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Abstract. The asymptotic rate $H_\mu(T)$ of an automorphism T, introduced by K.Winkelbauer in [7], works as its crucial characteristic (e.g. for the existence of finite generators). In case of a flow $\{T_t\}_{t\in\mathbf{R}}$ on a countably generated probability space $(\Omega,\mathcal{F},\mu)$, the relation $H_\mu(T_t)=|t|\cdot H_\mu(T_1)$ (for any $t\in\mathbf{R}\setminus\{0\}$) is derived in the present paper. The asymptotic rate of a flow, defined by $H_\mu(\{T_t\}_{t\in\mathbf{R}})=H_\mu(T_1)$, equals the essential supremum of the entropies of its ergodic components, if such a decomposition exists (provided the separability of $\mathcal{F}$).

Keywords: flow, entropy, asymptotic rate, ergodic measure

Classification: 28D10,28D20

INTRODUCTION

Let $\mathbf{N}$ be the positive integers, $\mathbf{I}$ the integers and $\mathbf{R}$ the reals.

$(\Omega,\mathcal{F},\mu)$ always denotes a probability space. By a partition of Ω we mean any collection $\zeta=\{Z_\alpha,\alpha\in A\}$ of mutually disjoint $\mathcal{F}$–measurable sets with $\Omega=\bigcup_{\alpha\in A} Z_\alpha$. The class of all finite and all at most countable partitions of Ω will be denoted by $\wp_f$ and $\wp$, respectively.

Define a real function η by $\eta(t)=-t\cdot\log t$ for $0<t<1$, $\eta(t)=0$ otherwise, $\log=\log_e$. Recall the conditional entropy of such a partition $\zeta\in\wp$ with respect to a σ–algebra $\mathcal{E}\subset\mathcal{F}$ is given by $h_\mu(\zeta|\mathcal{E})=\int\mathcal{H}_\mu(\zeta|\mathcal{E})(\omega)d\mu(\omega)$, where $\mathcal{H}_\mu(\zeta|\mathcal{E})(\omega)=\sum_{\alpha\in A}\eta(\mu(Z_\alpha|\mathcal{E})(\omega))$ and $\mu(Z_\alpha|\mathcal{E})$ means the conditional probability. The entropy of ζ is defined by $h_\mu(\zeta)=h_\mu(\zeta|\{\emptyset,\omega\})$ and, given $0<\varepsilon<1$, the $\mathcal{E}$–length by

$$L_\mu(\varepsilon,\zeta)=\min\{\mathrm{card}(A'):A'\subset A,\sum_{\alpha\in A'}\mu(Z_\alpha)>1-\varepsilon\}.$$

Put $\wp_\mu=\{\zeta\in\wp:h_\mu(\zeta)<\infty\}$.

Let T be an automorphism of $(\Omega,\mathcal{F},\mu)$ (invertible measure–preserving transformation of Ω onto itself). If $\zeta=\{Z_\alpha,\alpha\in A\}$ is a partition of Ω, then $T^k\zeta=\{T^kZ_\alpha,\alpha\in A\}(k\in\mathbf{I})$ is a partition, too. Another partition $\xi=\{X_\beta,\beta\in B\}$ is a refinement of ζ $(\zeta\le\xi)$ if, for any $\beta\in B, X_\beta\subset Z_\alpha$ for some $\alpha\in A$. Put $\zeta_T^-=\bigvee_{k=1}^{\infty}T^{-k}\zeta$ and, for any $n\in\mathbf{N},\zeta_T^n=\bigvee_{k=0}^{n-1}T^k\zeta$ ($\bigvee$ means the customary operation of the roughest common refinement). The entropy of T is given by

$$h_\mu(T)=\sup_{\zeta\in\wp_\mu}h_\mu(T,\zeta)=\sup_{\zeta\in\wp_f}h_\mu(T,\zeta) \tag{1}$$

where, for $\zeta \in \wp_\mu, h_\mu(T,\zeta) = h_\mu(\zeta|\sigma\zeta_T^-) = \lim_\mu \frac{1}{n} h_\mu(\zeta_T^n)$ (see e.g.[2]; $\sigma\mathcal{M}$ means the smallest σ-algebra over a set-system $\mathcal{M} \subset \exp\Omega$). The asymptotic rate of T, introduced by K.Winkelbauer, is given by

$$H_\mu(T) = \sup_{\zeta\in\wp_\mu} H_\mu(T,\zeta) = \sup_{\zeta\in\wp_f} H_\mu(T,\zeta) \tag{2}$$

where $H_\mu(T,\zeta) = \lim_{\varepsilon\to 0+} \operatorname{limsup}_\mu \frac{1}{n} \log L_\mu(\varepsilon, \zeta_T^n)$ for $\zeta \in \wp_\mu$;the limit $H_\mu(T,\zeta)$ always exists. (The validities of the second equations in (1) and (2) were shown in [2],[9], too.)

