Commentationes Mathematicae Universitatis Carolinae

Viacheslav I. Malykhin
On countable Fréchet-Urysohn spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 4, 695--701

Persistent URL: http://dml.cz/dmlcz/106686

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,4 (1988)

ON COUNTABLE FRÉCHET-URYSOHN SPACES

V.I. MALYKHIN

Dedicated to Professor Miroslav Katětov on his seventieth birthday

Abstract: Modifications of Fréchet-Urysohn property, introduced as ⟨i-FU⟩-properties by A.V. Arhangelskii, are examined. It is shown that ⟨1-FU⟩ and ⟨5-FU⟩-properties are similar to the countability character but differ from it.

Key words: Fréchet-Urysohn property, ⟨i-FU⟩-properties, filter.

Classification: 54A25, 54A35

---

0. Recall that a point x of a topological space is said to be Fréchet-Urysohn point if whenever x is in the closure of a set there is a sequence from this set converging to x.

The Fréchet-Urysohn property is pointwise, i.e. it is determined by a neighbourhood filter of a given point. The character, the pseudocharacter are also pointwise properties, characteristics like the π-character is not. The sequentiality and many kinds of compactness are not pointwise.

There are some modifications of Fréchet-Urysohn property. They can be divided into three groups:

1. The bisequentiality, strong Fréchet property and so on.

These are characterized naturally (see, for example, [2]): by means of maps, by their behaviour under multiplication and so on.

2. The Preiss-Simon property (see [3]), Φ-space in Popov-Ranchin's sense [4] and some others.

3. The ⟨i-FU⟩ - properties introduced by A.V. Arhangelskii [1, 2].

Let us recall the relevant definitions.

A point x of a topological space is called an ⟨i-FU⟩-point, $i=1,2,3,4,5$ if it is a Fréchet-Urysohn point and if for every countable family $\mathcal{L}$ of mutually disjoint sequences converging to x, there exists a sequence ξ converging to x for which the following condition holds:

1) $|\ell \setminus \xi| < \aleph_0$ for every $\ell \in \mathcal{L}$ [1];
2) $|\ell \setminus \xi| < \aleph_0$ for infinitely many $\ell \in \mathcal{L}$;
3) $|\xi \cap \ell| = \aleph_0$ for infinitely many $\ell \in \mathcal{L}$;
4) $\xi \cap \ell \neq \emptyset$ for infinitely many $\ell \in \mathcal{L}$;
5) $|\xi \cap \ell| = \aleph_0$ for every $\ell \in \mathcal{L}$.

Let us note that our definition 5) is equivalent to the definition 5) of [2]. The definitions in [5] and in [7] differ from those given in [2].

All ⟨i-Fu⟩-properties are pointwise. In the sequel, the filter of deleted neighbourhoods of an ⟨i-FU⟩-point is called also ⟨i-FU⟩-filter.

The main results of this paper show that ⟨1-FU⟩ - and ⟨5-FU⟩-properties are similar to the countability character (see Theorem 1 and its corollaries) and, on the other hand, differ from it (see Theorems 2, 3).

First of all on analogies. The following statements are well known.

Statement 1. On a countable set there exist at most $2^{\aleph_0}$ different filters of countable character (i.e. with countable base).

Statement 2. There exist at most $2^{\aleph_0}$ mutually non-homeomorphic Hausdorff countable spaces of countable character.

Let us take up Theorem 1 and its corollaries.

Theorem 1. Let $2^{\aleph_0}=k$ in a model $\mathfrak{M}$, and $\mathfrak{M}'$ be obtained by adding to $\mathfrak{M}$ any number of new Cohen reals. Then in $\mathfrak{M}'$ any ⟨5-FU⟩-filter has a base of power not greater than k.

Corollary 1. It is impossible to define in ZFC a ⟨5-FU⟩-filter of the character $\mathfrak{c}$.

Corollary 2. It is impossible to construct in ZFC a family of mutually non-homeomorphic Hausdorff countable ⟨5-FU⟩-spaces of power greater than $2^{\aleph_0}$.

