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METRIZABILITY, GENERALIZED METRIC SPACES AND g-FUNCTIONS

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Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: Characterizations of metrizable and generalized metric spaces in terms of g-function will be discussed.

Key words: g-function, metrizable space, generalized metric space, k-network, cs-network, $\aleph$-space.

Classification: 54E99

The purpose of this note is to summarize recent results on characterization of metrizable and generalized metric spaces in terms of g-function. In this note all spaces are at least T_1, and N denotes the set of all natural numbers. (See J. Nagata [7] for standard terminologies in general topology.)

Definition 1. A function $g: N\times X \longrightarrow \tau$ is called a g-function, where τ denotes the topology of X. (Actually in most of the following discussions we may assume only that g(n,x) is a not necessarily open nbd (=neighborhood) of x.)

We shall begin with a survey of some of the results obtained by Z. Gao and the author in this aspect, which were published in Questions and Answers in General Topology since the Journal has a relatively small circulation. The proofs will be omitted. Then we shall proceed to further results, where proofs will be given, though their methods are not necessarily very new.

Theorem 1 (J. Nagata [8]). A space X is metrizable iff X has a g-function satisfying

(1) for any $x\in X$ and any nbd P of x there is $n\in N$ for which

$x \notin [\cup\{g(n,y) | y\in X - P\}]^-$,

(2) for any $Y \subset X$,

$\bar{Y} \subset \cup\{g(n,y) | Y \in X\}$.

Corollary. A space X is metrizable iff X has a g-function satisfying (1) of Theorem 1 and

(2) $y \in g(n,x)$ implies $x \in g(n,y)$.

Many other metrization theorems follow from Theorem 1, which will be discussed in [6].

Theorem 2 (J. Nagata [8]). A regular space X is Lašnev (= the closed continuous image of a metric space) iff X is Fréchet and has a g-function satisfying

(1) if $\{x_n | n \in N\} \to p \in X$ and if $x_n \in g(n,y_n)$ for all $n \in N$, then $\{y_n | n \in N\} \to p$, where $\to$ denotes convergence,

(2) if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$,

(3) if a sequence $\{x_i | i \in N\}$ satisfies for some $n \in N$

$x_i \notin g(n,x_j)$ or $x_j \notin g(n,x_i)$ whenever $i \neq j$,

then $\{x_i\}$ is discrete in X (i.e. $\{x_i\}$ has no cluster point in X).

Remark. It is obvious that (1) of Theorem 2 is weaker than (1) of Theorem 1, and (3) of Theorem 2 is weaker than (2) of Theorem 1.

Theorem 3 (Z. Gao [3]). A regular space X is k-semistratifiable iff it has a g-function satisfying (1) of Theorem 2.

Remark. It is well-known that a space is stratifiable iff it has a g-function satisfying (1) of Theorem 1.

In reply to questions posed by J. Nagata [8], Z. Gao [4] obtained the following results.

Theorem 4 (Z. Gao [4]). A space X is metrizable iff it is Fréchet and has a g-function g satisfying (1) of Theorem 1 and (3) of Theorem 2.

Remark. It is not known if Fréchet, (1) and (3) of Theorem 2 characterize Lašnev spaces. However, if "Fréchet" is strengthened to "strongly Fréchet", then metrizability follows.

Definition 2. X is called <u>strongly Fréchet</u> if, whenever $\{F_n | n \in N\}$ is a decreasing sequence of subsets of X with a cluster point x, then there are x_n F_n, $n \in N$ such that $\{x_n | n \in N\} \to x$.

Theorem 5 (Z. Gao [4]). A space X is metrizable iff X is strongly Fréchet and has a g-function satisfying (1) and (3) of Theorem 2.

The following theorem improves Theorem 1.

Theorem 6 (Z. Gao [4]). A space X is metrizable iff X has a g-function satisfying (1) of Theorem 2 and (2) of Theorem 1.