By $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ we mean a flow on the probability space $(\Omega, \mathcal{F}, \mu)$, i.e. $\{T_t\}_{t\in\mathbf{R}}$ is a group of its automorphisms with respect to the composition $\circ$ such that

(a) $T_{t+s} = T_t \circ T_s$ for any $t, s \in \mathbf{R}$,
(b) $\varphi(\omega, t) = T_t\omega \quad (\omega \in \Omega, t \in \mathbf{R})$ is an $\mathcal{F} \times \mathcal{B}_\mathbf{R} - \mathcal{B}_\mathbf{R}$ measurable mapping ($\mathcal{B}_\mathbf{R}$ means the Borel sets of $\mathbf{R}$).

If $\mathcal{D}, \mathcal{E} \subset \mathcal{F}$ are sub-σ-algebras such that, for any $D \in \mathcal{D}$ there is $E \in \mathcal{E}$ with $\mu(D \triangle E) = 0$ ($\triangle$ denotes the symmetrical difference), we write $\mathcal{D} \overset{\circ}{\subset} \mathcal{E}$; $\mathcal{D} \overset{\circ}{=} \mathcal{E}$ means $\mathcal{D} \overset{\circ}{\subset} \mathcal{E}$ and $\mathcal{E} \overset{\circ}{\subset} \mathcal{D}$ at the same time. The space $(\Omega, \mathcal{F}, \mu)$ is said to be countably generated if $\mathcal{F} \overset{\circ}{=} \sigma(\{F_n\}_{n=1}^\infty)$ for some sequence $\{F_n\}_{n=1}^\infty$ in $\mathcal{F}$. In fact by this supposition, it has been shown for the flow that

$$h_\mu(T_t) = |t| \cdot h_\mu(T_1) \tag{3}$$

for any $t \in \mathbf{R} \setminus \{0\}$, see [1],[3], compare with Lemma 3.

The aim of the present paper is, first of all, to prove a corresponding relation for the asymptotic rate, too.

Theorem 1. *If $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ is a flow on a countably generated probability space then, for any $t \in \mathbf{R} \setminus \{0\}$,*

$$H_\mu(T_t) = |t| \cdot H_\mu(T_1). \tag{4}$$

Afterwards, the definition below is justified.

Definition. The entropy $h_\mu(\{T_t\}_{t\in\mathbf{R}})$ and the asymptotic rate $H_\mu(\{T_t\}_{t\in\mathbf{R}})$ of a flow $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ on a countably generated probability space is defined by

$$h_\mu(\{T_t\}_{t\in\mathbf{R}}) = h_\mu(T_1) \tag{5}$$

and

$$H_\mu(\{T_t\}_{t\in\mathbf{R}}) = H_\mu(T_1), \tag{6}$$

respectively.

If T is an automorphism of $(\Omega, \mathcal{F}, \mu)$, $\mathcal{I}_T$ denotes the σ–algebra $\{F \in \mathcal{F} : TF = F\}$ of T–invariant measurable sets. For a flow $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$, the σ–algebra of flow–invariant sets is taken as $\mathcal{I} = \bigcap_{t\in\mathbf{R}} \mathcal{I}_{T_t}$. The measure μ is called T–ergodic (ergodic) if there is no $E \in \mathcal{I}_T(E \in \mathcal{I})$ with $0 < \mu(E) < 1 \cdot \mathcal{M}(T)$ denotes the class of all T–invariant probability measures π on $(\Omega, \mathcal{F})$ (i.e. $\pi \circ T^{-1} = \pi$). Put $\mathcal{M}(\{T_t\}_{t\in\mathbf{R}}) = \bigcap_{t\in\mathbf{R}} \mathcal{M}(T_t)$.

Let us consider for a moment an example $\Omega = A^{\mathbf{I}}$, $\mathcal{F} = \sigma\mathcal{V}_A$, where A is an at most countable set and $\mathcal{V}_A$ the class of all elementary cylinders $[\overline{\alpha}]_i^j = \{x \in A^{\mathbf{I}} : (x_i, x_{i+1}, \ldots, x_j) = \overline{\alpha}\}, \overline{\alpha} \in A^{j-i+1}, i \leq j, i, j \in \mathbf{I}$. Put $\gamma_A = \{[\alpha]_0^0, \alpha \in A\}$; it is a measurable partition of $A^{\mathbf{I}}$. Further, define a 1:1 bimeasurable mapping S_A of $A^{\mathbf{I}}$ onto itself (the shift) by $(S_A x)_i = x_{i+1}, x = \{x_i\}_{-\infty}^{\infty}, x \in A^{\mathbf{I}}$. As known, $(A^{\mathbf{I}}, \sigma\mathcal{V}_A)$ is a Polish space when a suitable metric is introduced ($\sigma\mathcal{V}_A$ is the σ–algebra of its Borel sets), so the family of regular conditional probabilities induced by $\mathcal{I}_{S_A}$ with respect to a given S_A–invariant probability measure ϑ on $(A^{\mathbf{I}}, \sigma\mathcal{V}_A)$ always exists. In this special space, denote it by $(\vartheta_x, x \in A^{\mathbf{I}})$. For almost all $x[\vartheta]$, the measures ϑ_x are S_A–invariant. As it has been shown in [4] and [8], the following assertion holds.

