belonging to some dense subset of $B(u_0, \varepsilon)$. Then $\|A(u_0)\| \leq r$.*

PROOF: It is possible to assume $u_0 = 0$ without any loss of generality. Suppose $\|A(0)\| > r$. The density assumption implies the existence of $u \in B(0,\varepsilon) \setminus \{0\}$ such that $\|A(u)\| \leq r$ and

$$\left\|\frac{u}{\|u\|} - \frac{A(0)}{\|A(0)\|}\right\| < \frac{\|A(0)\| - r}{\|A(0)\|}.$$

Then (see a note after Definition 2)
$\|u - 0\| \leq \|(u-0) + t(A(u) - A(0))\|$ for any $t > 0$, or equivalently $1 \leq \|\frac{u}{\|u\|} + t(A(u) - A(0))\|$ for any $t > 0$. Putting $t = 1/\|A(0)\|$ we get

$$1 \leq \left\|\frac{u}{\|u\|} - \frac{A(0)}{\|A(0)\|} + \frac{A(u)}{\|A(0)\|}\right\| < \frac{\|A(0)\| - r}{\|A(0)\|} + \frac{r}{\|A(0)\|} = 1.$$

This is a contradiction. ■

Lemma 7. *Let $A : X \to 2^X$ be an accretive operator with* $\operatorname{int} D(A)$ *nonempty. Then A is locally bounded on some dense open subset of* $\operatorname{int} D(A)$.

PROOF: Let $G \subset \operatorname{int} D(A)$ be any nonempty open set. Denote $G_n = \{u \in G : A(u) \cap \overline{B(0,n)} \neq \emptyset\}$ for $n = 1, 2, \dots$. Then $G = \cup G_n$ and consequently there exists n_0 such that G_{n_0} is dense in some nonempty open subset V of G. Then for any $v \in V$ $\|A(v)\| \leq n_0$. Indeed, it suffices to use Lemma 6 for a proper singlevalued selection of A on some $B(v,\varepsilon) \subset V$. ■

Lemma 8. *Let X be reflexive and Fréchet smooth, and let $A : X \to 2^X$ be a maximal accretive mapping. If A is bounded on some neighborhood of $u_0 \in \operatorname{int} D(A)$ then A is norm–weak u.s.c. at u_0.*

PROOF: Let A be not norm weak u.s.c. at u_0. Then there exist a weak open set W and a sequence $\{(u_n, x_n)\} \subset G(A)$ such that $A(u_0) \subset W$, $\lim_{n\to\infty} u_n = u_0$ and $x_n \notin W$ for $n = 1, 2, \dots$. The assumptions imply that the sequence $\{x_n\}$ is bounded. Hence there exists a subsequence $\{x_k\}$ of $\{x_n\}$ weakly converging to some $x_0 \in X$. It is clear that $x_0 \notin W$. Accretiveness of A implies $\langle x_k - y, J(u_k - v)\rangle \geq 0$ for any $(v, y) \in G(A)$ and any k. J is norm continuous; hence, limiting k to infinity, we get

$$\langle x_0 - y, J(u_0 - v)\rangle \geq 0 \text{ for any } (v,y) \in G(A).$$

Consequently $x_0 \in A(u_0) \subset W$ because of maximality of A, and this is the needed contradiction. ■

As a corollary of Lemma 7, Lemma 8 and Theorem 3 we state the following

Theorem 4. *Let X be a reflexive Fréchet smooth Banach space and let $A : X \to 2^X$ be an accretive operator with* $\operatorname{int} D(A) \neq \emptyset$. *Then A is norm continuous on a residual subset of* $\operatorname{int} D(A)$.

PROOF: Let $\widetilde{A} : X \to 2^X$ be a maximal accretive operator with $G(A) \subset G(\widetilde{A})$ ($\widetilde{A}$ exists by Zorn's lemma) and let $U \subset \operatorname{int} D(A)$ be a dense open subset such that $\widetilde{A}$ is locally bounded on U (Lemma 7). Then $\widetilde{A}$ is norm–weak u.s.c. on U by Lemma 8. Consequently A is norm continuous on a dense G_δ subset H of U (Theorem 3). Evidently, H is residual in $\operatorname{int} D(A)$. ∎

4. Convexity and weak closedness of values of maximal accretive mappings.

The following two propositions are well–known and we give a sketch of proofs only.

Proposition 1. *Let L be a real linear space and f, g be linear functionals on L. Suppose g is not identically equal to zero and for any $x \in L$ the following implications holds:*

$$f(x) \geq 0 \Rightarrow g(x) \geq 0. \tag{6}$$

Then there exists $\alpha > 0$ such that $g = \alpha f$.

