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An isomorphical classification of function spaces of zero-dimensional locally compact separable metric spaces

Jan Baars and Joost de Groot

Abstract: In this paper we derive an isomorphical classification of the function spaces $C_p(X)$ and $C_0(X)$, for zero-dimensional locally compact separable metric spaces X.

Key words: Function space.

AMS Classification: 54C35, 57N17.

0 Introduction

By a space we mean a separable metric space.

For a space X let $C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous }\}$. If we endow $C(X)$ with the topology of pointwise convergence then we write $C_p(X)$ (i.e. we regard $C(X)$ to be a subspace of $\mathbb{R}^X$). We can also endow $C(X)$ with the compact-open topology and then we write $C_0(X)$.

In [2] Bessaga and Pelczyński presented an isomorphical classification of the spaces $C_0(X)$, where X is a zero-dimensional compact space. In this note we prove that a similar classification can be derived if we replace $C_0(X)$ by $C_p(X)$.

Furthermore we give a complete isomorphical classification of the spaces $C_p(X)$ and $C_0(X)$, where X is a locally compact zero-dimensional space. The classification is such that for two locally compact zero-dimensional spaces X and Y it follows that $C_p(X)$ is linearly homeomorphic to $C_p(Y)$ if and only if $C_0(X)$ is linearly homeomorphic to $C_0(Y)$. In general this is not the case, since in [5] Pestov proved that if $C_p(X)$ is linearly homeomorphic with $C_p(Y)$, then $\dim X = \dim Y$. From this we have $C_p(C)$ is not linearly homeomorphic to $C_p(\mathrm{I})$ (here C denotes the Cantor discontinuum and I denotes the unit interval). However by Miljutin's theorem ([6, page 379]) we have $C_0(C)$ is linearly homeomorphic to $C_0(\mathrm{I})$.

1 Preliminaries

Let X be a space. For every ordinal α we define $X^{(\alpha)}$, the ***α-th derivative***, by transfinite induction as follows:

a) $X^{(0)} = X$ and $X^{(1)} = \{x \in X \mid x$ is an accumulation point of $X\}$.

b) If α is a successor, say $\alpha = \beta + 1$, then $X^{(\alpha)} = (X^{(\beta)})^{(1)}$.

c) If α is a limit ordinal then $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$.

For every countable compact space X there is an ordinal $\alpha < \omega_1$, such that $X^{(\alpha)} = \emptyset$ [5, page 149]. So for a countable compact space X we can define the ***scattered height*** $\varkappa(X)$ by the smallest ordinal α such that $X^{(\alpha)} = \emptyset$. Because X is compact, $\varkappa(X)$ is a successor.

Notice that the isolated points of a countable compact space X are dense in X.

For every pair of ordinals α, β let $[\alpha, \beta] = \{\gamma \mid \alpha \leqq \gamma \leqq \beta\}$ and $[\alpha, \beta) = \{\gamma \mid \alpha \leqq \gamma < \beta\}$, provided with the order-topology.

We have the well-known [6, page 155]:

1.1 **THEOREM:** (Sierpiński-Mazurkiewicz) *Let X be a countable compact space. Then $X \approx [1, \omega^{\alpha} \cdot m]$ if and only if $\varkappa(X) = \alpha + 1$ and $X^{(\alpha)}$ contains m points (m finite).*

An ordinal α is a ***prime component*** whenever $\alpha = \beta + \delta$ for ordinals β and δ, then $\delta = 0$ or $\delta = \alpha$.

For every ordinal α denote by α' the largest prime component which is less than or equal to α.

Well-known facts about ordinals (see [6] and [7]) are formulated in:

1.2 **PROPOSITION:** *Let α be an ordinal.*

a) α is a prime component iff there is an ordinal μ such that $\alpha = \omega^{\mu}$.

b) $\alpha = \alpha' \cdot n + \gamma$ for certain $n < \omega$ and $\gamma < \alpha'$.

c) $\alpha = \beta + \omega^{\tau}$ for certain β and τ with $\beta = 0$ or $\beta \geqq \omega^{\tau}$.

1.3 **LEMMA:** *Let α and β be ordinals such that $\alpha \geqq \omega$ and $\alpha \leqq \beta < \alpha^{\omega}$. Then*

$$\alpha' \leqq \beta < (\alpha')^{\omega}.$$

PROOF: Since $\alpha \leqq \beta < \alpha^{\omega}$, there is $n \in \mathbb{N}$ such that $\alpha \leqq \beta < \alpha^{n}$. Furthermore by proposition 1.2a $(\alpha')^2$ is a prime component, from which we may conclude $\alpha' \leqq \alpha < (\alpha')^2$. So it follows that $\alpha' \leqq \beta < (\alpha')^{2n} < (\alpha')^{\omega}$. □

For more information about ordinals we refer to [4].

2 Compact zero-dimensional spaces

In this section we derive an isomorphical classification of the spaces $C_p(X)$, where X denotes any compact zero-dimensional space. In [2] Bessaga and Pelczyński derived a similar classification for the spaces $C_0(X)$.

First we fix some notation. Let X be a space and $A \subset X$ closed. By $C_{p,A}(X)$ we denote the subspace of $C_p(X)$ of all functions vanishing on A.

Let $Y_{X,A}$ be the quotient space obtained from X by identifying A to a single point, say ∞. Let $p: X \to Y_{X,A}$ be the quotient map between X and $Y_{X,A}$. Clearly $Y_{X,A}$ need not be separable metric, violating our convention that all spaces are separable metric. However if X is compact then p is perfect, so $Y_{X,A}$ is a space. If it is obvious which space X and subset A of X we mean, we simply write Y instead of $Y_{X,A}$. Furthermore let $C_{p,0}(Y_{X,A}) = \{f \in C_p(Y_{X,A}) \mid f(\infty) = 0\}$.

Finally, if X and Y are linear spaces then the symbol "$X \sim Y$" means that X is linearly homeomorphic to Y.

We now come to the following:

2.1 **LEMMA:** *Let X be a zero-dimensional space and let A be a closed subset of X. Then*

$$C_p(X) \sim C_{p,A}(X) \times C_p(A).$$

PROOF: Define $\varrho: C_p(X) \to C_p(A)$ by $\varrho(f) = f\,|A$. Notice that ϱ is a continuous linear map. Because X is zero dimensional, there is a retraction $r: X \to A$ (see [3]). Define $\xi: C_p(A) \to C_p(X)$ by $\xi(f) = f \cdot r$. Notice that ξ is a continuous linear map, and that $\varrho \cdot \xi = id_{C_p(A)}$.