Proposition 1. *If $h_\vartheta(\gamma_A) < \infty$ then*

$$h_\vartheta(S_A) = \int h_{\vartheta_x}(S_A) d\vartheta(x), \tag{7}$$

$$H_\vartheta(S_A) = \operatorname{ess.sup}_{[\vartheta]} h_{\vartheta_x}(S_A). \tag{8}$$

(ess. $\sup_{[\vartheta]}$ means the essential supremum modulo ϑ; the supposition $h_\vartheta(\gamma_A) < \infty$ can be omitted, c.f. Lemma 6.)

To obtain such a relation between the entropy and the asymptotic rate of a flow $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ in a more general case, the decomposition into ergodic components of μ is needed. To this end, we shall suppose that $\mathcal{F}$ is even separable, i.e. $\mathcal{F} = \sigma(\{E_n\}_{n=1}^{\infty})$ (strictly) for some sequence in $\mathcal{F}$, and that the family $(m_\omega^{\mathcal{I}}, \omega \in \Omega)$ of regular conditional probabilities induced by $\mathcal{I}$ with respect to μ exists. It represents just the desired ergodic decomposition because almost all $[\mu]$ measures $m_\omega^{\mathcal{I}}$ belong to $\mathcal{M}(\{T_t\}_{t\in\mathbf{R}})$ and are ergodic (Lemmas 7 and 8). Although in general $\mathcal{I} \subsetneq \mathcal{I}_{T_1}$ (compare with (5) and (6)), the next theorem is true.

Theorem 2. *Let $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ be a flow on a probability space whose σ–algebra $\mathcal{F}$ is separable. If there is the family $(m_\omega^{\mathcal{I}}, \omega \in \Omega)$, then*

$$h_\mu(\{T_t\}_{t\in\mathbf{R}}) = \int h_{m_\omega^{\mathcal{I}}}(\{T_t\}_{t\in\mathbf{R}}) d\mu(\omega) \tag{9}$$

and

$$H_\mu(\{T_t\}_{t\in\mathbf{R}}) = \operatorname{ess.sup}_{[\mu]} h_{m_\omega^{\mathcal{I}}}(\{T_t\}_{t\in\mathbf{R}}). \tag{10}$$

2. The conjugacy with the shift and further basic facts.

Let T be an automorphism of $(\Omega, \mathcal{F}, \mu)$. For $E \in \mathcal{F}$ with $\mu(E) > 0$ put $\mu_E(F) = \frac{\mu(E \cap F)}{\mu(E)}, F \in \mathcal{F}; \mu_E \in \mathcal{M}(T)$ if $E \in \mathcal{I}_T$. The measure μ is said to be T–aperiodic if, for any $n \in \mathbf{N}$, each set $F \in \mathcal{F}$ of positive measure contains some $E \subset F (E \in \mathcal{F})$ such that $\mu(E \bigtriangleup T^{-n}E) > 0$. On the contrary, μ is said to be T–purely periodic, if there is a partition $\xi = \{X_n, n \in \mathbf{N}\} \in \wp$ such that $\mu(E \bigtriangleup T^{-n}E) = 0$ whenever $E \subset X_n (E \in \mathcal{F})$ and $n \in \mathbf{N}$.

If μ is not T–purely periodic, there is a T–aperiodic part $\mu_a \in \mathcal{M}(T)$ of μ, i.e. a T–aperiodic measure of the form $\mu_a = \mu_E$ for a certain $E \in \mathcal{I}_T$ such that, if $\mu(\Omega \setminus E) > 0$, $\mu_p = \mu_{\Omega \setminus E}$ is T–purely periodic (it is a consequence of the above terms). Thus

$$\mu = v_\mu \cdot \mu_a + (1 - v_\mu) \cdot \mu_p \tag{11}$$

for $v_\mu = \mu(E)$ if μ_a and μ_p are defined.

Lemma 1. *If $\mu = \sum\limits_n v_n \mu_n$ is an at most countable convex combination in $\mathcal{M}(T)$ then $h_\mu(T) = \sum\limits_n v_n \cdot h_{\mu_n}(T)$ and $H_\mu(T) = \sup\{H_{\mu_n}(T) : v_n > o\}$.*

Lemma 2. *If μ is T–purely periodic then $h_\mu(T) = H_\mu(T) = 0$. In the opposite case, $h_\mu(T) = v_\mu \cdot h_{\mu_a}(T)$ and $H_\mu(T) = H_{\mu_a}(T)$.*

For the proof of Lemma 1 see [7]. The first part of Lemma 2 follows from (1) and (2) directly, the second one from (11).

For an arbitrary partition $\zeta = \{Z_\alpha, \alpha \in A\} \in \wp$ we define an S_A–invariant probability measure μ^ζ on $(A^{\mathbf{I}}, \sigma\mathcal{V}_A)$ by $\mu^\zeta([\overline{\alpha}]_i^j) = \mu(\bigcap\limits_{k=i}^{j} T^{-k} Z_{\alpha_k}), \overline{\alpha} = (\alpha_i, \ldots, \alpha_j) \in A^{j-i+1}, i \leq j, i, j \in \mathbf{I}$. By an examination of the definitions, we obtain that

$$h_\mu(T, \zeta) = h_{\mu^\zeta}(S_A, \gamma_A), \tag{12}$$

$$H_\mu(T, \zeta) = H_{\mu^\zeta}(S_A, \gamma_A) \tag{13}$$

for $\zeta \in \wp_\mu$.