The following theorems were motivated by the question:
What is the difference (in terms of g-function) between metrizable spaces and various generalized metric spaces, like semistratifiable, σ-, k-semistratifiable and stratifiable spaces ? E.g. Theorem 1 indicates that the condition (2) is the difference between metrizable spaces and stratifiable spaces. But (2) may be overacting its part. Namely we have suspected that a condition weaker than (2) plus the condition (1) may characterize metrizability. As a result we have obtained

Theorem 7. A space X is metrizable iff it has a g-function satisfying (1) of Theorem 1 and

(2) for any $Y \subset X$, $\bar{Y} \subset \cup\{g^2(n,y) \mid y \in Y\}$, where $g^2(n,y) = \cup\{g(n,z) \mid z \in g(n,y)\}$.

Proof. Necessity of the condition is obvious.
To prove sufficiency, assume $g(n,x) \subset g(n-1,x)$ without loss of generality. Then put

$U_{nm}(x) = X - \{y \mid x \notin g^2(n,y)\}^-$,

$V_{nm}(x) = X - [\cup\{g^2(m,z) \mid z \in X - U_{nm}(x)\}]^-$

if $x \notin [\cup\{g^2(m,z) \mid z \in \{y \mid x \notin g^2(n,y)\}^-\}]^-$.
Otherwise we put

$U_{nm}(x) = V_{nm}(x) = X$.

Now, assume

$x_0 \notin U_{nm}(x)$ for all $x \in Y$.

Then $x_0 \in \{y \mid x \notin g^2(n,y)\}^-$ for all $x \in Y$.
Hence $g(m,x_0) \cap V_{nm}(x) = \emptyset$ for all $x \in Y$.

Assume

$x_0 \in V_{nm}(x)$ for all $x \in Z$ and $U_{nm}(x) \neq X$.

Then, put

$W(x_0) = X - \{z \mid x_0 \notin g^2(m,z)\}^-$.

Then $W(x_0)$ is an open nbd of x_n satisfying

$X - W(x_0) = \{z \mid x_0 \notin g^2(m,z)\}^- \supset \{y \mid x \notin g^2(n,y)\}^- = X - U_{nm}(x)$.

Because $p\in\{y|x\notin g^2(n,y)\}^-$ implies $g^2(m,p)\subset X-V_{nm}(x)\not\ni x_0$, which implies $p\in\{z|x_0\notin g^2(m,z)\}\subset\{z|x_n\notin g^2(m,z)\}^-$. Thus $W(x_0)\subset U_{nm}(x)$ for all $x\in Z$.

Put

$$W_{nm}(x_0)=g(m,x_0)\cap W(x_0)\cap V_{nm}(x_0) \text{ for each } x_0\in X.$$

Then $y\notin U_{nm}(x)$ implies $W_{nm}(y)\cap W_{nm}(x)=\emptyset$, and

$$y\in W_{nm}(x) \text{ implies } W_{nm}(y)\subset U_{nm}(x).$$

Now, let P be an open nbd of x. Then by (1) (of Theorem 1) there is n for which

$$x\notin[\cup\{g^2(n,y)|y\in X-P\}]^-.$$

By (1) (of Theorem 1) and (2) there is n for which

$$x\notin[\cup\{g^2(m,z)|z\in\{y|x\notin g^2(n,y)\}^-\}]^-.$$

Then $U_{nm}(x)=X-\{y|x\notin g^2(n,y)\}^-\subset P$. Namely $\{U_{nm}(x)|n,m\in N\}$ is a nbd base of x. Thus by Theorem VI 2 of J. Nagata [7] X is metrizable.

Theorem 8. A space X is metrizable iff it has a g-function satisfying (1) of Theorem 2 and (2) of Theorem 7.

Proof. Necessity is obvious.

