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Roman Pol

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NOTE ON A THEOREM OF S. MERCOURAKIS ABOUT WEAKLY K-ANALYTIC BANACH SPACES

R. POL

Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: An alternative proof of a recent result of S. Mercourakis on Banach spaces which are K-analytic or countably determined in the weak topology is given.

Key words: Banach space, K-analytic space, countably determined space.

Classification: 54H05, 46B20

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1. Introduction. In this note we give an alternative proof of a very interesting recent result of S. Mercourakis [M1], [M2] on Banach spaces which are K-analytic or countably determined in the weak topology. Essentially, this proof is closely related to the original line of reasonings of S. Mercourakis, employing the "long sequence of projections" method introduced by Amir and Lindenstrauss [A-L]. However, while Mercourakis refined in an ingenious way the technique of Gul´ko [G1] in his proof, we apply directly some results of Gul´ko [G1] and Vašak [V]. We hope that this approach simplifies the proof; in a natural way, it yields also a proposition (Proposition 3.2) which seems slightly more general than the theorem of Mercourakis (Theorem 3.1).

2. Notation and terminology. We shall explain here only the terminology needed to follow the proof of Proposition 3.2; for other notions, we refer the reader to Negrepontis [N], Mercourakis [M2] or Talagrand [T].

2.1. Let R denote the real line, the cardinality of a set A is denoted by $|A|$ and we write $f \upharpoonright A$ for the restriction of a function $f:X \longrightarrow R$ to an $A \subset X$.

If $F \subset X \times Y$, for each $x \in X$ we let

$F^x = F \cap (\{x\} \times Y)$,

i.e. F^x is the vertical section of F at x. Given a topological space S, we denote by $C_p(S)$ the space of continuous real-valued functions on S endowed with the topology of pointwise convergence; if $f \in C_p(S)$ is bounded, we write $\|f\|$ for the sup-norm of f.

2.2. A space X is K-analytic (countably determined) if there exists an upper-semicontinuous map $\Phi: M \longrightarrow K(X)$ from irrationals (from a subset of irrationals) to the family of compact subsets of X such that $X = \bigcup\{\Phi(t): t \in M\}$.

We say that a subset A of a Banach space E is weakly K-analytic (countably determined) if A has the corresponding property with respect to the weak topology in E.

2.3. Let Γ be a set and let R^Γ be the Tychonoff product of the real line. Given an $x \in R^\Gamma$, let $\mathrm{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. The Σ-product $\Sigma(\Gamma)$ and the "long c_0-space" $c_0(\Gamma)$ are defined by the formulas:

$\Sigma(\Gamma) = \{x \in R^\Gamma : |\mathrm{supp}(x)| \leq \aleph_0\}$,

$c_0(\Gamma) = \{x \in R^\Gamma$: for each $\varepsilon > 0$ the set $\{\gamma : |x(\gamma)| > \varepsilon\}$ is finite$\}$.

Given an $x \in R^\Gamma$ and a $\Lambda \subset \Gamma$ we put

$$x|\Lambda(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in \Lambda, \\ 0 & \text{if } \gamma \notin \Lambda. \end{cases}$$

2.4. The space of Mercourakis $c_{\mathcal{K}}(M \times \Lambda)$. Let M be a separable metrizable space and let Λ be a set. Mercourakis [M1],[M2] defined the following subspace of the Tychonoff product $R^{M \times \Lambda}$ ($M \times \Lambda$ is the set of indices):

$c_{\mathcal{K}}(M \times \Lambda) = \{x \in R^{M \times \Lambda}$: for every compact set $C \subset M$ and every $\varepsilon > 0$ the set $\{(t, \lambda) \in C \times \Lambda : |x(t, \lambda)| \geq \varepsilon\}$ is finite$\}$
(in Mercourakis´ notation, $c_{\mathcal{K}}(M \times \Lambda) = C_1(M \times \Lambda^*)$, where Λ^* is the one-point compactification of Λ); notice that if M is a singleton, $c_{\mathcal{K}}(M \times \Lambda)$ is just $c_0(\Lambda)$.

3. The theorem of Mercourakis

3.1. Theorem ([M2]; Theorem 2.5). Let L be a weakly countably determined subset of a Banach space E of density τ. There exists then a separable metrizable space M, a set Λ of cardinality τ and a bounded linear continuous injection $T: C_p(L) \longrightarrow c_{\mathcal{K}}(M \times \Lambda)$. In case L is K-analytic, one can take as M the irrationals.