$\widetilde{\mathcal{F}}(\mu)$ denotes the measure–algebra associated with $(\Omega, \mathcal{F}, \mu)$. For any $F \in \mathcal{F}$, the equivalence class containing F will be denoted by $\widetilde{F}$. $\zeta \in \wp$ is a generator (for T, μ) if $\sigma\zeta_T \stackrel{\circ}{=} \mathcal{F}$ $\quad (\zeta_T = \bigvee\limits_{k \in \mathbf{I}} T^k \zeta)$. In such a case the automorphisms T and S_A are conjugated, it means there is a measure–algebra isomorphism $\widetilde{\Phi} : \widetilde{\mathcal{F}}(\mu) \to (\widetilde{\sigma\mathcal{V}_A})(\mu^\zeta)$ satisfying $\widetilde{\Phi} \circ \widetilde{T} = \widetilde{S}_A \circ \widetilde{\Phi}$ $\quad$ ($\widetilde{T}$ and $\widetilde{S}_A$ are induced transformations on the equivalence classes). The entropy and the asymptotic rate are invariant with respect to the conjugacy as we deduce from (1) and (2), so $h_\mu(T) = h_{\mu^\zeta}(S_A)$ and $H_\mu(T) = H_{\mu^\zeta}(S_A)$. If, moreover, $\zeta \in \wp_\mu$, then $h_\mu(T) = h_\mu(T, \zeta)$ and $H_\mu(T) = H_\mu(T, \zeta)$ ([2],[9]).

As it follows from (1),(2),(7),(8) and (12),(13), the inequality

$$H_\mu(T) \geq h_\mu(T) \tag{14}$$

always holds.

Proposition 2. *If $(\Omega, \mathcal{F}, \mu)$ is countably generated and μ is T-aperiodic, then there is a generator $\zeta \in \wp_\mu$ (for T, μ) whenever $h_\mu(T) < \infty$.*

For the proof see [2],[9].

Let $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ be a flow. For $F \in \mathcal{F}, \omega \in \Omega$ and $n \in \mathbf{N}$, we write in short $s_n(F,\omega) = \frac{1}{2n}\int_{-n}^{n} \chi_F(T_t\omega)d\lambda(t)$; χ_F denotes the characteristic function of F and λ the usual Lebesgue measure on $(\mathbf{R}, \mathcal{B}_\mathbf{R})$. The next statement is a consequence of the individual ergodic theorem (see e.g. [5]).

Proposition 3. *For any $F \in \mathcal{F}$,*

$$\lim_n s_n(F,\omega) = \mu(F|\mathcal{I})(\omega) \qquad \mu\text{-a.e.} \tag{15}$$

Proposition 4. *Let $(\Omega, \mathcal{F}, \mu)$ be countably generated. Then $\lim_{t\to 0} \mu(T_tF \triangle F) = 0$ for any $F \in \mathcal{F}$. If μ is ergodic then, for all (with an exception of at most countable set) $t \in \mathbf{R}$, μ is T_t-ergodic, too.*

The proof can be found in (e.g.) [3]. The second part is based on the known fact that μ need not be T_t–ergodic ($t \in \mathbf{R}$), only if $e^{i\theta t} = 1$ for some point θ of the discrete spectrum associated with the flow.

For the basic calculus of the entropic theory, which will be used below, we refer to [2].

3. The proof of Theorem 1.

($(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ is still a flow on a countably generated probability space.)

Lemma 3. *If $t \in \mathbf{R}\setminus\{0\}$ and $E \in \mathcal{F}$ such that $\mu(E) > 0$ and $T_1E = T_tE = E$, then*

$$h_{\mu_E}(T_t) = |t| \cdot h_{\mu_E}(T_1). \tag{16}$$

PROOF: As clearly $h_{\mu_E}(T_t) = h_{\mu_E}(T_{-t})$, it suffices to prove (16) for $t > 0$ only. For t being rational it follows directly from the definition (it holds namely $T_{1/q}E = E$ if $t = p/q$ and p, q are relative prime). Suppose that $t > 1$ is an irrational and put $C = \{i + jt : i, j \in \mathbf{I}\}$; C is dense in $\mathbf{R}$. Let $\zeta \in \wp_f$ and $\varepsilon > 0$. There is $\delta > 0$ such that $h_{\mu_E}(T_s\zeta|\sigma\zeta) < \varepsilon$ whenever $s \in C \cap (-\delta, \delta)$ by the first part of Proposition 4. Take a finite subset $D \subset C \cap \langle 0, 1)$ which is δ–dense in $\langle 0, 1)$, and put $\xi = \bigvee_{s\in D} T_s^{-1}\zeta$. For $n, p \in \mathbf{N}$ let $k = k(n) = [(n+1)t]$ (the integer–part) and $r(p) = \max\{i + s : i + s \le pt, i \in \mathbf{I}, s \in D\}$. By the usual calculus, for any $n \in \mathbf{N}$,