Now define $h: C_p(X) \to C_{p,A}(X) \times C_p(A)$ by

$$h(f) = (f - \xi\varrho(f),\ \varrho(f)).$$

We have to prove that h is well defined. Take an arbitrary $f \in C_p(X)$. It is obvious that $\varrho(f) \in C_p(A)$ and that $f - \xi\varrho(f) \in C_p(X)$. Furthermore

$$(f - \xi\varrho(f))\,|A = \varrho(f - \xi\varrho(f)) = \varrho(f) - \varrho\xi\varrho(f) = \varrho(f) - \varrho(f) \equiv 0,$$

so $f - \xi\varrho(f) \in C_{p,A}(X)$.

That h is continuous and linear is a triviality. We show that h is a linear homeomorphism. For that define $i: C_{p,A}(X) \times C_p(A) \to C_p(X)$ by

$$i(f, g) = f + \xi(g).$$

It is trivial that i is well defined, continuous and linear. Furthermore, $i \cdot h = id_{C_p(X)}$ and $h \cdot i = id_{C_{p,A}(X) \times C_p(A)}$, so h is a linear homeomorphism. □

2.2 **LEMMA:** *Let X be a compact space and let A be a closed subset of X. Then*

$$C_{p,A}(X) \sim C_{p,0}(Y).$$

PROOF: For every function $f: X \to \mathbb{R}$ which is constant on A there is a unique function $\tilde{f}: Y \to \mathbb{R}$ such that $\tilde{f} \cdot p = f$. By the quotient topology of Y with respect to p we have that f is continuous if and only if $\tilde{f}$ is continuous.

If $f\,|A \equiv 0$ then $\tilde{f}(\infty) = 0$, which implies that $f \in C_{p,A}(X)$ if and only if $\tilde{f} \in C_{p,0}(Y)$.

If we now define $\phi: C_{p,A}(X) \to C_{p,0}(Y)$ by $\phi(f) = \tilde{f}$, then ϕ is a well defined linear bijection. Let $f \in C_{p,A}(X)$, $y_1, \ldots, y_n \in Y$, $\epsilon > 0$ and

$$U(\phi(f), y_1, \ldots, y_n, \epsilon) = \{g \in C_{p,0}(Y) \mid |g(y_i) - \phi(f)(y_i)| < \epsilon \ (i \leq n)\}.$$

For every $i \leq n$ choose $x_i \in p^{-1}(y_i)$. Then $f \in V = V(f, x_1, \ldots, x_n, \epsilon)$ (with its obvious meaning) and $\phi(V) \subset U$, which proves that ϕ is continuous. Finally, let $f \in C_{p,0}(Y)$, $x_1, \ldots, x_n \in X$, $\epsilon > 0$ and for every $i \leq n$ let $y_i = p(x_i)$. Then $f \in U = U(f, y_1, \ldots, y_n, \epsilon)$ and $\phi^{-1}(U) \subset V(\phi^{-1}(f), x_1, \ldots, x_n, \epsilon)$, which proves that ϕ^{-1} is continuous. So ϕ is a linear homeomorphism. □

From the last lemmas we have the useful

2.3 **COROLLARY:** *Let X be a zero-dimensional compact space and let A be a closed subset of X. Then $C_p(X) \sim C_{p,0}(Y) \times C_p(A)$.* □

The next lemma gives a classification of the spaces $Y = Y_{X,A}$ if X and A are of a special form (cf corollary 2.5).

2.4 **LEMMA:** *Let X be a countable compact space and let $A = X^{(\alpha)}$ for some $\alpha < \varkappa(X)$. Then a) for every $\beta \leqq \alpha$, $p(X^{(\beta)}) = Y^{(\beta)}$,*

b) $Y^{(\alpha)} = \{\infty\}$.

PROOF: Notice that for every $\beta \leqq \alpha$ we have $p(X^{(\beta)}) = X^{(\beta)}\backslash A \cup \{\infty\}$.
We prove, by induction on β, that for every $\beta \leqq \alpha$, $Y^{(\beta)} = X^{(\beta)}\backslash A \cup \{\infty\}$.
For $\beta = 0$ this is a triviality, so let $0 < \beta \leqq \alpha$ and assume it is true for every $\gamma < \beta$.

Case 1: β is a successor, say $\beta = \gamma + 1$.
Then $Y^{(\gamma)} = X^{(\gamma)}\backslash A \cup \{\infty\}$. We first prove that $\infty \in Y^{(\beta)}$. Take therefore an isolated point x in A. Since $x \in X^{(\beta)}$, there is a sequence $(x_n)_{n\in \mathbb{N}}$ in $X^{(\gamma)}\backslash A$ which converges in X to x. This means that $(x_n)_{n\in\mathbb{N}}$ is a sequence in $Y^{(\gamma)}$, which converges in Y to ∞, from which it follows that $\infty \in Y^{(\beta)}$.
Now $Y^{(\beta)} = (X^{(\gamma)}\backslash A)^{(1)} \cup \{\infty\} = (X^{(\gamma)})^{(1)}\backslash A \cup \{\infty\} = X^{(\beta)}\backslash A \cup \{\infty\}$.

Case 2: β is a limit ordinal.
Then $Y^{(\beta)} = \bigcap_{\gamma<\beta} Y^{(\gamma)} = \bigcap_{\gamma<\beta} (X^{(\gamma)}\backslash A \cup \{\infty\}) = (\bigcap_{\gamma<\beta} X^{(\gamma)})\backslash A \cup \{\infty\} = X^{(\beta)}\backslash A \cup \{\infty\}$.

By a) we have $Y^{(\alpha)} = p(X^{(\alpha)}) = p(A) = \{\infty\}$, so this proves b). □

2.5 **COROLLARY:** *Let X be a countable compact space and let $A = X^{(\alpha)}$ for some $\alpha < \varkappa(X)$. Then $Y \approx [1,\omega^\alpha]$. In particular if $X = [1,\omega^\alpha\cdot n]$ for certain $n \in \mathbb{N}$, then $A = \{\alpha\cdot 1,\ldots,\alpha\cdot n\}$ and $Y \approx [1,\omega^\alpha]$.*

PROOF: This follows from theorem 1.1 and lemma 2.4b. □

In the following we prove some properties of function spaces of ordinals. We use the following notation. For an ordinal α we denote by $C_{p,0}([1,\alpha])$ the subspace of $C_p([1,\alpha])$ of all continuous functions vanishing at α.

Furthermore for two spaces X and Y we denote by $X \oplus Y$ the topological sum of X and Y and for spaces X_i $(i \in \mathbb{N})$ the topological sum is denoted by $\oplus_{i=1}^{\infty} X_i$.