Let us note now that E. Resnichenko [6] constructed a family of power $2^{\mathfrak{c}}$

---

[1] A sequence ℓ converging to x is the countable subset ℓ, such that $|\ell \setminus Ox| < \aleph_0$ for every neighbourhood Ox of x.

of mutually non-homeomorphic completely regular countable ⟨3-FU⟩ -spaces. In connection with this result the following question was raised:

Is this valid for ⟨1-FU⟩ - and ⟨5-FU⟩ -spaces?

(The general question about maximal power of families of mutually non-homeomorphic ⟨i-FU⟩ -spaces was raised by A.V. Arhangelskii.) The corollary 2 shows that the negative answer to the indicated question is consistent with ZFC.

The following Theorems 2, 3 expose a big difference between ⟨1-FU⟩ - or ⟨5-FU⟩ -properties and the character countability, and demonstrate the independence of corresponding statements from ZFC.

Theorem 2 [CH]. On a countable set, there exist $2^{\mathfrak{c}}$ different ⟨1-FU⟩ - filters and hence there exist $2^{\mathfrak{c}}$ mutually non-homeomorphic countable ⟨1-FU⟩-spaces with only one non-isolated point.

Theorem 3. On a countable set ω there exist two ⟨5-FU⟩-filters F_1, F_2 of uncountable character, such that the countable spaces N_{F_1}, N_{F_2} with only one non-isolated point associated with them have the following properties:

1) N_{F_1}, N_{F_2} are ⟨5-FU⟩ -spaces;

2) $\aleph_0 \notin Sp(N_{F_1})$, $\aleph_0 \notin Sp(N_{F_2})$;

3) for these spaces there exist no completely regular countable compact extensions of countable tightness;

4) the product $N_{F_1} \times N_{F_2}$ is not a Fréchet-Urysohn space;

5) the character of every space N_{F_1}, N_{F_2} equals $\mathfrak{c}$ under LB.

LB denotes Lemma of Booth - one of the most important consequences of Martin axiom MA.

Some additional remarks. Recently A. Dow proved that it is consistent with ZFC that each ⟨1-FU⟩-filter on a countable set has a countable base and also that it is consistent with ZFC that each ⟨5-FU⟩ -filter on a countable set is ⟨1-FU⟩ ([7]).

I. The ⟨i-FU⟩ -properties can be characterized in terms of Stone-Čech compactification of the corresponding discrete space. If we wish to consider only separable regular spaces, then we can consider only filters on a countable set and characterize them in terms of Stone-Čech compactification of the ω .

Let $[\omega]^{\omega} = \{A \subset \omega : |A| = \aleph_0\}$. For $A \in [\omega]^{\omega}$ let $A^* = [A]_{\beta\omega} \setminus \omega$, for $\mathcal{A} \subset [\omega]^{\omega}$ let $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$. Let Int X denote the interior of a set

$X \subset \omega^*$.

There exists a natural correspondence among non-empty closed subsets of ω^*, countable spaces with one non-isolated point only and free filters on ω :

$F \subset \omega^* \longleftrightarrow \dot{N} \cup \{F\} = N_F \longleftrightarrow \Phi = \{A \subset \omega : A^* \supset F\}$.

These objects are called associated.

This correspondence extends over some characteristics of these objects, for example, over the character F in ω^*, the character of the point $\{F\}$ in the space N_F and over the character of Φ .

Proposition 1. Let F be a non-empty closed subset of ω^*, then the associated filter Φ is

o) a Fréchet-Urysohn filter iff $F = [\mathrm{Int}\, F]$, i.e. F is the regular closed subset of ω^*;

1) a $\langle 1\text{-FU}\rangle$-filter iff $F = [\mathrm{Int}\, F]$ and for every countable family $\mathcal{E}^*$ of clopen subsets of ω^*, contained in F, there exists a clopen set $E^* \subset F$, such that $E^* \supset \bigcup \mathcal{E}^*$;

5) a $\langle 5\text{-FU}\rangle$-filter iff $F = [\mathrm{Int}\, F]$ and for every countable family $\mathcal{E}^*$ of clopen subsets of ω^*, contained in F, there exists a clopen set $A^* \subset F$, such that $A^* \cap E^* \neq \emptyset$ for every non-empty $E^* \in \mathcal{E}^*$.