Assume that g satisfies the said conditions. Put

$$U_n(x)=X-\{y|x\notin g^2(n,y)\}^-,\ n\in N,\ x\in X.$$

Then $U_n(x)$ is an open nbd of x by (2). Suppose P is an open nbd of x, and assume $U_n(x)\not\subset P$ for all $n\in N$. Then there are sequences $\{y_n|n\in N\}\subset X-P$ and $\{x_n||n\in N\}$ such that

$$x\in g(n,x_n) \text{ and } x_n\in g(n,y_n).$$

Thus $y_n\longrightarrow x$ follows from (1), which is impossible. Namely $\{U_n(x)|n\in N\}$ is a nbd base of x.

Now, to prove (1) of Theorem 1, assume the contrary that $x\in[\cup\{g(n,x)||x\in X-P\}]^-$ for all $n\in N$. Then pick $z_n\in U_n(x)\cap[\cup\{g(n,x)|x\in X-P\}]$. There is $p_n\in X-P$ such that $z_n\in g(n,p_n)$. Thus, since $z_n\longrightarrow x$, by (1) $p_n\longrightarrow x$, which is impossible. Hence (1) of Theorem 1 follows. Namely X is metrizable by Theorem 7.

Remark. By virtue of Theorem 3, we may say that (2) of Theorem 7 gives the difference between metrizable and k-semistratifiable spaces.

Problem. Find a condition which is weaker than (2) of Theorem 7 and combined together with (1) of Theorem 1 characterizes metrizability of X.

Theorem 9. A space X is metrizable iff it has a g-function satisfying (2) of Theorem 1 and

(1) if $p \in g(n,x_n)$ and $x_n \in g(n,y_n)$ for all $n \in N$, then $y_n \to p$.

Proof. It suffices to show that (1) of Theorem 1 is satisfied by such a g-function g. Suppose p is a point of X and put

$U_n(p)=X-\{y \mid g(n,y) \not\ni p\}$.

Then $U_n(p)$ is a nbd of p by (2). Assume $p \in [\cup\{g(n,y) \mid y \in X-P\}]^-$ for all $n \in N$, where P is an open nbd of p. Then there are $x_n \in U_n(p) \cap g(n,y_n)$ and $y_n \in X-P$, $n \in N$. Then $p \in g(n,x_n)$ and $x_n \in g(n,y_n)$. Hence by (1), $y_n \to p$, which is impossible because $y_n \notin P$, $n \in N$. Thus

$p \notin [\cup\{g(n,y) \mid y \in X-P\}]^-$ for some n,

proving (1) of Theorem 1. Thus metrizability of X is proved.

Remark. As proved essentially by R. Heath [5], the condition (1) characterizes σ-spaces. Thus (2) gives the difference between metrizable and σ-spaces.

Problem. Is it possible to replace (2) of Theorem 9 with the weaker condition (2) of Theorem 7 ?

Theorem 10. A space X is metrizable iff it has a g-function satisfying

(1) if $g(n,x_n) \ni p$ for all $n \in N$, then $x_n \to p$,

(2) for any $Y \subset X$, $[\cup\{g(n,x) \mid x \in Y\}]^- \subset \cup\{g(n-1,x) \mid x \in Y\}$.

Proof. Necessity of the condition is obvious.

Note that (2) of Theorem 1 follows from (2). Thus, by virtue of Theorem 6, it suffices to show (1) of Theorem 2.

Assume $x_n \to p$ and $x_n \in g(n,y_n)$, $n \in N$. Then for each n, $\{x_i\}$ is eventually in $\cup\{g(n,y_i) \mid i \geq i_0\}$ for each i_0. Hence

$p \in [\cup\{g(n,y_i) \mid i \geq i_0\}]^- \subset \cup\{g(n-1,y_i) \mid i \geq i_0\}$.

Hence we can select $n_1 < n_2 < \dots$, a subsequence of N such that $p \in g(i,y_{n_i})$, $i \in N$. Hence $y_{n_i} \to p$.