We shall derive this theorem of Mercourakis from the following proposition:

3.2. Proposition. Let M be a separable metrizable space, let Γ be a a set and let $F \subset M \times \Sigma(\Gamma)$ be a closed subspace such that for every compact set $C \subset M$ and every $\gamma \in \Gamma$, the set of real numbers $\{x(\gamma):(t,x) \in F$ for some $t \in C\}$ is bounded. There exists then a set Λ with $|\Lambda|=|\Gamma|$ and a collection of continuous linear injections

$$(*) \quad T_t : C_p(F^t) \longrightarrow c_0(\{t\} \times \Lambda), \ \|T_t\| \leq 1, \ t \in M,$$

F^t being the vertical section of F at t, such that the operator

$$(**) \quad T(f)(t,\lambda)=T_t(f \restriction F^t)(t,\lambda)$$

is a bounded linear continuous injection

$$(***) \quad T:C_p(F) \longrightarrow c_{\mathcal{K}}(M \times \Lambda).$$

3.3. Remark. If M is a singleton, this proposition is a particular case of a result of Gul´ko [G1] that for each closed subspace F of $\Sigma(\Gamma)$ there is a linear continuous injection of $C_p(F)$ into $c_0(\Lambda)$. Our proof of Proposition 3.2 is in fact a repetition of Gul´ko´s arguments "uniformly" with respect to $t \in M$ - what emerges from this procedure is just the operator T we are looking for.

4. Proof of Proposition 3.2. The following lemma is a particular case of a result of Gul´ko [G2]; it can be proved easily by a standard reasoning going back to Gleason, cf. Isbell [I; VII, Th. 19], Kuratowski [K; § 49, IX, Th.2] - we give such a simple proof in Sec. 6.2 for the sake of completeness.

Throughout this section F is a subset of $M \times \Sigma(\Gamma)$ satisfying the assumptions of Proposition 3.2.

4.1. Lemma. Let $|\Gamma| > \aleph_0$. There exists an increasing sequence $\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_\xi \subset \ldots \subset \Gamma_\tau = \Gamma$ with $|\Gamma_\xi| < |\Gamma|$ for $\xi < \tau$ and $\Gamma_\xi = \bigcup_{\alpha<\xi} \Gamma_\alpha$ for any limit $\xi \leq \tau$ such that for each ξ, the set F is invariant under the retraction

$$(1) \qquad r_\xi(t,x)=(t,x_\xi \mid \Gamma_\xi),$$

i.e. $r_\xi(F) \subset F$ for $\xi \leq \tau$.

Step (I). Given the retractions $\langle r_\xi : \xi < \tau \rangle$ defined in Lemma 4.1, let

$$(2) \qquad F_\xi = r_\xi(F)=\{(t,x \mid \Gamma_\xi):(t,x) \in F\}.$$

For each section F^t and each $\xi < \tau$ define an operator

$$R_{\xi t}:C_p(F^t)\longrightarrow C_p(F^t_{\xi+1})$$

by the formula

(3) $\quad R_{\xi t}(f)(t,x)={}^1/2\,[f(t,x)-f(t,x|\Gamma_\xi)]$

and let

$$R_\xi :C_p(F)\longrightarrow C_p(F_{\xi+1})$$

be the combination of the operators $R_{\xi t}$, i.e.

(4) $\quad R_\xi(f)(t,x)=R_{\xi t}(f\restriction F^t)(t,x).$

The collection of the operators $\langle R_\xi : \xi<\tau\rangle$ defines the diagonal operator

$$R:C_p(F)\longrightarrow C_p(\bigoplus_{\xi<\tau} F_{\xi+1})\cong \prod_{\xi<\tau} C_p(F_{\xi+1})$$

by the formula

(5) $\quad R(f)=\langle R_\xi(f):\ \xi<\tau\rangle.$

4.2. Lemma. The operator R defined by (5) has the following properties:

(i) R is a linear continuous injection and

$|R(f)(t,x)|\le \|f\restriction F^t_{\xi+1}\|$ for each $(t,x)\in F_{\xi+1}$, $f\in C_p(F)$,

(ii) if $t_n\longrightarrow t_o$ in M, $\xi_1<\xi_2<\ldots<\tau$, $(t_n,x_n)\in F_{\xi_n+1}$, then for every $f\in C_p(F)$ and $\varepsilon>0$ the set $\{n:|R_{\xi_n}(f)(t_n,x_n)|\ge\varepsilon\}$ is finite.