SKETCH OF PROOF: It is easy to prove that (6) implies $f^{-1}(0) \subset g^{-1}(0)$. Since g is not identically zero, the sets $f^{-1}(0), g^{-1}(0)$ are subspaces of codimension 1 in L. Thus $f^{-1}(0) = g^{-1}(0)$. Take arbitrary $x_0 \in X \setminus f^{-1}(0) = X \setminus g^{-1}(0)$ such that $f(x_0) > 0$. Then also $g(x_0) > 0$ and it is easy to prove

$$g = \frac{g(x_0)}{f(x_0)} f.$$

Proposition 2. *Let S be a closed nonempty proper subset of X such that both S and $S^c = X \setminus S$ are convex. Then $S = \{x \in X : \langle x, y_0^* \rangle \geq \beta\}$ for some $y_0^* \in X^* \setminus \{0\}$ and $\beta \in R$.*

SKETCH OF PROOF: S, S^c are disjoint nonempty convex sets and S^c is open. By Hahn–Banach Theorem, there exist $y_0^* \in Y^*$ and $\beta \in R$ such that $S \subset \{x \in X : \langle x, y_0^* \rangle \geq \beta\}$ and $S^c \subset \{x \in X : \langle x, y_0^* \rangle < \beta\}$. Clearly $y_0^* \neq 0$. Since $S \cup S^c = X$, the inclusions are in fact equalities.

For simplicity, we shall denote (see (2) in first section) $F_u = F_{u,0} = \{x \in X : \langle x, u^* \rangle \geq 0 \text{ for some } u^* \in J(u)\}$.

Lemma 9. *Let $u \in X$. Then the following two assertions are equivalent:*

(i) *$J(u)$ is a singleton;*
(ii) *F_u is convex.*

PROOF: For $u = 0$ the equivalence is trivial. Let $u \neq 0$. If $J(u)$ is a singleton then the set $F_u = \{x \in X : \langle x, J(u) \rangle \geq 0\}$ is a halfspace and hence convex.

Let F_u be convex. Then $tu \in F_u$ for any $t \geq 0$ and $tu \in F_u^c = X \setminus F_u$ for $t < 0$. Lemma 2 (ii) implies that F_u is closed. It is obvious that F_u^c is convex since $F_u^c = \bigcap_{u^* \in J(u)} \{x \in X : \langle x, u^* \rangle < 0\}$. By Proposition 2, there exist $y_0^* \in X^* \setminus \{0\}$ and $\beta \in R$ such that

$$F_u = \{x \in X : \langle x, y_0^* \rangle \geq \beta\}. \tag{7}$$

$\beta = 0$ since 0 is a boundary point of F_u. Without any loss of generality, it is possible to suppose $\|y_0^*\| = \|u\|$. Choose an arbitrary $u^* \in J(u)$. The definition of F_u and (7) imply that $\{x \in X : \langle x, u^* \rangle \geq 0\} \subset \{x \in X : \langle x, y_0^* \rangle \geq 0\}$. Hence $y_0^* = \alpha u^*$ for some $\alpha > 0$ (Proposition 1). But $\|y_0^*\| = \|u\| = \|u^*\|$, thus $y_0^* = u$. Consequently $J(u) = \{y_0^*\}$. ■

In the following lemma, $\dim J(u)$ means the dimension of a linear hull of $J(u)$.

Lemma 10. *Let $u \in X$. Then the following assertions are equivalent:*

(i) $\dim J(u) < \infty$;

(ii) *F_u is weak closed.*

PROOF: Lemma 10 is trivial for $u = 0$. Let $u \neq 0$. Let $\{v_1^*, \dots, v_n^*\}$ be a basis of the linear space $L = \operatorname{lin} J(u)$. Let $x_0 \in F_u^c = X \setminus F_u = \bigcap_{u^* \in J(u)} \{x \in X : \langle x, u^* \rangle < 0\}$. Then $m := \sup\{\langle x_0, u^* \rangle : u^* \in J(u)\} = \max\{\langle x_0, u^* \rangle : u^* \in J(u)\} < 0$, since $J(u)$ is weak* compact. Any $u^* \in J(u)$ can be written in the form $u^* = \sum_{i=1}^{n} a_i(u^*) v_i^*$ where $a_i(u^*) \in R, i = 1, \dots, n$. Since all norms on finite–dimensional space L are equivalent, there must exist $c_1 > 0$ such that $0 < \max\{|a_i(u^*)| : i = 1, \dots, n\} \leq c_1\|u^*\| = c_1\|u\| =: c$ for any $u^* \in J(u)$. Define $W = \{y \in X : |\langle y - x_0, v_i^* \rangle| < \frac{-m}{2nc}$ for $i = 1, \dots, n\}$. W is a weak neighborhood of x_0. It suffices to prove $W \subset F_u^c$. Let $y \in W$ and $u^* \in J(u)$. Then $\langle y, u^* \rangle = \langle x_0, u^* \rangle + \langle y - x_0, u^* \rangle \leq \langle x_0, u^* \rangle + \sum_{i=1}^{n} |a_i(u^*)| \cdot |\langle y - x_0, v_i^* \rangle| < m + nc(\frac{-m}{2nc}) = m/2 < 0$. Hence $y \in F_u^c$ and the implication (i)⇒(ii) is proved.

