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Abstract. The method for proving the existence of second order ordinary linear differential equations with prescribed (in certain sense) oscillatory properties in simply connected domains is given. As an example we show the existence of second order linear equation every solution of which has infinitely many zeros in the unit disk.

Keywords: Second order ordinary linear differential equation, Schwarzian derivative, interior mapping, Riemannian surface

Classification: 34A20, 30G15

1. Introduction. While the oscillatory behaviour on the real line of solutions of second order linear differential equations has been studied for more than 150 years and plenty of oscillatory criteria have been proved, in the complex plane the zero-point problem has been investigated much less. There are some results concerning special equations (for a survey see [4]), the other results are mostly non-oscillatory criteria. In several papers Z. Nehari and his successors (e.g. [2], [5], [6], [7], [8]) have dealt with the equation

$$y'' + q(z)y = 0 \tag{1}$$

in simply connected domains and have found various conditions on the coefficient $q(z)$ for every solution of (1) not to have more than one zero. On the other hand, in the case that $q(z) \not\equiv 0$ is holomorphic in the whole complex plane it can be proved that there are at most (up to a constant multiple) two solutions of (1) which have a finite number of zero points (see [9]).

So the question if there is a simply connected domain and an equation (1) in it every solution of which has infinitely many zeros, seems to be natural and the aim of this note is to show the existence of such an equation in the unit disk. Our method is based on the well known relation between the equation (1) and the Schwarzian derivative of the ratio of its solutions and it essentially uses topological characterization of holomorphic mappings between Riemannian surfaces derived by S. Stoilow in [10]. The theorem proved in the next section brings a general tool for proving the existence of equations with prescribed oscillatory properties. For instance, the existence of an equation every non-trivial solution of which has zeros but only a finite number or the existence of an equation every solution of which has infinitely many zeros with the exception of one that has a prescribed finite number of zeros can be shown in the way similar to our example.

2.Local homeomorphisms and oscillatory properties. We start with some notation. The complex plane and the extended complex plane (Riemannian sphere) are denoted C and C^*, respectively. The letter K stands for the open unit disk and D is an arbitrary simply connected domain in the complex plane.

A meromorphic function $f : D \to C^*$ is called simple if it is conformal in a neighbourhood of every point in D, especially $f'(z) \neq 0$ for $f(z) \neq \infty$. To such a function we can assign the holomorphic function $\{f, z\} : D \to C$ called the Schwarzian derivative and defined by the formula

$$\{f, z\} = \frac{1}{2}\frac{f'''}{f'} - \frac{3}{4}\left(\frac{f''}{f'}\right)^2$$

for all $z \in D$ such that $f(z) \neq \infty$. The following statement describes the well known relation between the Schwarzian derivative and the equation (1) with a holomorphic coefficient $q(z)$ in D.

Proposition 1. (see [4]) *Let u_1, u_2 be two linearly independent solutions of the equation (1) in a simply connected domain D. Then*

$$f(z) = \frac{u_1(z)}{u_2(z)} \tag{2}$$

is a simple function and

$$\{f, z\} = q(z) \tag{3}$$

Conversely, if f is a solution of (3) then there are two linearly independent solutions u_1, u_2 of the equation (1) such that (2) holds.

Next we show how the function f defined by (2) determines oscillatory properties of the equation (1). Let us decompose all non-trivial solutions of equation (1) into classes such that two solutions belong to the same class exactly if they differ by a constant multiple. The class containing a solution v will be denoted as $[v]$.