$$\begin{aligned} h_{\mu_E}(\bigvee_{p=1}^{n} T_{pt}^{-1}\zeta) &\le h_{\mu_E}(\bigvee_{i=0}^{k} T_i^{-1}\xi) + h_{\mu_E}(\bigvee_{p=1}^{n} T_{pt}^{-1}\zeta | \bigvee_{i=0}^{k} T_i^{-1}\xi) \le \\ &\le h_{\mu_E}(\bigvee_{i=0}^{k} T_i^{-1}\xi) + \sum_{p=1}^{n} h_{\mu_E}(T_{pt}^{-1}\zeta|T_{r(p)}^{-1}\zeta) = \\ &= h_{\mu_E}(\bigvee_{i=0}^{k} T_i^{-1}\xi) + \sum_{p=1}^{n} h_{\mu_E}(T_{pt-r(p)}^{-1}\zeta|\zeta) < h_{\mu_E}(\bigvee_{i=0}^{k} T_i^{-1}\xi) + n\varepsilon. \end{aligned}$$

As $\lim_n \frac{k(n)}{n} = t$, it holds $h_{\mu_E}(T_t) \le t \cdot h_{\mu_E}(T_1) + \varepsilon$, and so $h_{\mu_E}(T_t) \le t \cdot h_{\mu_E}(T_1)$ because ε was chosen arbitrary. If $0 < t < 1$ (irrational), we can use the relation $h_{\mu_E}(T_t) = \frac{1}{k} \cdot h_{\mu_E}(T_{kt}) \le \frac{kt}{k} \cdot h_{\mu_E}(T_1)$, where $k \in \mathbf{N}$ is taken as to $kt > 1$. The converse follows by the exchange t for 1. ■

For the proof of Theorem 1, let us suppose that $H_\mu(T_t) > |t| \cdot H_\mu(T_1)$ for some $t \in \mathbf{R} \setminus \{0\}$. Denote the T_t–aperiodic part of μ by μ'. According to (14), (16) and Lemma 2, $h_{\mu'}(T_t) < \infty$, and so there is a generator $\zeta = \{Z_\alpha, \alpha \in A\} \in \wp_{\mu'}$ (for T_t, μ') by Proposition 2. Write in short $(\mu')^\zeta = \vartheta$, let $\widetilde{\Phi} : \widetilde{\mathcal{F}}(\mu') \to (\widetilde{\sigma \mathcal{V}_A})(\vartheta)$ be the corresponding measure–algebra isomorphism under which T_t and S_A are conjugated. As $H_\mu(T_t) = H_{\mu'}(T_t) = H_\vartheta(S_A)$, it holds that $\vartheta(F_0) > 0$ for $F_0 = \{x \in A^{\mathbf{I}} : h_{\vartheta_x}(S_A) > |t| \cdot H_\mu(T_1)\}$ by Proposition 1 (since $h_\vartheta(\gamma_A) = h_{\mu'}(\zeta) < \infty$). Take $E_0 \in \widetilde{\Phi}^{-1}\widetilde{F}_0$ such that $E_0 \in \mathcal{I}_{T_t}$; it is possible since $F_0 \in \mathcal{I}_{S_A}$. Further, put for any $k \in \mathbf{N}$ $E_k = T_k E_0 \setminus \bigcup_{j=-k+1}^{k-1} T_j E_0$ and $E_{-k} = T_{-k} E_0 \setminus \bigcup_{j=-k+1}^{k} T_j E_0$, and $E = \bigcup_{k \in \mathbf{I}} E_k$. Thus $T_1 E = T_t E = E$ and $T_t E_k = E_k$ for any k. Let $\mathbf{I}_0 = \{k \in \mathbf{I} : \mu'(E_k) > 0\}$ and, for $k \in \mathbf{I}_0$, take $F_k \in \widetilde{\Phi}(\widetilde{T}_{-k}\widetilde{E}_k)$. The measures $\mu_k = \mu'_{E_k}$ and $\vartheta_k = \vartheta_{F_k}$ are T_t– and S_A–invariant, the automorphisms T_t and S_A (of $(\Omega, \mathcal{F}, \mu_k)$ and $(A^{\mathbf{I}}, \sigma\mathcal{V}_A, \vartheta_k)$, respectively) are under $\widetilde{\Phi} \circ \widetilde{T}_{-k}$ conjugated, and $\vartheta(F_k \setminus F_0) = 0$, so

$$h_{\mu_k}(T_t) = h_{\vartheta_k}(S_A) = \frac{1}{\vartheta(F_k)} \int_{F_k} h_{\vartheta_x}(S_A) d\vartheta(x) > |t| \cdot H_\mu(T_1)$$

by (7) (since clearly $h_{\vartheta_k}(\gamma_A) < \infty$ and $(\vartheta_x, x \in A^{\mathbf{I}})$ corresponds to ϑ_k, too). But it further implies $h_{\mu'_E}(T_t) > |t| \cdot H_\mu(T_1)$ by Lemma 1, which gives a contradiction as $H_\mu(T_1) \ge H_{\mu'_E}(T_1) \ge h_{\mu'_E}(T_1)$. Proof of the converse is the same.