Let $\dim J(u)$ be infinite. It is evident that $(-u) \in F_u^c$. We shall show that $-u$ is not in the weak–interior of F_u^c. Let $\{v_1^*, \dots, v_n^*\}$ be an arbitrary finite subset of $X^* \setminus \{0\}$ and $\varepsilon > 0$. Define $W = \{y \in X : |\langle y + u, v_i^* \rangle| < \varepsilon$ for $i = 1, \dots, n\}$ and $L = \operatorname{lin}\{v_1^*, \dots, v_n^*\}$. There exists $u_0^* \in J(u) \setminus L$. Let $w \in X$ be such that $\langle w, u_0^* \rangle > \|u\|^2$ and $\langle w, v^* \rangle = 0$ for any $v^* \in L$. Put $y = w - u$. Clearly $y \in W$ since $\langle y + u, v_i^* \rangle = \langle w, v_i^* \rangle = 0$ for $i = 1, \dots, n$. But $\langle y, u_0^* \rangle = \langle w, u_0^* \rangle - \langle u, u_0^* \rangle = \langle w, u_0^* \rangle - \|u\|^2 > 0$, thus $y \notin F_u^c$ and F_u^c does not contain W. Consequently $-u$ is not a weak interior point of F_u^c, since the sets W form a base of weak neighborhoods of $-u$. ■

Remark. It is possible to prove that the condition (i) from Lemma 10 is equivalent to:

$$u = 0 \text{ or } \operatorname{codim} L_u < \infty, \text{ where } L_u\{v \in X : q'(u, v) = -q'(u, -v)\}$$

(L_u is the linear space of all vectors v such that the norm on X is differentiable in the direction v at u).

The following two theorems will be proved simultaneously.

Theorem 5. *The following assertions are equivalent for any Banach space X:*

(i) *X is Gâteaux smooth;*
(ii) *$A(u)$ is convex for any maximal accretive operator $A : X \to 2^X$ and any $u \in D(A)$.*

Theorem 6. *The following assertions are equivalent for any Banach space X:*

(i) *$\dim J(u) < \infty$ for any $u \in X$;*
(ii) *$A(u)$ is weak closed for any maximal accretive operator $A : X \to 2^X$ and any $u \in D(A)$.*

PROOF: of Theorem 5 and Theorem 6 Let (i) hold. Let $A : X \to 2^X$ be a maximal accretive operator and $u \in D(A)$. The maximality of A implies

$$A(u) = \{x \in X : \forall (v,y) \in G(A) \quad \exists w^* \in J(u-v) \qquad \langle x-y, w^* \rangle \geq 0\} =$$
$$= \bigcap_{(v,y)\in G(A)} \{x \in X : \langle x-y, w^*\rangle \geq 0 \text{ for some } w^* \in J(u-v)\} =$$
$$(8) \qquad = \bigcap_{(v,y)\in G(A)} (y + F_{u-v})$$

and hence (ii) holds by Lemma 9, respectively Lemma 10.

Let (i) not hold. There exists $u \in X$ such that $J(u)$ is not a singleton, respectively $\dim J(u)$ is infinite. Obviously $u \neq 0$. Then F_u is not convex by Lemma 9, respectively F_u is not weakly closed by Lemma 10. Put $A_1(0) = \{0\}$, $A_1(u) = F_u$ and $A_1(v) = \emptyset$ for $v \in X \setminus \{0,u\}$. Then $A_1 : X \to 2^X$ is an accretive operator with $D(A_1) = \{0,u\}$. Let now A be a maximal accretive operator such that $G(A_1) \subset G(A)$. Let $x \in A(u)$. Then there exists $u^* \in J(u-0)$ such that $\langle x-0, u^*\rangle \geq 0$. Hence $x \in F_u = A_1(u)$. Consequently $A(u) = F_u$ and thus (ii) does not hold.

The theorems are proved. ■

Remark. Note that the formula (8) from the proof, and Lemma 2, (ii) immediately imply that $A(u)$ is norm closed for any maximal accretive operator $A : X \to 2^X$ and any $u \in D(A)$.

It would be interesting to know a characterization of Banach spaces X with the following property:
for any maximal accretive $A : X \to 2^X$ the set $A(u)$ is convex (resp. weak closed) for $u \in \operatorname{int} D(A)$.
Does a general Banach space satisfy this property? These problems seem to be open.

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