Proposition 2 (see [6]. *Let u_1, u_2 be fixed linearly independent solutions of the equation (1) in a simply connected domain D and let f be defined by (2). Then there is a bijection F between C^* and all classes of solutions of the equation (1) such that if $F(a) = [v]$ then*

$$v(z) = 0 \quad \textit{iff} \quad f(z) = a. \tag{4}$$

PROOF: For $a \in C$ define $F(a) = [u_1 - au_2]$ and for $a = \infty$ let $F(a) = [u_2]$. It is clear that (4) is satisfied in both cases. If $v(z) = u(z) = 0$ for some z then u and v belong to the same class. Consequently, the defined mapping is injective. Finally, every solution is a constant multiple either of u_2 or $u_1 - au_2$ for some $a \in C$, which completes the proof. ∎

Further we turn to the topological characterization of holomorphic mappings between Riemannian surfaces. For this purpose we need the notion of the interior mapping (see [10]). Let M and P be topological spaces. A continuous mapping $h : M \to P$ is called interior if it is open and the image of any continuum which differs from a single point is not a single point. The following proposition plays a central role in our considerations.

Proposition 3. (see[**10**], chapter 5) *Let M be a two-dimensional manifold and let $h : M \to C^*$ be an interior mapping. Then there is a Riemannian surface R with a homeomorphism k of R onto M such that $g = h \circ k$ is a holomorphic mapping of R into C^*.*

Now, combining Propositions 1,2 and 3 and the Riemann Mapping Theorem, we obtain the statement which enables us to show the existence of equations with prescribed oscillatory properties.

Theorem. *Let h be a continuous mapping of a simply connected domain D into C^* which is a homeomorphism in some neighbourhood of every point in D. Then there is an equation (1) in K or C and a bijection F between C^* and all classes of solutions of this equation such that if $F(a) = [v]$ then the number of zeros of the solution v is eq   o the cardinality of $h^{-1}(a)$.*

Corollary.

Under the same assumptions there is an equation (1) in K and a bijection F between C^* and all classes of solutions of this equation such that for $F(a) = [v]$ the solution v has infinitely many zeros if $h^{-1}(a)$ is not empty and v has no zero if $h^{-1}(a)$ is empty with the exception of one class $[v]$ where v has infinitely many zeros if $h^{-1}(a)$ contains at least two points and v has no zero otherwise.

PROOF: Let $h : D \to C^*$ be a local homeomorphism. Then it is an interior mapping and, according to Proposition 3, there exists a Riemannian surface R with a homeomorphism $k : R \to M$ such that $g = h \circ k : R \to C^*$ is a holomorphic mapping. R is open and simply connected that is why, according to the Riemann Mapping Theorem (see e.g.[**3**]), there is a conformal mapping l of R onto K or C. Now we put

$$f = g \circ l^{-1}. \tag{5}$$

f is meromorphic and a local homeomorphism in K or C, which means that it is simple. So we can take the equation (1) with a coefficient $q(z) = \{f, z\}$. Due to Propositions 1 and 2 we get a bijection F between C^* and classes of solutions of this equation satisfying (4) for $F(a) = [v]$.That especially means that the number of zeros of v is equal to the cardinality of $f^{-1}(a)$. Now, we realize that k and l are bijections, which completes the proof of Theorem.

We proceed to prove Corollary. First let us suppose that R is conformally mapped on C by l. Then f is defined in C. Let L be a conformal mapping of K onto the halfplane $\{z \in C, \operatorname{Im} z > 0\}$. Then we put

$$F = \bar{f} \circ \exp \circ L, \tag{6}$$

where $\bar{f}$ is the restriction of f to $C \setminus \{0\}$. It is clear that $F : K \to C^*$ is simple. For $a \neq f(0)$ the set $F^{-1}(a)$ is empty if $f^{-1}(a)$ is empty and it is infinite otherwise. Further $F^{-1}(f(0))$ is empty if $f^{-1}(f(0)) = \{0\}$, otherwise it is infinite.