4. The decomposition.

In what follows, the σ–algebra $\mathcal{F}$ is assumed to be separable. Let T be an automorphism of $(\Omega, \mathcal{F}, \mu)$. Recall that $h_\pi(T, \zeta_n) \uparrow h_\pi(T), n \to \infty$, for an arbitrary $\pi \in \mathcal{M}(T)$, if $\{\zeta_n\}_{n=1}^\infty$ is a nondecreasing sequence (with respect to $\le$) in $\wp_f$ satisfying $\sigma(\bigvee_{n=1}^\infty \zeta_n) = \mathcal{F}$ (that exists just by the separability). Let us make the following convention: a sub–σ–algebra $\mathcal{D} \subset \mathcal{F}$ has r.c.p. if there is the family $(m_\omega^{\mathcal{D}}, \omega \in \Omega)$ of regular conditional probabilities induced by $\mathcal{D}$ with respect to μ. The next two assertions we obtain by the use of standard methods (c.f. [4],[2]) employing the calculus of conditional probabilities. The proofs are the same as those of Lemma 2 and Theorem 6 in [6].

Lemma 4. *Let $\mathcal{D}, \mathcal{E}$ be sub–σ–algebras of $\mathcal{F}$ such that $\mathcal{D}$ has r.c.p., $\mathcal{E}$ is separable and $\mathcal{D} \overset{\circ}{\subset} \mathcal{E}$. Then, given $F \in \mathcal{F}$,*

$$m_\omega^{\mathcal{D}}(\{z : \mu(F|\mathcal{E})(z) = m_z^{\mathcal{D}}(F|\mathcal{E})(z)\}) = 1 \tag{17}$$

for almost all $\omega[\mu]$.

Lemma 5. *If* $\mathcal{D} \subset \mathcal{I}_T$ *has r.c.p. then* $m_\omega^{\mathcal{D}} \in \mathcal{M}(T)$ μ*-a.e. and, given* $\zeta \in \wp_f$,

$$h_\mu(T,\zeta) = \int h_{m_\omega^{\mathcal{D}}}(T,\zeta)d\mu(\omega). \tag{18}$$

Corollary. *If* $\mathcal{D} \subset \mathcal{I}_T$ *has r.c.p. then*

$$h_\mu(T) = \int h_{m_\omega^{\mathcal{D}}}(T)d\mu(\omega). \tag{19}$$

Lemma 6. *If* $\mathcal{D} \subset \mathcal{I}_T$ *has r.c.p. and, moreover, if*

$$h_{\mu_E}(T) = \int h_{m_\omega^{\mathcal{D}}}(T)d\mu_E(\omega)$$

whenever $E \in \mathcal{I}_T$ *with* $\mu(E) > 0$*, then*

$$H_\mu(T) = \text{ess.}\sup\nolimits_{[\mu]} h_{m_\omega^{\mathcal{D}}}(T). \tag{20}$$

PROOF: Write $s = \text{ess.}\sup_{[\mu]} h_{m_\omega^{\mathcal{D}}}(T)$. Let $H_\mu(T) > s$ and denote the T-aperiodic part of μ by μ'. It is $h_{\mu'}(T) < \infty$, as $h_\mu(T) < \infty$ by (19), so a generator $\zeta = \{Z_\alpha, \alpha \in A\} \in \wp_{\mu'}$ (for T, μ') exists: the corresponding measure–algebra isomorphism assign by $\widetilde{\Phi}$. $H_\mu(T) = H_{\mu'}(T) = H_\vartheta(S_A) = \text{ess.}\sup_{[\vartheta]} h_{\vartheta_x}(S_A)$, where $\vartheta = (\mu')^\zeta$, by Proposition 1. So $\vartheta(F) > 0$ for $F = \{x \in A^{\mathbf{I}} : h_{\vartheta_x}(S_A) > s\}, \vartheta_F \in \mathcal{M}(S_A)$ and $h_{\vartheta_F}(S_A) > s$ by (7). But this is impossible as taking $E \in \widetilde{\Phi}^{-1}\widetilde{F} \cap \mathcal{I}_T$ (recall $F \in \mathcal{I}_{S_A}$) such that $\mu_E = \mu'_E$, we get $h_{\vartheta_F}(S_A) = h_{\mu_E}(T) = \int h_{m_\omega^{\mathcal{D}}}(T)d\mu_E(\omega) \le s$ by the supposition. On the other hand, $H_\mu(T) < s$ is impossible, too, otherwise $h_{\mu_E}(T) > H_\mu(T) \ge H_{\mu_E}(T)$ for $E = \{\omega : h_{m_\omega^{\mathcal{D}}}(T) > H_\mu(T)\}(E \in \mathcal{I}_T$ as $\mathcal{D} \subset \mathcal{I}_T)$. ■

The proof of Theorem 2.

Let $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in\mathbf{R}})$ be a flow ($\mathcal{F}$ is still assumed to be separable). From now on we always assume that $\mathcal{I}$ has r.c.p. Notice that for any $\mathcal{I}$-measurable function g, $m_\omega^{\mathcal{I}}(\{z : g(z) = g(\omega)\}) = 1$ for almost all $\omega[\mu]$.

Lemma 7. *For almost all* $\omega[\mu], m_\omega^{\mathcal{I}} \in \mathcal{M}(\{T_t\}_{t\in\mathbf{R}})$.