2.6 LEMMA: *Let* α *and* β *be ordinals. Then*

$$C_p([1,\alpha+\beta]) \sim C_p([1,\alpha]) \times C_p([1,\beta]) \sim C_p([1,\beta]) \times C_p([1,\alpha]) \sim C_p([1,\beta+\alpha])$$

and

$$C_{p,0}([1,\alpha+\beta]) \sim C_p([1,\alpha]) \times C_{p,0}([1,\beta]).$$

PROOF: Since $[1,\alpha+\beta] \approx [1,\alpha] \oplus [1,\beta]$ we have

$$C_p([1,\alpha+\beta]) \sim C_p([1,\alpha] \oplus [1,\beta]) \sim C_p([1,\alpha]) \times C_p([1,\beta])$$

and

$$C_{p,0}([1,\alpha+\beta]) \sim C_p([1,\alpha]) \times C_{p,0}([1,\beta]). \square$$

2.7 LEMMA: *Let* $\alpha \geqq \omega$ *be an ordinal. Then* $C_p([1,\alpha]) \sim C_{p,0}([1,\alpha])$.

PROOF: By first applying corollary 2.3 and then lemma 2.6, we obtain

$$C_p([1,\alpha]) \sim C_p(\{\alpha\}) \times C_{p,0}([1,\alpha]) \sim C_p(\{1\}) \times C_{p,0}([1,\alpha]) \sim C_{p,0}([1,\alpha]). \square$$

2.8 LEMMA: *Let* $\alpha \geqq \omega$ *be a prime component and* $n \in \mathbb{N}$. *Then*

$$C_p([1,\alpha\cdot n]) \sim C_p([1,\alpha]).$$

PROOF: By proposition 1.2a there is an ordinal μ such that $\alpha = \omega^{\mu}$. Then

$C_p([1,\alpha\cdot n])$	$\sim$	$C_p(\{\alpha\cdot 1,\dots,\alpha\cdot n\}) \times C_{p,0}([1,\alpha])$	corollary 2.3 and 2.5
	$\sim$	$C_{p,0}([1,\alpha])$	lemma 2.6
	$\sim$	$C_p([1,\alpha])$	lemma 2.7 $\square$

2.9 LEMMA: *Let* $\alpha \geqq \omega$ *be an ordinal. Then* $C_p([1,\alpha]) \sim C_p([1,\alpha'])$.

PROOF: $\alpha = \alpha'\cdot n + \gamma$ for some $n < \omega$ and $\gamma < \alpha'$ (proposition 1.2b). Notice that for every $\gamma < \alpha'$, $\gamma + \alpha' = \alpha'$, which implies that $\gamma + \alpha'\cdot n = \gamma + \alpha' + \alpha'\cdot(n-1) = \alpha'\cdot n$. So

$C_p([1,\alpha])$	$=$	$C_p([1,\alpha'\cdot n + \gamma])$	
	$\sim$	$C_p([1,\gamma + \alpha'\cdot n])$	lemma 2.6
	$=$	$C_p([1,\alpha'\cdot n])$	
	$\sim$	$C_p([1,\alpha'])$	lemma 2.8 $\square$

We now come to the following result:

2.10 LEMMA: *Let $\omega \leqq \alpha < \omega_1$ be an ordinal and let $\alpha \leqq \beta < \alpha^\omega$. Then*

$$C_p([1,\alpha]) \sim C_p([1,\beta]).$$

PROOF: By lemma 1.3 and lemma 2.9 we may assume that α is a prime component. By proposition 1.2a there is an ordinal μ such that $\alpha = \omega^\mu$.

We prove the lemma by transfinite induction on β. If $\beta = \alpha$ it is a triviality, so suppose the lemma is true for every ordinal γ such that $\omega^\mu \leqq \gamma < \beta < \alpha^\omega$. Let $X = [1,\beta]$ and $A = X^{(\mu)}$. Notice that $\mu < \varkappa(X)$ because $[1,\omega^\mu] \subset X$. In particular A is nonempty. By corollary 2.5, $Y_{X,A} \approx [1,\omega^\mu] = [1,\alpha]$. There is $n \in \mathbb{N}\backslash\{1\}$ such that $\omega^{\mu(n-1)} < \beta \leqq \omega^{\mu \cdot n}$. So by theorem 1.1 and by the fact that $\beta \leqq \omega^{\mu \cdot n}$, $A^{(\mu(n-1))} = [1,\beta]^{(\mu \cdot n)}$ contains at most one point. Again by theorem 1.1 there is an ordinal γ such that $A \approx [1,\gamma]$. If $\gamma > \omega^{\mu(n-1)}$, then by the special form of γ (cf. theorem 1.1), $A^{(\mu(n-1))}$ contains more then one point, so $\gamma \leqq \omega^{\mu(n-1)}$. Furthermore by corollary 2.3 and lemma 2.7,

$$C_p([1,\beta]) \sim C_p([1,\alpha]) \times C_p([1,\gamma]).$$

If $\gamma < \omega^\mu = \alpha$ then $\gamma + \alpha = \alpha$, so by lemma 2.6,

$$C_p([1,\beta]) \sim C_p([1,\gamma+\alpha]) = C_p([1,\alpha]).$$

If $\gamma \geqq \omega^\mu = \alpha$ then by the inductive hypothesis, lemma 2.6 and lemma 2.8,

$$C_p([1,\beta]) \sim C_p([1,\alpha]) \times C_p([1,\alpha]) \sim C_p([1,\alpha \cdot 2]) \sim C_p([1,\alpha]). \square$$

Now we can easily derive the following:

2.11 COROLLARY: *Let $\omega \leqq \alpha \leqq \beta < \omega_1$ be ordinals. Then $C_p([1,\alpha]) \sim C_p([1,\beta])$ iff $\beta < \alpha^\omega$. (In particular if $\alpha = \omega^\mu$ and $\beta = \omega^\nu$ with $\nu \leqq \mu$, then $\mu < \nu \cdot \omega$).*

PROOF: If $\beta < \alpha^\omega$ then apply lemma 2.10. Suppose $C_p([1,\alpha]) \sim C_p([1,\beta])$. By Arhangelskiĭ, [1, theorem 2a] it follows that $C_0([1,\alpha]) \sim C_0([1,\beta])$. By Bessaga and Pelczyński [2, theorem 1] this implies $\beta < \alpha^\omega$. □

We are now able to prove the classification we mentioned at the beginning of this section. First we state the classification of Bessaga and Pelczyński ([2]):

2.12 THEOREM: *Let X and Y be zero-dimensional compact spaces. Then* $C_0(X) \sim C_0(Y)$ *iff one of the following holds:*

- *(i) X and Y are finite and have the same number of elements.*
- *(ii) There are countable infinite ordinals* α *and* β *such that* $X \approx [1,\alpha]$, $Y \approx [1,\beta]$ *and* $\max(\alpha,\beta) < [\min(\alpha,\beta)]^{\omega}$.
- *(iii) X and Y are uncountable.*

2.13 THEOREM: *Let X and Y be zero-dimensional compact spaces. Then* $C_p(X) \sim C_p(Y)$ *iff one of the following holds:*

- *(i) X and Y are finite and have the same number of elements.*
- *(ii) There are countable infinite ordinals* α *and* β *such that* $X \approx [1,\alpha]$, $Y \approx [1,\beta]$ *and* $\max(\alpha,\beta) < [\min(\alpha,\beta)]^{\omega}$.
- *(iii) X and Y are uncountable.*

PROOF: Let X and Y be zero-dimensional compact spaces.
If $C_p(X) \sim C_p(Y)$ then by Arhangelskiĭ [1, theorem 2a] we have $C_0(X) \sim C_0(Y)$. So by theorem 2.12, (i), (ii) or (iii) holds.
Now suppose that (i), (ii) or (iii) holds.