Proof of (i). If $f,g\in C_p(F)$ are distinct, consider the first ordinal η with $f\restriction F_\eta\ne g\restriction F_\eta$; since for every limit α $F_\alpha=\overline{\bigcup_{\beta<\alpha}F_\beta}$, η is non-limit, i.e. $\eta=\xi+1$. Thus $f\restriction F_\xi=g\restriction F_\xi$ and hence $R_\xi(f)\ne R_\xi(g)$, by (3) and (4). Formula (3) yields also the second part of (i).

Proof of (ii). Assume on the contrary that the set $I=\{n:|R_{\xi_n}(f)(t_n,x_n)|\ge\varepsilon\}$ is infinite. Since $C=\{t_n:n=0,1,\ldots\}$ is compact, the assumptions about F guarantee that the set $F\cap C\times\Sigma(\Gamma)$ is sequentially compact and hence the sequence $\langle(t_n,x_n):n\in I\rangle$ has in F a convergent subsequence $(t_{n_i},x_{n_i})\longrightarrow(t_o,x_o)$.

Now, since $\Gamma_{\xi_1}\subset\Gamma_{\xi_1+1}\subset\Gamma_{\xi_2}\subset\Gamma_{\xi_2+1}\subset\ldots$ and $\operatorname{supp}(x_n)\subset\Gamma_{\xi_n+1}$, it follows that also $x_{n_i}|\Gamma_{\xi_{n_i}}\longrightarrow x_o$, and therefore, by continuity of f,

$0=\lim_i 1/2\,[f(t_{n_i},x_{n_i})-f(t_{n_i},x_{n_i}|\Gamma_{\xi_{n_i}})]=\lim_i R_{\xi_{n_i}}(f)(t_{n_i},x_{n_i})$ - a contradiction with the definition of I.

Step (II). We are now in a position to carry out the proof of Proposition 3.2. by transfinite induction with respect to the cardinality of the set Γ.

If $|\Gamma| = \aleph_0$, choose a countable dense subset $\{(t_n,x_n): n \in \omega\}$ of F and let

$$T(f)(t,i) = \begin{cases} 0 & \text{if } t \neq t_i, \\ 1/i\, f(t_i,x_i) & \text{if } t=t_i; \end{cases}$$

then $T: C_p(F) \longrightarrow c_0(M \times \omega)$ is a continuous linear injection, defined "sectionwise", as required.

Let $|\Gamma| = \tau > \aleph_0$ and let us assume that for each $\xi < \tau$ there exists a collection of maps

$$T_{\xi t}: C_p(F^t_{\xi+1}) \longrightarrow c_0(\{t\} \times \Lambda_\xi),\ t \in M, \text{ with } \|T_{\xi t}\| \leq 1,$$

where $|\Lambda_\xi| = |\Gamma_{\xi+1}|$, satisfying the assertions of Proposition 3.2, i.e.

$$T_\xi(f)(t,\lambda) = T_{\xi t}(f \restriction F^t_{\xi+1})(t,\lambda)$$

is a linear continuous injection

$$T_\xi: C_p(F_{\xi+1}) \longrightarrow c_{\varkappa}(M \times \Lambda_\xi)$$

We shall combine the maps $\langle T_{\xi t}: \xi < \tau, t \in M \rangle$ using the operator R defined in Step (I).

Let $\Lambda = \bigoplus_{\xi < \tau} \Lambda_\xi$. Define

$$T_t: C_p(F^t) \longrightarrow \prod_{\xi < \tau} c_0(\Lambda_\xi)$$

by the formula (see (3))

$$T_t(f) = \langle T_{\xi t} \circ R_{\xi t}(f): \xi < \tau \rangle,$$

and let, for $(t,\lambda) \in M \times \Lambda$, $f \in C_p(F)$,

$$T(f)(t,\lambda) = T_t(f \restriction F^t)(t,\lambda).$$

We shall check that

$$T: C_p(F) \longrightarrow c_{\varkappa}(M \times \Lambda),$$

so, in particular,

$T_t : C_p(F^t) \longrightarrow c_0(\Lambda)$ for all $t \in M$.