PROOF: There is $N \subset \Omega$ $(N \in \mathcal{F})$ such that $\mu(N) = 0$ and $m_\omega^{\mathcal{I}} \in \mathcal{M}(T_s)$ whenever $\omega \in \Omega \setminus N$ and $s \in Q$ (the rationals). Thus, for such ω, s and arbitrary $t \in \mathbf{R}, F \in \mathcal{F}, m_\omega^{\mathcal{I}}(T_{s+t}F) = m_\omega^{\mathcal{I}}(T_tF)$. According to the definition of a flow, given $F \in \mathcal{F}$ and $\omega \in \Omega \setminus N$, the function $m_\omega^{\mathcal{I}}(T_tf)$ of t is $\mathcal{B}_{\mathbf{R}}$-measurable. Thus, for every $a < b$ $(a, b \in \mathbf{R})$ and $s \in Q$,

$$\int_a^b m_\omega^{\mathcal{I}}(T_tF)d\lambda(t) = \int_{a-s}^{b-s} m_\omega^{\mathcal{I}}(T_{s+t}F)d\lambda(t) = \int_{a-s}^{b-s} m_\omega^{\mathcal{I}}(T_tF)d\lambda(t) \tag{21}$$

by the translation–invariance of the Lebesgue measure λ, which implies $m_\omega^{\mathcal{I}}(T_tF) =$ const. λ-a.e. By application of (21) to each $F \in \mathcal{F}_0$, where $\mathcal{F}_0$ means a countable algebra generating $\mathcal{F}$, we get a $t_\omega \in \mathbf{R}$ such that $m_\omega^{\mathcal{I}} \circ T_t = m_\omega^{\mathcal{I}} \circ T_{t_\omega}$ λ-a.e. But an easy examination shows that $G = \{t \in \mathbf{R} : m_\omega^{\mathcal{I}} \circ T_{t_\omega + t} = m_\omega^{\mathcal{I}} \circ T_{t_\omega}\}$ is an additive subgroup of $\mathbf{R}$, so $G = \mathbf{R}$ (because $\lambda(G) > 0$). ■

Lemma 8. *For almost all $\omega[\mu], m_\omega^{\mathcal{I}}$ is ergodic.*

PROOF: $\mathcal{F} = \sigma\mathcal{F}_0$ for a certain countable algebra $\mathcal{F}_0$. It suffices to show that, given $F \in \mathcal{F}_0, m_\omega^{\mathcal{I}}(F|\mathcal{I})(z) = m_\omega^{\mathcal{I}}(F)\ m_\omega^{\mathcal{I}}$–a.e. for almost all $\omega[\mu]$. Due to (15), $\lim_n s_n(F,z) = \mu(F|\mathcal{I})(z)\ \mu$–a.e. and $\lim_n s_n(F,z) = m_\omega^{\mathcal{I}}(F|\mathcal{I})(z)\ m_\omega^{\mathcal{I}}$–a.e. if $m_\omega^{\mathcal{I}} \in \mathcal{M}(\{T_t\}_{t\in\mathbf{R}})$. The first equality gives $\lim_n s_n(F,z) = \mu(F|\mathcal{I})(z)\ m_\omega^{\mathcal{I}}$–a.e. for almost all $\omega[\mu]$, which implies the assertion because $\mu(F|\mathcal{I})(z) = m_\omega^{\mathcal{I}}(F)\ m_\omega^{\mathcal{I}}$–a.e. for almost all $\omega[\mu]$. ∎

Let $\mathcal{I}'$ be a fixed separable σ–algebra such that $\mathcal{I}' \subset \mathcal{I}_{T_1}$ and $\mathcal{I}_{T_1} \overset{\circ}{\subset} \mathcal{I}'$; notice that $h_\mu(T_1,\zeta) = h_\mu(\zeta|\mathcal{I}' \vee \sigma\zeta_{T_1}^-)$ for $\zeta \in \wp_f$. Put

$$f_\mu(\zeta,\omega) = \mathcal{H}_\mu(\zeta|\mathcal{I}' \vee \sigma\zeta_{T_1}^-)(\omega) \text{ and } f_\mu^*(\zeta,\omega) = \limsup_n \frac{1}{n}\sum_{k=0}^{n-1} f_\mu(\zeta, T_k\omega);$$

$$\int f_\mu^*(\zeta,\omega)d_\mu(\omega) = h_\mu(T_1,\zeta)$$

by the ergodic theorem. Further observe that, for $E \in \mathcal{I}_{T_1}$ with $\mu(E) > 0$ and for $F \in \mathcal{F}$, $\mu_E(F|\mathcal{I}' \vee \sigma\zeta_{T_1}^-) = \mu(F|\mathcal{I}' \vee \sigma\zeta_{T_1}^-)\ \mu_E$–a.e. Fix a nondecreasing sequence $\{\zeta_n\}_{n=1}^\infty$ in $\wp_f$ which satisfy $\sigma(\bigvee_{n=1}^\infty \zeta_n) = \mathcal{F}$. For any $n \in \mathbf{N}$ it holds that $\mu(E) = 0$ for $E = \{\omega\colon f_\mu^*(\zeta_n,\omega) > f_\mu^*(\zeta_{n+1},\omega)\}$, otherwise (since $\mu_E \in \mathcal{M}(T_1)$) $h_{\mu_E}(T_1,\zeta_n) = \int f_{\mu_E}^*(\zeta_n,\omega)d\mu_E(\omega) = \int f_\mu^*(\zeta_n,\omega)d\mu_E(\omega) > \int f_\mu^*(\zeta_{n+1},\omega)d\mu_E(\omega) = h_{\mu_E}(T_1,\zeta_{n+1})$, which is impossible.