Case 1: (i) holds.
Suppose X and Y both contain m points. Then $C_p(X) \sim \mathbb{R}^m \sim C_p(Y)$.

Case 2: (ii) holds.
By theorem 1.1 there are ordinals α and β such that $X \approx [1,\alpha]$ and $Y \approx [1,\beta]$. By corollary 2.11 we have the desired inequality.

Case 3: (iii) holds.
It is enough to prove that for every uncountable zero-dimensional compact space X we have $C_p(X) \sim C_p(C)$ where C is the Cantor discontinuum.

CLAIM: For every zero-dimensional compact space X we may assume that X is a closed subspace of C such that $Y_{C,X} \approx C$.

For take an embedding $g: X \to C$ and let $h: X \to C \times C$ be defined by $h(x) = (g(x), 0)$. It is clear that this is the required embedding.

By the claim and corollary 2.3 we have for every zero-dimensional compact space X,

$$C_p(C) \sim C_p(X) \times C_{p,0}(C). \qquad (*)$$

Now let X be an uncountable zero-dimensional compact space. By the Cantor-Bendixson theorem we may assume that C is a closed subspace of X, since C is the unique compact zero-dimensional space without isolated points. So

$$\begin{aligned} C_p(X) &\sim C_p(C) \times C_{p,0}(Y_{X,C}) && \text{by corollary 2.3} \\ &\sim C_p(C) \times C_{p,0}(C) \times C_{p,0}(Y_{X,C}) && \text{take } X = C \text{ in } (*) \\ &\sim C_p(X) \times C_{p,0}(C) \\ &\sim C_p(C) && \text{by } (*). \ \square \end{aligned}$$

REMARK: The proofs of Bessaga and Pelczyński in [2] used properties of Banach spaces, for example the fact that if a Banach space B is the direct sum of two closed subspaces E and F, then it is isomorphic to $E \times F$. However we were able to prove our result for the spaces $C_p(X)$ in a similar way as they did for $C_0(X)$, avoiding Banach spaces.

The main difference between the proof in [2] and ours, is that we use transfinite induction. It is also possible to avoid transfinite induction in our proof and follow from a certain point the construction of Bessaga and Pelczyński: From lemma 2.9 and the fact that for two ordinals α and β we have $(\alpha \cdot \beta)' = \alpha' \cdot \beta'$, it is possible to prove directly by the method of corollary 2.3, that if $\beta \leqq \alpha < \omega_1$ then

$$C_p([1, \alpha \cdot \beta]) \sim C_p([1, \alpha]).$$

Then we are in a position from which we can derive lemma 2.10 with the same arguments Bessaga and Pelczyński use (cf. the proof of lemma 1 in [2]). However our proof seems to be a bit easier.

3 Locally compact zero-dimensional spaces

In this section we present an isomorphical classification of the function spaces $C_p(X)$ and $C_0(X)$ for locally compact zero-dimensional spaces X.

First we state some definitions and a proposition from [1]. Let ϕ: $C(X) \to C(Y)$ be a linear mapping, where X and Y are spaces. For every $y \in Y$, the ***support*** of y in X is defined to be the set $\text{supp}(y)$ of all $x \in X$ satisfying the condition that for every neighborhood U of x, there is an $f \in C(X)$ such that $f(X \backslash U) = \{0\}$ and $\phi(f)(y) \neq 0$. For a subset A of Y, we denote $\bigcup\{\text{supp}(y) \mid y \in A\}$ by supp$A$. Whenever ϕ is a linear homeomorphism, we can consider the support of a point in Y with respect to ϕ and the support of a point in X with respect to ϕ^{-1}. In this section it will always be clear which "support" we mean. Furthermore ϕ is said to be ***effective*** if for every $f, g \in C(X)$ and $y \in Y$, such that f and g coincide on a neighborhood of supp(y), $\phi(f)(y) = \phi(g)(y)$.

3.1 **PROPOSITION:** ([1] Arhangel'skiĭ) *Let X and Y be spaces, and let ϕ: $C_0(X) \to C_0(Y)$ be a linear homeomorphism. Then*

a) ϕ is effective

b) If A is a compact subset of Y, then $\overline{\text{supp}A}$ is compact in X.

Another result in [1] is that if ϕ: $C_p(X) \to C_p(Y)$ is a linear homeomorphism, then ϕ considered as a map from $C_0(X)$ to $C_0(Y)$ is also a linear homeomorphism. In the sequel we shall not mention this result, but whenever we have a linear homeomorphism between $C_p(X)$ and $C_p(Y)$ we consider it also as a linear homeomorphism between $C_0(X)$ and $C_0(Y)$.

3.2 **LEMMA:** *Let X and Y be spaces such that $X = X_1 \oplus X_2 \oplus X_3$ and $Y = Y_1 \oplus Y_2 \oplus Y_3$. Suppose ϕ: $C_0(X) \to C_0(Y)$ is a linear homeomorphism such that* supp$X_1 \subset Y_1$ *and* supp$Y_2 \subset X_1 \oplus X_2$. *Then there is a linear embedding θ: $C_0(Y_2) \to C_0(X_2)$.*

PROOF: For each $f \in C_0(Y_2)$ we define $f^* \in C_0(Y)$ by $f^*(y) = f(y)$ if $y \in Y_2$ and $f^*(y) = 0$ elsewhere. In a similar way we define for every $g \in C_0(X_2)$, $g^+ \in C_0(X)$.
Define θ: $C_0(Y_2) \to C_0(X_2)$ by $\theta(f) = \phi^{-1}(f^*) \mid X_2$ and ψ: $C_0(X_2) \to C_0(Y_2)$ by $\psi(g) = \phi(g^+) \mid Y_2$.