Let C be a compact subset of M and let us assume that for some $f \in C_p(F)$ and $\varepsilon > 0$ the set

$$H = \{(t,\lambda) \in C \times \Lambda : |T(f)(t,\lambda)| \geq \varepsilon\}$$

is infinite. Since $T_\xi \circ R_\xi(f) \in c_{\varkappa}(M \times \Lambda_\xi)$ for every $\xi < \tau$, each set $H \cap (M \times \Lambda_\xi)$ is finite and therefore one can find a sequence $(t_n, \lambda_n) \in H \cap (C \times \Lambda_{\xi_n})$, where $\xi_1 < \xi_2 < \ldots$. Since $\| T_{\xi_n t_n} \circ R_{\xi_n t_n}(f) \| \geq |T(f)(t_n, \lambda_n)| \geq \varepsilon$ and $\| T_{\xi_n t_n} \| \leq 1$, there exist points $(t_n, x_n) \in F_{\xi_n + 1}$ such that $|R_{\xi_n t_n}(f)(t_n, x_n)| \geq \varepsilon/2$, i.e. $|R_{\xi_n}(f)(t_n, x_n)| \geq \varepsilon/2$. This, however, contradicts the property (ii) established in Lemma 4.2, since the compactness of C allows one to assume, passing if necessary to a subsequence, that the sequence $\langle t_n \rangle_n$ is convergent. Clearly, $\| T_t \| \leq 1$ for each $t \in M$.

5. Proof of Mercourakis' Theorem 3.1. Let L be contained in a Banach space E; one can assume that E is a closed linear span of L, so E is also weakly countably determined, cf. [T]. Denote by E_W the Banach space E equipped with the weak topology. Since L is weakly countably determined, there exists a closed set $H \subset M \times E_W$, where M is a subset of irrationals, such that for each compact set $C \subset M$ the intersection $H \cap (C \times E_W)$ is compact and the projection of H onto the second axis is equal to L (one can take as H the graph of a function Φ described in Sec. 2.2); if L is K-analytic, one can take as M the irrationals. By a theorem obtained by Gul'ko [G1] and Vašak [V], there exists a linear continuous injection $V : E_W — c_0(\Gamma)$, where $|\Gamma|$ is equal to the density of E_W. The map $W(t,u) = (t, V(u))$ embeds H homeomorphically onto a closed subspace $F = W(H)$ of the product $M \times c_0(\Gamma)$ such that for each compact set C in M the intersection $F \cap (C \times c_0(\Gamma))$ is compact. In particular, F satisfies the assertions of Proposition 3.2 and so the required injection into $c_{\varkappa}(M \times \Lambda)$ exists for $C_p(F)$, and hence for $C_p(H)$. Now, L being a continuous image of H, $C_p(L)$ embeds into $C_p(H)$ by a linear isometry preserving the pointwise topology, and this completes the proof.

6. Comments

6.1. Among various applications of his Theorem 5.1, Mercourakis gave some characterizations of compact sets K such that the function space $C_p(K)$

is K-analytic or countably determined [M1; Theorems 3.1, 3.2, 3.3], [M2]. These characterizations are closely related to some characterizations of such compacta obtained independently by Sokolov [S]. The methods used by Sokolov (though, essentially, based also on results of Gul´ko) seem different from the reasoning of Mercourakis or the approach presented in this note.

6.2. For completeness sake, we give here a proof of Lemma 4.1. It is enough to show that, given an infinite set $\Lambda_0 \subset \Gamma$, there exists a set Λ_∞ in Γ containing Λ_0 with $|\Lambda_\infty| = |\Lambda_0|$ such that F is invariant under the retraction $(t,x) \longrightarrow (t,x|\Lambda_\infty)$. Define inductively sets $\Lambda_0 \subset \Lambda_1 \subset \dots$ with $|\Lambda_{i+1}| = |\Lambda_i|$ in the following way: if Λ_i is defined, choose a set $A_{i+1} \subset F$ of cardinality $|\Lambda_i|$ such that the set $\{(t,x|\Lambda_i):(t,x)\in A_{i+1}\}$ is dense in the set $\{(t,x|\Lambda_i):(t,x)\in F\}$, then take $\Lambda_{i+1} = \bigcup\{\text{supp}(x):(t,x)\in A_{i+1}$ for some $t\in M\}$ and finally, put $\Lambda_\infty = \bigcup_i \Lambda_i$. Let $A_\infty = \bigcup_i A_i$, where the sets A_i have been chosen during the construction. One easily checks that A_∞ is dense in the set $\{(t,x|\Lambda_\infty):(t,x)\in F\}$ and since the supports supp(x) of points $x\in A_\infty$ are in Λ_∞, it follows that $(t,x|\Lambda_\infty)\in F$ for every $(t,x)\in F$, the set F being closed in $M \times \Sigma(\Gamma)$.

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Institute of Mathematics, University of Warsaw, 00-901 Warsaw, PKiN IX.p., Poland

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