In fact the above argument shows that any subsequence of $\{y_n\}$ contains a subsequence converging to p. Hence $y_n \to p$, proving (1) of Theorem 2, and thus metrizability of X follows.

Remark. It is well-known that the condition (1) characterizes semistratifiability of X. Thus (2) gives the difference between metrizable and semistratifiable spaces. It is impossible to replace (2) of Theorem 10 with the weaker condition (2) of Theorem 1. In fact it is easy to see that (1) of Theorem 10 plus (2) of Theorem 1 characterizes semimetrizable spaces.

As is well-known, a space X is called an $\aleph$-space if it has a σ-locally finite K-network, where a collection $\mathcal{B}$ of subsets of X is a k-network if for any compact set C and its open nbd C there is a finite subcollection $\mathcal{B}'$ of $\mathcal{B}$ such that $C \subset \bigcup\mathcal{B}' \subset U$. The following implication is well-known. Fréchet $\aleph \longrightarrow$ Lašnev $\longrightarrow$ stratifiable $\longrightarrow$ k-semistratifiable $\longrightarrow$ σ $\longrightarrow$ semistratifiable. The implication, k-semistratifiable $\longrightarrow$ σ follows directly from Theorem 3. Z. Gao [3] also proved that under the continuum hypothesis a Lašnev space X is $\aleph$ iff the character $\chi(X)$ of X does not exceed $\aleph_1$. Observe that all conditions of g-functions in the above theorems except (2) of Theorem 2 are satisfied by $g(n,x)=S_{1/n}(x)=\{y\in X \mid \rho(x,y)<1/n\}$ in case that $\langle X,\rho\rangle$ is a metric space.

Problem. Characterize $\aleph$-spaces and Lašnev spaces in terms of g-function (preferably by use of conditions satisfied by $S_{1/n}(x)$ when X is a metric space). Then characterize metrizability by adding some more conditions so that one can see the difference between metrizability and $\aleph$- (Lašnev) spaces clearly.

Z. Gao [1] gave some characterization of $\aleph$-spaces in terms of g-function. Z. Gao-Y. Hattori [9] characterized regular Fréchet $\aleph$-spaces as the closed s-images of metric spaces. We can also give some characterizations in terms of g-function, though conditions not satisfied by $S_{1/n}(x)$ are involved.

Theorem 11. A regular Fréchet space X is an $\aleph$-space iff it has a g-function satisfying

(1) $y\in g(n,x)$ implies $g(n,y)\subset g(n,x)$,

(2) if $x_i \longrightarrow x\in U$, where U is open, then there are n, $i_0\in N$ such that $x\in X-\bigcup\{g(n,y)\mid y\in X-U\}$ and such that $x\in g(n,x_i)$ for all $i\geq i_0$.

Proof. Necessity. Let $\{\mathcal{F}_i \mid i\in N\}$ be a σ-discrete closed cs-network of X. (See Z. Gao [2].) For each $x\in X$, $n\in N$ we put

$g(n,x)=X-\bigcup\{F\mid x\notin F\in \mathcal{F}'_n\}$.

Then it is easy to see that g satisfies (1) and (2).

Sufficiency. Put

$H_n(x)=X-\bigcup\{g(n,y)\mid x\notin g(n,y)\}$, $n\in N$, $x\in X$.

Then $\{H_n(x)\mid x\in X\}=\mathcal{H}_n$ is closure-preserving because of (1). We claim that $\mathcal{H}=\bigcup_{n=1}^{\infty}\mathcal{H}_n$ is a cs-network of X. To prove it, let $x_i\to x\in U$, where U is open. By (2) we select n such that

$x\in X-\bigcup\{g(n,y)\mid y\in X-U\}$, and

$x\in g(n,x_i)$ for all $i\geq i_0$.

If $x\notin g(n,y)$ for some y, then $x_i\notin g(n,y)$ for all $i\geq i_0$. Because otherwise by (1)

$g(n,x_i)\subset g(n,y)\not\ni x$, which is a contradiction.