There exist analogous characterizations for $\langle i\text{-FU}\rangle$-filters for i=2,3,4 (by Arhangelskii´s result [2], a $\langle 4\text{-FU}\rangle$-filter is strongly Fréchet, the characterization of which is given in [8].)

II. Proofs of Theorems

The proof of Theorem 1. Add m new Cohen reals using a partially ordered set $\mathcal{F}_m$ consisting of functions p, for which range $p \subseteq \{0,1\}$, dom $p \subset m$, $|\mathrm{dom}\, p| < \aleph_0$ and $p \leq q$ iff $p \supset q$. Let $\mathfrak{M}$ be any ground model, and $\mathfrak{M}'' = \mathfrak{M}[G]$, where G is any $\mathfrak{M}$-generic subset of $\mathcal{F}_m$. It is known that for every $E \in \mathfrak{M}$, $E \subset m$ the set $G_E = G \cap \mathcal{F}_E$ is the $\mathfrak{M}$-generic subset of $\mathcal{F}_E$ and $\mathfrak{M}'' = (\mathfrak{M}[G_E])[G_{m \setminus E}]$, where $G_{m \setminus E}$ is some $\mathfrak{M}[G_E]$-generic subset of $\mathcal{F}_{m \setminus E}$. It is known also that cardinals and their confinalities are preserved by adding Cohen reals, and if not greater than $\mathfrak{c}$ new Cohen reals are added, then arithmetic in $\mathfrak{M}''$ and $\mathfrak{M}$ are the same.

So, let $\mathfrak{M}'$ be obtained by adding m new Cohen reals to a model $\mathfrak{M}$, in which $2^{\aleph_0} = k$.

Let, in $\mathfrak{M}'$, Φ be any $\langle 5\text{-FU}\rangle$-filter on ω and F a closed subset of ω^*

associated with it, i.e. $F=\bigcap\Phi^*$. Let $\mathcal{A}=\{A\in[\omega]^{\omega}:A^*\subset F\}$. It is clear that $[\bigcup\mathcal{A}^*]=F$.

Working in $\mathfrak{M}$, find the set $E\subset m$, $|E|\leq k$ by transfinite induction, such that the following conditions 1), 2) are fulfilled (see below).

Let us denote the model $\mathfrak{M}[G_E]$ for brevity through $\mathfrak{M}_E$. In $\mathfrak{M}'$ let $\Phi_E=\Phi\cap\mathfrak{M}_E$, $\mathcal{A}_E=\mathcal{A}\cap\mathfrak{M}_E$, then $\Phi_E,\mathcal{A}_E\in\mathfrak{M}_E$ and in $\mathfrak{M}_E$ the conditions 1), 2) should be fulfilled:

1) $[\bigcup\mathcal{A}^*_E]=F_E$ ($=\bigcap\Phi^*_E$);

2) Φ_E is the ⟨5-FU⟩-filter.

The construction of the set E is a standard method for finding an intermediate model with necessary properties.

It was shown that the last model $\mathfrak{M}'$ is obtained by adding Cohen reals to $\mathfrak{M}_E$ by means of the partially ordered set $\mathcal{F}_{m\setminus E}$.

So, let us consider the generic extension $\mathfrak{M}_E\xrightarrow{\mathcal{F}_{m\setminus E}}\mathfrak{M}'$.

Let $1\Vdash$ "$|\underset{\cdot}{A}\cap\check{K}|=\aleph_0$" for every $K\in\Phi_E$. We can assume that $\underset{\cdot}{A}\subseteq\omega\times\mathcal{F}_s$ for some countable set $s\subset m\setminus E$. Therefore we can consider in the proof only the case of a countable partially ordered set $\mathcal{P}$ instead of $\mathcal{F}_{m\setminus E}$.

So, let $1_{\mathcal{P}}\Vdash$ "$|\underset{\cdot}{A}\cap K|=\aleph_0$" for every $K\in\Phi_E$.