Now, let us consider the case that R is conformally equivalent to K. Let H be a conformal mapping of K onto the halfplane $\{z \in C, \operatorname{Re} z < 0\}$. Taking the function

$$F = \bar{f} \circ \exp \circ H, \tag{7}$$

where $\bar{f}$ is the restriction of f from (5) to $K \setminus \{0\}$, it can be easily shown that it has the same properties as the function defined by (6). Using Propositions 1 and 2 and considering F from (6) or (7) instead of f we can complete the proof of Corollary in the same way as the proof of Theorem. ■

Remark. It is well known that every equation (1) corresponds to Riccati's equation

$$w' + w^2 + q(z) = 0 \tag{8}$$

in such a way that if y is a non-trivial solution of (1) then $\frac{y'}{y}$ is a solution of (8) and vice versa, every solution of (8) can be expressed in this form. So Theorem and Corollary remain true if we replace the words "equation (1)", "classes of solutions" and "zeros" by the words "equation (8)", "solutions" and "poles", respectively.

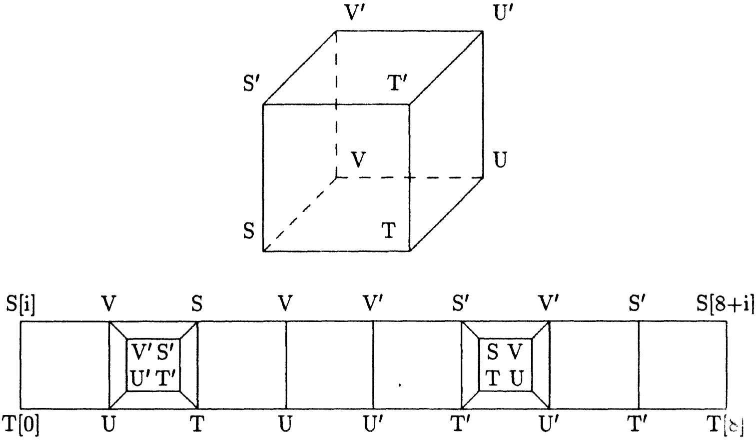
3. Example. Due to the above results, for proving the existence of an equation every solution of which has infinitely many zeros in the unit disk, it is sufficient to construct a local homeomorphism of some simply connected domain D onto the two-dimensional sphere such that the inverse image of every point contains at least two points. As a simply connected domain D we take the strip region $\{z \in C,\ 0 < \text{Im}\ z < 1\}$. First we describe a mapping h which maps the rectangle $P = \{z \in C,\ 0 \le \text{Im} \le 1,\ 0 \le \text{Re}\ z \le 8\}$ onto the surface of the cube $STUVS'T'U'V'$, see the picture.We divide the rectangle P into 16 quadrangles in the way shown in the picture. Let us define h on the rectangle such that

(i) h maps vertices of the quadrangles on the vertices of the cube in the way indicated by the picture in which the vertices of the quadrangles are denoted according to their images,
(ii) h is a homeomorphism on every side XY of some quadrangle and maps it onto the edge XY of the cube,
(iii) h is a homeomorphism on every quadrangle $XYWZ$ and maps it onto the face $XYWZ$ of the cube.

Now, we extend h on the strip $\{z \in C,\ 0 \le \text{Im} \le 1\}$ periodically with the period 8 (it is the length of the side of the rectangle P. It is not difficult to show that the mapping h restricted to the open strip $D = \{z \in C \quad 0 < \text{Im}\, z < 1\}$ is a local homeomorphism and that the inverse image of every point on the surface of the cube is an infinite set with respect to this restricted mapping. Composing h on D with a homeomorphism between the surface of the cube and the two–dimensional sphere we obtain the mapping of the required properties.

Remark. According to [1], every couple of linearly independent solutions of the equation (1) in the real domain I can be written in the form $|\alpha'(t)|^{-\frac{1}{2}} \sin(\alpha(t))$ and $|\alpha'(t)|^{-\frac{1}{2}} \cos(\alpha(t))$, where α is defined on the interval I and $\alpha'(t) \neq 0$. This is not possible in the complex domain. If u_1 and u_2 are arbitrary linearly independent solutions of the equation from the above example, then u_1/u_2 maps K onto the whole C^* while the function tan maps C only on $C^* \setminus \{i, -i\}$.

Picture.



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