Thus $x_i\in H_n(x)$ for all $i\geq i_0$, and $H_n(x)\subset U$. ... (a)
Namely $\mathcal{H}$ is a cs-network.

Now, for each $(n,m)\in N\times N$ and $H\in\mathcal{H}_n$, we put

$\mathcal{G}_{nm}(H)=\{H,\bigcup\{H'\in\mathcal{H}_m\mid H'\cap H=\emptyset\}\}$.

Then it is obvious that

$\mathcal{G}_{nm}=\wedge\{\mathcal{G}_{nm}(H)\mid H\in\mathcal{H}_n\}$
is a discrete closed collection. We claim that $\mathcal{G}=\bigcup_{n,m=1}^{\infty}\mathcal{G}_{nm}$ is a cs-network of X. To prove it, let $x_i\to x\in U$, where U is open. Select $n\in N$ for which (2) and accordingly (a) holds. Then, if $x\in H\in\mathcal{H}_n$, $x_i\in H$ for all $i\geq i_0$. ... (b)
follows. Because, if $x_i\notin H=H_n(y)$ for some $i\geq i_0$, then there is $z\in X$ such that $g(n,z)\not\ni y$ and $g(n,z)\ni x_i$. Hence by (1) and (2) $x\in g(n,x_i)\subset g(n,z)$. Hence $x\notin H_n(y)$, a contradiction.

Now, by use of (a), select $m\in N$ and $H'\in\mathcal{H}_m$ such that

$x_i\in H'$ for all $i\geq i_1$ and $H'\subset X-\bigcup\{H\in\mathcal{H}_n\mid x\notin H\}$.

If $H\in\mathcal{H}_n$ and $x\notin H$, then

$x_i\in H'\subset\bigcup\{H''\in\mathcal{H}_m\mid H''\cap H=\emptyset\}\ni x$ for all $i\geq i_1$.

Combine this fact with (b) to conclude that $x_i\in G$ for all $i\geq\max(i_0,i_1)$, where

$G=[\cap\{H\in\mathcal{H}_n\mid x\in H\}]\cap[\cap\{\bigcup\{H''\in\mathcal{H}_m\mid H''\cap H=\emptyset\}\mid x\notin H\in\mathcal{H}_n\}]\in\mathcal{G}_{nm}$.
On the other hand $G\subset H_n(x)\subset U$ follows from (a). Therefore $\mathcal{G}$ is a σ-discrete cs-network, and hence X is an $\aleph$-space (by Z. Gao [2]).

Although the condition (2) of the following theorem which is stronger than (3) of Theorem 2, is not so beautiful, it explains the difference between $\aleph$-spaces and Lašnev spaces if the theorem is compared with Theorem 2.

Theorem 12. A regular space X is an $\aleph$-space iff it has a g-function satisfying (1), (2) of Theorem 2 and

(3) for each $x \in X$ and $n \in N$ $|\{g(n,y) \mid y \in g(n,x),\ x \notin g(n,y)\}| < \aleph_0$.

Proof. Necessity. Let $\{\mathcal{F}_n \mid n \in N\}$ be a σ-locally finite closed cs-network of X. Then $g(n,x)=X-\cup\{F \in \mathcal{F}_n \mid x \notin F\}$ satisfies the said conditions.

Sufficiency. Put

$$H_n(x)=[\cap\{g(n,y) \mid x \in g(n,y)\}] \cap [\cap\{X-g(n,y) \mid x \notin g(n,y)\}],$$

$$\mathcal{H}_n=\{H_n(x) \mid x \in X\}, \quad \mathcal{H}=\bigcup_{n=1}^{\infty} \mathcal{H}_n.$$

Then it is easy to see that $\mathcal{H}$ is a σ-locally finite wcs-network of X. Hence X is an $\aleph$-space. (See Z. Gao [2].)

Problem. Is it possible to drop (2) from Theorem 12 ?

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