For every $p\in\mathcal{P}$ let $L_p=\{k\in\omega:\exists q\leq p,\ q\Vdash\text{"}\check{k}\in\underset{\cdot}{A}\text{"}\}$. As it can easily be seen, $|L_p\cap K|=\aleph_0$ for every $K\in\Phi_E$. As Φ_E is ⟨5-FU⟩-filter and the family $\{L_p:p\in\mathcal{P}\}$ is countable, so there exists a sequence L converging to Φ_E, such that $|L\cap L_p|=\aleph_0$ for every L_p. Therefore, $1_{\mathcal{P}}\Vdash$ "$|\underset{\cdot}{A}\cap\check{L}|=\aleph_0$". Note that $L\in\mathcal{A}_E$.

If in $\mathfrak{M}'$ A is such that $|A\cap K|=\aleph_0$ for every $K\in\Phi_E$, then there exists some $L\in\mathcal{A}_E$, such that $|L\cap A|=\aleph_0$. It follows that Φ_E is the base of Φ . Let us note that in $\mathfrak{M}'$ the power of Φ_E is not greater than k. This completes the proof of Theorem 1.

The proof of Theorem 2. As it is known under CH, there exist $2^{\mathfrak{c}}$ of different P-points in ω^* . As it was noted in [7], for every P-point $p\in\omega^*$ there exists an open set V_p in ω^*, having only one boundary point p which is also the unique accumulation point of $\omega^*\setminus V_p$. Hence, $[V_p]=V_p\cup\{p\}$ is the closed subset such that the filter Φ_p associated with it is ⟨1-FU⟩. If p, q are different P-points in ω^*, then Φ_p, Φ_q are also different ⟨1-FU⟩-filters, hence there exist $2^{\mathfrak{c}}$ of different ⟨1-FU⟩-filters on ω . This completes the proof of Theorem 2.

The proof of Theorem 3. F. Hausdorff (see [9]) and N.N. Luzin [10] constructed in ZFC two families $\mathcal{A}$, $\mathcal{B}$ of infinite subsets of ω with the following properties:

1) $\mathcal{A}=\{A_\alpha:\alpha\le\omega_1\}$, $\mathcal{B}=\{B_\beta:\beta\le\omega_1\}$;

2) $A^*_\alpha\subset A^*_\beta$, $B^*_\alpha\subset B^*_\beta$ for any $\alpha<\beta<\omega_1$;

3) $(\bigcup\mathcal{A}^*)\cap(\bigcup\mathcal{B}^*)=\emptyset$;

4) $[\bigcup\mathcal{A}^*]\cap[\bigcup\mathcal{B}^*]\neq\emptyset$.

Now such pair is called the Hausdorff-Luzin gap.

Let $F_1=[\bigcup\mathcal{A}^*]$, $F_2=[\bigcup\mathcal{B}^*]$. The filters Φ_1, Φ_2 associated with F_1, F_2 are ⟨5-FU⟩-filters. Let us consider the associated spaces $N_{F_1}=\omega\cup\{F_1\}$, $N_{F_2}=\omega\cup\{F_2\}$. These are ⟨5-FU⟩-spaces. As $F_1\cap F_2\neq\emptyset$ but $\mathrm{Int}(F_1\cap F_2)=\emptyset$, one has $\langle\{F_1\},\{F_2\}\rangle\in[\{\langle n,n\rangle:n\in\omega\}]$ in the product $N_{F_1}\times N_{F_2}$. However, there exists no sequence of the set $\{\langle n,n\rangle:n\in\omega\}$ which converges to the point $\langle\{F_1\},\{F_2\}\rangle$, hence the product $N_{F_1}\times N_{F_2}$ is not a Fréchet-Urysohn space.

Let us consider now the space N_{F_1} (the arguments for the space N_{F_2} are identical). The space N_{F_1} has a compact extension bN_1, which is obtained from $\beta\omega$ by collapsing the closed set F_1 to a point $\{F_1\}$. As it is easy to see, the tightness of this point $\{F_1\}$ in bN_1 is uncountable, from which it follows that $\varkappa_0\notin\mathrm{Sp}(N_{F_1})$. Recall that Sp(X) is the spectrum of frequences, a special characteristic of a space X which was introduced by A.V. Arhangelskii [1] to investigate the behaviour of tightness by multiplication of the space X with other spaces.