Then θ and ψ are linear continuous mappings. Furthermore for every $h \in C_0(Y_2)$, we have $\psi(\theta(h)) = h$. Indeed, suppose the contrary, i.e. $\phi(\theta(h)^+) \mid Y_2 \neq h^* \mid Y_2$. This implies $\theta(h)^+ \mid X_1 \oplus X_2 \neq \phi^{-1}(h^*) \mid X_1 \oplus X_2$, since $X_1 \oplus X_2$ is a neighborhood of suppY_2 and ϕ is effective. Now $h^* \equiv 0$ on Y_1, so $\phi^{-1}(h^*) \equiv 0$ on X_1, since Y_1 is a neighborhood of suppX_1 and ϕ^{-1} is effective. Furthermore $\theta(h)^+ \equiv 0$ on X_1, so that $\theta(h)^+ = \phi^{-1}(h^*)$ on X_1. This implies $\theta(h)^+ \mid X_2 \neq \phi^{-1}(h^*) \mid X_2$. But now we have $\theta(h) \neq \theta(h)$. Contradiction.
We conclude that θ is a linear embedding. □

3.3 **COROLLARY:** *Let X and Y be zerodimensional spaces such that $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Suppose $\phi\colon C_0(X) \to C_0(Y)$ is a linear homeomorphism such that* $\operatorname{supp} Y_1 \subset X_1$. *Then there is a linear embedding $\theta\colon C_0(Y_1) \to C_0(X_1)$.*

PROOF: Let $X_1 = Y_1 = \emptyset$ in lemma 3.2. □

Let $\mu > 0$ be a prime component. We will assign to μ a fixed sequence $(\mu_i)_i$ of ordinals. If $\mu = 1$, let $\mu_i = 0$ for each $i \in \mathbb{N}$. If $\mu = \tau \cdot \omega$ for some τ, let $\mu_i = \tau \cdot i$ for each $i \in \mathbb{N}$, and in other cases let $(\mu_i)_i$ be a fixed strictly increasing sequence of ordinals such that $\mu_i \to \mu$ and $1 \leq \mu_i < \mu$ for each $i \in \mathbb{N}$.

We define the following classes of spaces:

$\mathscr{A} = \{X \mid X = [1,\omega^\mu] \oplus [1,\omega^\tau),\ \mu, \tau \text{ prime components}, \mu \geq \tau \geq 1\}$,

$\mathscr{B} = \{X \mid X = [1,\omega^\tau),\ \tau \geq 1 \text{ a prime component}\}$.

Observe the following:

(1) if $X \in \mathscr{A}$, say $X = [1,\omega^\mu] \oplus [1,\omega^\tau)$, then since for every i, ω^{τ_i} is a prime component, $X = [1,\omega^\mu] \oplus [1,\omega^{\tau_1}] \oplus [1,\omega^{\tau_2}] \oplus \ldots\ldots$

(2) if $X \in \mathscr{B}$, say $X = [1,\omega^\tau)$, then $X = [1,\omega^{\tau_1}] \oplus [1,\omega^{\tau_2}] \oplus \ldots.$

Whenever we have X in one of the above classes and discuss a "decomposition" $X = \bigoplus_{i=1}^{\infty} X_i$, we mean the decomposition fixed as above.

3.4 **LEMMA:** *Let τ be an ordinal, which is not a prime component. Then there is a decomposition $\bigoplus_{i=1}^{\infty} X_i$ of $[1,\omega^\tau)$, such that for every i, $C_p(X_i) \sim C_p([1,\omega^{\tau'\cdot i}])$. In particular $C_p([1,\omega^\tau)) \sim C_p([1,\omega^{\tau'\cdot\omega}))$*

PROOF: *case 1:* τ is a successor, say $\tau = \nu + 1$.
Then $[1,\omega^\tau) = [1,\omega^\nu\cdot\omega) \approx \bigoplus_{i=1}^{\infty}[1,\omega^\nu]$. Since $\tau' = \nu'$, we have $\nu < \tau'\cdot\omega$ and therefore $\omega^\nu < (\omega^{\tau'})^\omega$. So by corollary 2.11,

$$C_p([1,\omega^\nu]) \sim C_p([1,\omega^{\tau'}]) \sim C_p([1,\omega^{\tau'\cdot i}]),$$

which proves the first part of the lemma. Furthermore, as is easily seen,

$$C_p([1,\omega^\tau)) \sim \prod_{i=1}^{\infty} C_p([1,\omega^\nu]) \sim \prod_{i=1}^{\infty} C_p([1,\omega^{\tau'\cdot i}]) \sim C_p([1,\omega^{\tau'\cdot\omega})).$$

case 2: τ is a limit ordinal.

There is a strictly increasing sequence $\tau_i \to \tau$ such that $\tau' < \tau_i < \tau$ for every i. Furthermore let $\tau_0 = 0$. Then

$$[1, \omega^\tau) = \bigoplus_{i=1}^{\infty} [1, \omega^{\tau_i}].$$

Since $\tau_i' = \tau'$, by corollary 2.11,

$$C_p([1, \omega^{\tau_i}]) \sim C_p([1, \omega^{\tau'}]) \sim C_p([1, \omega^{\tau' \cdot i}]),$$

which proves the first part of the lemma. The proof of the second part is the same as in case 1. □

3.5 **LEMMA:** *Let X be a countable space which is locally compact but not compact. Then there is a decomposition $\oplus_{i=1}^{\infty} X_i$ of X and a space $Y \in \mathcal{A} \cup \mathcal{B}$ such that $C_p(X_i) \sim C_p(Y_i)$ (where Y_i is the i^{th} term in the above decomposition of Y). In particular $C_p(X) \sim C_p(Y)$.*

PROOF: Since X is a countable infinite space, which is locally compact but not compact, there is a limit ordinal α such that $X \approx [1, \alpha)$ (take the one-point compactification of X and apply theorem 1.1). Without loss of generality we may assume that $X = [1, \alpha)$. By proposition 1.2c there are ordinals β and τ such that $\alpha = \beta + \omega^\tau$, with $\tau > 0$ and $\beta = 0$ or $\beta \geqq \omega^\tau$.

case 1: $\beta = 0$, $\tau' = \tau$. Then $X = [1, \omega^\tau) \in \mathcal{B}$ and we are done.

case 2: $\beta = 0$, $\tau' \neq \tau$. Then apply lemma 3.4, which gives a space $Y = [1, \omega^{\tau' \cdot \omega})$ and a decomposition for X. Since $Y \in \mathcal{B}$, we are done.

case 3: $\beta \geqq \omega^\tau$, $\tau' = \tau$. Then there is an ordinal μ such that $\beta' = \omega^\mu$. Since $X = [1, \beta] \oplus [1, \omega^\tau)$ and $C_p([1, \beta]) \sim C_p([1, \omega^{\mu'}])$ we can take $Y = [1, \omega^{\mu'}] \oplus [1, \omega^\tau)$. Then $Y \in \mathcal{A}$, since $\mu' \geqq \tau \geqq 1$. That the desired decomposition of X exists is a triviality.

case 4: $\beta \geqq \omega^\tau$, $\tau' \neq \tau$. Again there is an ordinal μ such that $\beta' = \omega^\mu$. So if $\mu' > \tau'$ by lemma 3.4 and corollary 2.11, we can take $Y = [1, \omega^{\mu'}] \oplus [1, \omega^{\tau' \cdot \omega}) \in \mathcal{A}$. If $\mu' = \tau'$ then let $Y = [1, \omega^{\tau' \cdot \omega}) \in \mathcal{B}$. By lemma 3.5 we have the desired decomposition. □

The following definition and lemma can be found in [2]. Two topological vector spaces X and Y have the same ***linear dimension*** (notation $X \circleddash Y$) if X is linearly homeomorphic to a subspace of Y, and Y is linearly homeomorphic to a subspace of X.