Lemma 9. *If μ is ergodic then $\lim_n f_\mu^*(\zeta_n,\omega) = h_\mu(T_1)\ \mu$-a.e.*

PROOF: If $\mu(F_n) > 0$ for some $F_n = \{\omega : f_\mu^*(\zeta_n,\omega) > h_\mu(T_1)\}$ then $\mu(\bigcup_{k\in\mathbf{I}} T_t^k F_n) = 1$ for a certain $t \in \mathbf{R}$ by Proposition 4. Put $E_0 = F_n$ and, for $k \in \mathbf{N}, E_k = T_t^k E_0 \setminus \bigcup_{j=-k+1}^{k-1} T_t^j E_0, E_{-k} = T_t^{-k}E_0 \setminus \bigcup_{j=-k+1}^{k} T_t^j E_0$; it is still $E_k \in \mathcal{I}_{T_1}$ since $f_\mu^*(\zeta_n,.)$ is an $\mathcal{I}_{T_1}$–measurable function. If $\mu(E_k) > 0$ $(k \in \mathbf{I})$, put $\mu_k = \mu_{E_k}$ and $\mu_{0,k} = \mu_{T_{-kt}}E_k$. We get $h_{\mu_k}(T_1) \geq h_{\mu_k}(T_1, T_t^k\zeta_n) = h_{\mu_{0,k}}(T_1,\zeta_n) = \int f_\mu^*(\zeta_n,\omega)d\mu_{0,k}(\omega) > h_\mu(T_1)$, which is a contradiction by Lemma 1. Thus $\lim_n f_\mu^*(\zeta_n,\omega) \leq h_\mu(T_1)\ \mu$–a.e.

If $\mu(F) > 0$ for $F = \{\omega : \lim_n f_\mu^*(\zeta_n,\omega) < a\}$ for some $a < h_\mu(T_1)$, we have $h_{\mu_F}(T_1) = \lim_n h_{\mu_F}(T_1,\zeta_n) = \lim_n \int f_\mu^*(\zeta_n,\omega)d\mu_F(\omega) < a$. Further, for $E \subset T_t^k F$ $(k \in \mathbf{I})$ with $\mu(E) > 0$ and $E \in \mathcal{I}_{T_1}$ it holds $h_{\mu_E}(T_1) = \lim_n h_{\mu_E}(T_1, T_t^k\zeta_n) = \lim_n h_{\mu_{E'}}(T_1,\zeta_n)$ (where $E' = T_t^{-k}E$; we use the fact that $\sigma(\bigvee_{n=0}^\infty T_t^k\zeta_n) = \mathcal{F}$, too). This is equal to $\lim_n \int f_\mu^*(\zeta_n,\omega)d\mu_{E'}(\omega) < a$. So by an analogous argument as in the first part, we obtain a contradiction $h_\mu(T_1) < a$ by Lemma 1. ∎

Lemma 10. $\lim_n f^*_\mu(\zeta_n, \omega) = h_{m^\mathcal{I}_\omega}(T_1)$ *μ-a.e.*

PROOF: For almost all $\omega[\mu]$ it is $m^\mathcal{I}_\omega(\{z : \lim_n f^*_\mu(\zeta_n, z) = h_{m^\mathcal{I}_z}(T_1)\}) = m^\mathcal{I}_\omega(\{z : \lim_n f^*_{m^\mathcal{I}_\omega}(\zeta_n, z) = h_{m^\mathcal{I}_\omega}(T_1)\})$ by the use of Lemma 4 ($\mathcal{D} = \mathcal{I}$ and $\mathcal{E} = \mathcal{I}' \vee \sigma\zeta_T^-$) and by the $\mathcal{I}$-measurability of $h_{m^\mathcal{I}_z}(T_1)$. The last term is equal to one by Lemma 9, which implies the assertion due to the decomposition of μ. ∎

The proof of Theorem 2 will be complete if $h_{\mu_E}(T_1) = \int h_{m^\mathcal{I}_\omega}(T_1) d\mu_E(\omega)$ for an arbitrary $E \in \mathcal{I}_{T_1}$ with $\mu(E) > 0$ (compare with Lemma 6 and the Definition in §1). But it is true:

$$h_{\mu_E}(T_1) = \lim_n h_{\mu_E}(T_1, \zeta_n) = \lim_n \int f^*_\mu(\zeta_n, \omega) d\mu_E(\omega) =$$
$$= \int \lim_n f^*_\mu(\zeta_n, \omega) d\mu_E(\omega) = \int h_{m^\mathcal{I}_\omega}(T_1) d\mu_E(\omega).$$

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