It follows directly from Proposition 2 of [8] that every space N_{F_1}, N_{F_2} has no countably compact completely regular extension of countable tightness.

Let us prove the conclusion 5) of Theorem 3. It is known under LB that if $\mathcal{E}\subset[\omega]^\omega$, $\mathcal{E}^*$ is a centered family and $|\mathcal{E}|<\mathfrak{c}$, then $\mathrm{Int}(\bigcap\mathcal{E}^*)\neq\emptyset$. Let us suppose that $\chi(F_1,\omega^*)=\lambda<\mathfrak{c}$; then $\omega^*\setminus F_1=\bigcup\mathcal{K}^*$, where $\mathcal{K}\subset[\omega]^\omega$, $|\mathcal{K}|=\lambda$. For our situation, the family $\mathcal{E}=\{\omega\setminus A_\alpha:\alpha\in\omega_1\}\cup\{\omega\setminus K:K\in\mathcal{K}\}$ has the power $\lambda<\mathfrak{c}$ and $\mathcal{E}^*$ is a centered family, however, it is easy to see that $\mathrm{Int}(\bigcap\mathcal{E}^*)=\emptyset$. This contradiction completes the proof of Theorem 3.

Bibliography

[1] A.V. ARHANGELSKII: Spektr chastot topologicheskogo prostranstva i klassi-

fikaciya prostranstv, Doklady AN SSSR, 206(1972), 265-268.

(A.V. ARHANGELSKII: The frequency spectrum of a topological space and the classification of spaces, Sov. Math. Dokl. 13(1972), 265-268)

[2] A.V. ARHANGELSKII: Spektr chastot topologicheskogo prostranstva i operaciya proizvedeniya, Trudy Moskov. Matem. Ob-va 40(1979).

(A.V. ARHANGELSKII: The frequency spectrum of a topological space and the product operation, Trans. Moscow Math. Soc. (1981), Issue 2, 163-200)

[3] A.V. ARHANGELSKII, V.V. TKAČUK: Prostranstva funkcii i topologicheskie invarianty, Moskva, izd-vo MGU, 1985.

(A.V. ARHANGELSKII, V.V. TKAČUK: Function spaces and topological invariants)

[4] V.V. POPOV, D.B. RANCHIN: Ob odnom usilenii svoïstva Freshe-Urysona, Vestnik Mosk. un-ta, (1979), No 2, 75-80.

(V.V. POPOV, D.V. RANCHIN: About some strengthening of Fréchet-Urysohn property)

[5] T. NOGURA: The product of $\langle \alpha_i$-spaces$\rangle$, Topol. Appl. [illegible]

[6] E.A. REZNICHENKO: O kolichestve schetnykh prostranstv Freshe-Urysona, v sb. "Nepreryvnye funkcii na topologicheskikh prostranstvakh", Riga, izd-vo LGU, 1986, s. 147-154.

(E.A. REZNICHENKO: On the number of countable Fréchet-Urysohn spaces)

[7] A. DOW: Two classes of Fréchet-Urysohn spaces, Preprint, 1988.

[8] V.I. MALYKHIN: O schetnykh prostranstvakh, ne imeyushchikh bikompaktifikacii schetnoï tesnoty, Doklady AN SSSR 206(1972),1293-1296.

(V.I. MALYKHIN: On countable spaces having no compactifications of countable tightness, Sov. Math. Dokl. 13(1972))

[9] F. ROTHBERGER: On some problem of Hausdorff and Sierpinski, Fund. Math. 35(1948), 29-45.

[10] N.N. LUZIN: O chastyakh natural´nogo ryada, Doklady AN SSSR 40(1943), 195-199.

(N.N. LUZIN: About parts of the set of natural numbers)

Vešnjakovskaja 19-255, Moscow 111539, USSR

(Oblatum 15.6. 1988)