3.6 **LEMMA:** *Let $\omega \leqq \alpha \leqq \beta < \omega_1$. Then*

$$C_0([1,\alpha]) \sim C_0([1,\beta]) \text{ iff } C_0([1,\alpha]) \ominus C_0([1,\beta]) \text{ iff } \beta < \alpha^\omega.$$

In particular if $\alpha = \omega^\mu$ and $\beta = \omega^\nu$ with $\mu \leqq \nu$ then $\nu < \mu \cdot \omega$..

3.7 **LEMMA:** *Suppose θ: $C_0([1,\omega^\mu]) \to C_0([1,\omega^\nu])$ is a linear embedding with $\mu, \nu \geqq 1$.*

a) If μ is a prime component, then $\mu \leqq \nu$,

b) if $\nu \leqq \mu$, then $\mu < \nu \cdot \omega$.

PROOF: Observe that we have a linear embedding ϕ: $C_0([1,\omega^\mu]) \to C_0([1,\omega^{\nu'}])$ since by lemma 3.6, we have $C_0([1,\omega^\nu]) \sim C_0([1,\omega^{\nu'}])$.

For a) suppose that $\nu' < \mu$. Then there is a linear embedding ψ: $C_0([1,\omega^{\nu'}]) \to C_0([1,\omega^\mu])$. So $C_0([1,\omega^\mu]) \ominus C_0([1,\omega^{\nu'}])$. So by lemma 3.6, $\nu' < \mu < \nu' \cdot \omega$. But this is impossible since ν' and μ are prime components. So $\mu \leqq \nu' \leqq \nu$.

For b) we can derive as under a) that $C_0([1,\omega^\mu]) \ominus C_0([1,\omega^\nu])$, so by lemma 3.6, $\nu \leqq \mu < \nu \cdot \omega$. □

3.8 **LEMMA:**

a) Let $X = Z \oplus [1,\omega)$ with Z a compact zero-dimensional space and $Y = \oplus_{i=1}^\infty Z_i$ where each Z_i is an infinite compact zero-dimensional space. Then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.

b) Let $X = Z_1 \oplus Z_2$ be zero-dimensional such that Z_1 is an infinite compact subspace. Then $C_0(X)$ is not linearly homeomorphic with $C_0([1,\omega))$.

PROOF: For a) suppose that $C_0(X)$ is linearly homeomorphic with $C_0(Y)$. Then by proposition 3.1 there is $n \in \mathbb{N}$ such that $\operatorname{supp} Z \subset \oplus_{i=1}^n Z_i$. Again by proposition 3.1 there is $m \in \mathbb{N}$ such that $\operatorname{supp} Z_{n+1} \subset Z \oplus [1,m]$. By lemma 3.2, there is a linear embedding θ: $C_0(Z_{n+1}) \to C_0([1,m]) = \mathbb{R}^m$. Since Z_{n+1} is infinite we have a contradiction, since the algebraic dimension of $C_0(Z_{n+1})$ is infinite.

For b) suppose that $C_0(X)$ is linearly homeomorphic with $C_0([1,\omega))$. Then by proposition 3.1, there is $m \in \mathbb{N}$ such that $\operatorname{supp} Z_1 \subset [1,m]$. By corollary 3.3, there is a linear embedding θ: $C_0(Z) \to C_0[1,m] = \mathbb{R}^m$. Again we have a contradiction. □

3.9 **LEMMA:** *Let* $X=[1,\omega^{\mu}]\oplus[1,\alpha)$ *and* $Y=[1,\omega^{\sigma}]\oplus[1,\beta)$, *where* $\mu\geqq 1$ *and* $\sigma\geqq 1$ *are prime components,* $\alpha\leqq\omega^{\mu}$ *and* $\beta\leqq\omega^{\sigma}$.
If $C_0(X)\sim C_0(Y)$, *then* $\mu=\sigma$

PROOF: Suppose $C_0(X)\sim C_0(Y)$ and $\mu\neq\sigma$ We may assume $\mu<\sigma$. By proposition 3.1, there is $\gamma<\alpha$ such that $\operatorname{supp}[1,\omega^{\sigma}]\subset[1,\omega^{\mu}]\oplus[1,\gamma]$. Since $\gamma+\omega^{\mu}=\omega^{\mu}$ we have $[1,\omega^{\mu}]\oplus[1,\gamma]\approx[1,\omega^{\mu}]$. Therefore by corollary 3.3, there is a linear embedding $\theta\colon C_0([1,\omega^{\sigma}])\to C_0([1,\omega^{\mu}])$. Then by lemma 3.7a we have $\sigma\leqq\mu$. Contradiction. □

3.10 **LEMMA:** *Let* $X=Z_1\oplus[1,\omega^{\delta})$ *and* $Y=Z_2\oplus[1,\omega^{\tau\cdot\omega})$, *where* Z_1 *and* Z_2 *are compact zero-dimensional spaces,* δ,τ *are prime components and* $1\leqq\delta\leqq\tau$.
Then $C_0(X)$ *is not linearly homeomorphic to* $C_0(Y)$.

PROOF: Suppose the contrary. Then by lemma 3.8a, $\delta>1$. By proposition 3.1 there is $n\in\mathbb{N}$ such that $\operatorname{supp}Z_1\subset Z_2\oplus[1,\omega^{\tau\cdot n}]$. Again by proposition 3.1, there is $\delta_i<\delta$ such that $\operatorname{supp}[1,\omega^{\tau\cdot(n+1)}]\subset Z_1\oplus[1,\omega^{\delta_i}]$. By lemma 3.2 and the fact that $C_0([1,\omega^{\tau\cdot(n+1)}])\sim C_0([1,\omega^{\tau}])$ (lemma 3.6), there is a linear embedding $\theta\colon C_0([1,\omega^{\tau}])\to C_0([1,\omega^{\delta_i}])$. Then by lemma 3.7a, $\tau\leqq\delta_i<\delta$. Contradiction. □

3.11 **LEMMA:** *Let* $X=Z_1\oplus[1,\omega^{\tau})$ *and* $Y=Z_2\oplus[1,\omega^{\delta})$, *where* Z_1 *and* Z_2 *are compact zero-dimensional spaces, and* $\tau\geqq 1$ *and* $\delta\geqq 1$ *are prime components.*
If $C_0(X)\sim C_0(Y)$, *then* $\tau=\delta$.

PROOF: Suppose the contrary. We may assume $\tau<\delta$. By lemma 3.8a, $\tau>1$. By proposition 3.1, there is $\tau<\delta_i<\delta$ such that $\operatorname{supp}Z_1\subset Z_2\oplus[1,\omega^{\delta_i}]$.

Let $j>i$. Again by proposition 3.1, there is $\tau_k<\tau$ such that $\operatorname{supp}[1,\omega^{\delta_j}]\subset Z_1\oplus[1,\omega^{\tau_k}]$. Notice that there is a linear embedding $\theta\colon C_0([1,\omega^{\tau_k}])\to C_0([1,\omega^{\tau}])$. So by lemma 3.2, lemma 3.7b, we have $\delta_j<\tau\cdot\omega$. This implies $\tau<\delta\leqq\tau\cdot\omega$. So since δ and τ are prime components, we have $\delta=\tau\cdot\omega$. But this contradicts lemma 3.10. □

3.12 **COROLLARY:**

a) Let $X\in\mathscr{A}$ *and* $Y\in\mathscr{A}$. *Then* $C_p(X)\sim C_p(Y)$ *iff* $C_0(X)\sim C_0(Y)$ *iff* $X=Y$.

b) Let $X\in\mathscr{B}$ *and* $Y\in\mathscr{B}$. *Then* $C_p(X)\sim C_p(Y)$ *iff* $C_0(X)\sim C_0(Y)$ *iff* $X=Y$.

PROOF: a) follows by lemma 3.9 and lemma 3.11, b) follows from lemma 3.11 □

3.13 **LEMMA:** *Let* $X = [1,\omega^\mu] \oplus [1,\omega^\delta) \in \mathscr{A}$ *and* $Y = [1,\omega^\tau) \in \mathscr{B}$.
Then $C_0(X)$ *is not linearly homeomorphic with* $C_0(Y)$.

PROOF: Suppose the contrary. By lemma 3.8, $\tau > 1$ and $\delta > 1$. There is $\tau_k < \tau$ such that $\text{supp}\,[1,\omega^\mu] \subset [1,\omega^{\tau_k}]$. By corollary 3.3 and lemma 3.7b, $\mu \leq \tau_k < \tau$.

Let $j > k$. There is $\delta_i < \delta$ such that $\text{supp}\,[1,\omega^{\tau_j}] \subset [1,\omega^\mu] \oplus [1,\omega^{\delta_i}]$. By corollary 3.3 and lemma 3.7b, $\mu < \tau_j < \mu\cdot\omega$. So $\mu < \tau \leq \mu\cdot\omega$. This implies $\mu\cdot\omega = \tau$. But this contradicts lemma 3.10. □

3.14 **THEOREM:** *Let X and Y be countable spaces which are both locally compact but not compact. Then the following statements are equivalent:*

a) $C_p(X) \sim C_p(Y)$

b) $C_0(X) \sim C_0(Y)$

c) There are compacta X_i *and* Y_i $(i \in \mathbb{N})$ *such that* $X = \oplus_{i=1}^\infty X_i$, $Y = \oplus_{i=1}^\infty Y_i$ *and for every* $i \in \mathbb{N}$, $C_p(X_i) \sim C_p(Y_i)$.

d) There are compacta X_i *and* Y_i $(i \in \mathbb{N})$ *such that* $X = \oplus_{i=1}^\infty X_i$, $Y = \oplus_{i=1}^\infty Y_i$ *and for every* $i \in \mathbb{N}$, $C_0(X_i) \sim C_0(Y_i)$.

PROOF: Apply lemma 3.6, corollary 3.12 and lemma 3.13 to conclude that a) and c) are equivalent and b) and d) are equivalent. In both cases the decompositions are the same and since for countable compact spaces we have the same isomorphical classification (theorem 2.12 and theorem 2.13), c) and d) are equivalent.□

REMARK: By the obtained classification theorems we have that $C_p([1,\alpha)) \sim C_p([1,\beta))$ does not always imply $C_p([1,\alpha]) \sim C_p([1,\beta])$ and $C_p([1,\alpha]) \sim C_p([1,\beta])$ does not always imply $C_p([1,\alpha)) \sim C_p([1,\beta))$.

For example $C_p([1,\omega^\omega))$ is linearly homeomorphic with $C_p([1,\omega^2))$, however $C_p([1,\omega^\omega])$ is not linearly homeomorphic with $C_p([1,\omega^2])$. Furthermore $C_p([1,\omega])$ is linearly homeomorphic with $C_p([1,\omega^2])$, but $C_p([1,\omega))$ is not linearly homeomorphic with $C_p([1,\omega^2))$.

The same remark applies for the compact-open topology.

We shall now consider the case of uncountable locally compact zero dimensional spaces which are not compact.

3.15 LEMMA: *Let X be an uncountable zero-dimensional space which is locally compact, but not compact. Then there is a decomposition $\oplus_{i=1}^{\infty} X_i$ of X consisting of compacta such that either every X_i is uncountable or X_1 is the only uncountable X_i.*

PROOF: Let $X = \oplus_{i=1}^{\infty} Z_i$ be a decomposition of X consisting of compacta (this is possible because X is zero-dimensional).
case 1: Only finitely many Z_i's are uncountable.
Let $n = \max\{i \mid Z_i \text{ is uncountable}\}$. Let $X_1 = Z_1 \oplus \ldots \oplus Z_n$ and $X_i = Z_{n+i-1}$ $(i \geqq 2)$.
case 2: Infinitely many Z_i's are uncountable.
Suppose $Z_{i_1}, Z_{i_2}, \ldots$ are uncountable. Let $X_n = Z_{i_{n-1}+1} \oplus \ldots \oplus Z_{i_n}$ $(i_0 = 0)$. Since X_n is compact and uncountable we are done. □

Now we define the following classes of spaces:

$\mathscr{C} = \{X \mid X = C \oplus [1, \omega^\tau), C$ is the Cantor set and $\tau \geqq 1$ is a prime component$\}$,

$\mathscr{D} = \{X \mid X = \oplus_{i=1}^{\infty} C_i$, C_i is a copy of the Cantor set$\}$.

Observe the following:

If $X \in \mathscr{C}$, say $X = C \oplus [1, \omega^\tau)$, then $X = C \oplus [1, \omega^{\tau_1}] \oplus [1, \omega^{\tau_2}] \oplus \ldots$. Whenever we have $X \in \mathscr{C}$ and discuss a "decomposition" $X = \oplus_{i=1}^{\infty} X_i$, we mean this fixed decomposition. If $X \in \mathscr{D}$ we have the fixed decomposition $\oplus_{i=1}^{\infty} C$.

3.16 LEMMA: *Let X be an uncountable zero-dimensional space which is locally compact but not compact. Then there is a decomposition $\oplus_{i=1}^{\infty} X_i$ of X and a space $Y \in \mathscr{C} \cup \mathscr{D}$ such that $C_p(X_i) \sim C_p(Y_i)$ (where Y_i is the i-th component of the decomposition of Y stated as above). In particular $C_p(X) \sim C_p(Y)$.*

PROOF: By lemma 3.15 there is a decomposition $\oplus_{i=1}^{\infty} X_i'$ such that either X_1' is the only uncountable space or every X_i' is uncountable.
case 1: X_1' is the only uncountable space in the decomposition $\oplus_{i=1}^{\infty} X_i'$.
Since $X' = \oplus_{i=2}^{\infty} X_i'$ is a countable space which is locally compact, by lemma 3.5 there is a space $Y' \in \mathscr{A} \cup \mathscr{B}$ and a decomposition $\oplus_{i=1}^{\infty} Z_i$ of X' such that $C_p(Z_i) \sim C_p(Y_i')$. If $Y' \in \mathscr{A}$, say $Y' = [1, \omega^\mu] \oplus [1, \omega^\tau)$, then $C_p(Z_1) \sim C_p([1, \omega^\mu])$. Let $X_1 = X_1' \oplus Z_1$ and $X_i = Z_i$ $(i \geqq 2)$. Then $C_p(X_1) \sim C_p(C)$ because $X_1' \oplus Z_1$ is uncountable and compact (theorem 2.13). If we let $Y = C \oplus [1, \omega^\tau)$ we are done. If $Y' \in \mathscr{B}$, say $Y' = [1, \omega^\tau)$, then

let $Y = C \oplus [1,\omega^\tau)$, $X_1 = X_i'$ and $X_i = Z_i (i \geqq 2)$.

case 2: Every X_i' is uncountable.

Let $Y = \oplus_{i=1}^{\infty} C_i$. By theorem 2.13 $C_p(X_i') \sim C_p(C_i)$, so let $X_i = X_i'$. □

3.17 LEMMA:

a) If $X, Y \in \mathscr{C}$, then $C_p(X) \sim C_p(Y)$ iff $C_0(X) \sim C_0(Y)$ iff $X = Y$,

b) If $X \in \mathscr{C}$ and $Y \in \mathscr{D}$, then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.

PROOF: a) follows directly from lemma 3.11.

For b) suppose that $X = C \oplus [1,\omega^\tau)$ and $Y = \oplus_{i=1}^{\infty} C_i$. Suppose $C_0(X) \sim C_0(Y)$. There is $n \in \mathbb{N}$ such that $\operatorname{supp} C \subset C_1 \oplus \dots \oplus C_n$. There is $i \in \mathbb{N}$ such that $\operatorname{supp} C_{n+1} \subset C \oplus [1,\omega^{\tau_i}]$. So by lemma 3.2, there is an embedding ϕ: $C_0(C) \to C_0([1,\omega^{\tau_i}])$. Find a copy of $[1,\omega^{\tau\cdot\omega}]$ in C Since $[1,\omega^{\tau\cdot\omega}]$ is a retract of C, we have an embedding θ: $C_0([1,\omega^{\tau\cdot\omega}]) \to C_0([1,\omega^{\tau_i}])$. But then by lemma 3.7b, $\tau\cdot\omega < \tau_i\cdot\omega$. Contradiction. □

3.18 THEOREM: *Let X and Y be uncountable zero-dimensional spaces which are both locally compact but not compact. Then the following statements are equivalent:*

a) $C_p(X) \sim C_p(Y)$

b) $C_0(X) \sim C_0(Y)$

c) There are compacta X_i and Y_i ($i \in \mathbb{N}$) such that $X = \oplus_{i=1}^{\infty} X_i$, $Y = \oplus_{i=1}^{\infty} Y_i$ and $C_p(X_i) \sim C_p(Y_i)$.

d) There are compacta X_i and Y_i ($i \in \mathbb{N}$) such that $X = \oplus_{i=1}^{\infty} X_i$, $Y = \oplus_{i=1}^{\infty} Y_i$ and $C_0(X_i) \sim C_0(Y_i)$.

PROOF: From lemma 3.16 and lemma 3.17 it follows that a) and c) are equivalent and b) and d) are equivalent. Since the decompositions in c) and d) are the same we have again by theorem 2.12 and theorem 2.13 that c) and d) are equivalent.□

REMARK: In view of the remark after theorem 3.14 we have the following: Let X and Y be spaces as in theorem 3.18. Let X^* and Y^* be the one point compactifications of X and Y. By theorem 2.13 iii, $C_p(X^*)$ is linearly homeomorphic with $C_p(Y^*)$. This is independent from the question whether $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.

Again the same remark applies for the compact-open topology.

We almost completed the isomorphical classification of the function spaces $C_p(X)$ and $C_0(X)$ of locally compact zero-dimensional spaces X. Therefore we only need to distinguish between "compact" and "non-compact" and between "countable" and "uncountable". It is known that for spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, we have X is compact if and only if Y is compact and X is countable if and only if Y is countable. However for the compact open topology we need the following proposition.

3.19 **PROPOSITION:** *Let X and Y be locally compact zero-dimensional spaces such that $C_0(X)$ and $C_0(Y)$ are linearly homeomorphic. Then*

a) X is compact iff Y is compact,

b) X is countable iff Y is countable.

PROOF: For a) suppose that X is compact and Y is not compact. Then $Y = \bigoplus_{i=1}^{\infty} Y_i$ where each Y_i is nonempty. There is $n \in \mathbb{N}$ such that $\operatorname{supp} X \subset \bigoplus_{i=1}^{n} Y_i$. Let $y \in Y_{n+1}$. There are $f, g \in C_0(Y)$ such that $f(\bigoplus_{i=1}^{n} Y_i) = g(\bigoplus_{i=1}^{n} Y_i) = 0$, $f(y) = 1$ and $g(y) = 2$. Then $f = g$ on a neighborhood of $\operatorname{supp} X$. So by proposition 3.1a, $\phi^{-1}(f) = \phi^{-1}(g)$, which implies $f = g$. Contradiction.

For b) suppose that X is countable and Y is uncountable. By a) and theorem 2.12 we may assume that X and Y are not compact. By lemma 3.5 and lemma 3.16 we may assume that $X \in \mathscr{A} \cup \mathscr{B}$ and $Y \in \mathscr{C} \cup \mathscr{D}$. There is a clopen copy of C in Y. Then $\operatorname{supp} C$ is contained in a clopen copy $[1,\alpha]$ in X for some countable ordinal α. Find $\beta > \alpha^{\omega}$ and a copy of $[1,\beta]$ in C. Since $[1,\beta]$ is retract of C we have a linear embedding $\theta: C_0([1,\beta]) \rightarrow C_0([1,\alpha])$. But this is impossible by lemma 3.6. □

Notice that by theorem 2.12, 2.13, 3.14, 3.18 and proposition 3.19 we have as announced in the introduction that for locally compact zero dimensional spaces X and Y, $C_p(X)$ is linearly homeomorphic to $C_p(Y)$ if and only if $C_0(X)$ is linearly homeomorphic to $C_0(Y)$.

REMARK: It is now natural to consider the class of topologically complete zero-dimensional spaces. However this seems to be much more complicated.

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