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Abstract: For a locally compact space X and a positive integer n, denote $B_n(X)=\{f\in C(X)$: there is a compact $K\subset X$ such that $|f(X\setminus K)|\leq n\}$. Then the diagonal mapping $e_n=\Delta\{f:f\in B_n(X)\}$ is a homeomorphic embedding and the closure of $e_n(X)$ is a compactification of X denoted by e_nX. It is shown here that $|e_nX\setminus X|=n$ if and only if X has exactly one n-point compactification which holds if and only if $B_n(X)$ is a subalgebra of $C^*(X)$ but $B_m(X)$ is not whenever $1<m<n$. A number of other necessary and sufficient conditions for X to have only one n-point compactification are given.

Key words: n-point compactifications, locally compact spaces, sets generating compactifications, algebras of functions.

Classification: 54D35, 54D40, 54C20

Throughout this paper, X denotes a locally compact Hausdorff space. The algebra of all real-valued continuous functions on X is denoted by C(X) and its subalgebra of bounded functions - by $C^*(X)$.

For a compactification αX of X, let C_α denote the set of all functions $f\in C^*(X)$ continuously extendable to αX. For $f\in C_\alpha$, let f^α be the continuous extension of f to αX and, for $F\subset C_\alpha$, let $F^\alpha=\{f^\alpha:f\in F\}$.

Let $\mathcal{E}(X)$ be the family of all sets $F\subset C^*(X)$ such that the diagonal mapping $e_F=\underset{f\in F}{\Delta} f$ is a homeomorphic embedding. If $F\in\mathcal{E}(X)$, then the closure of $e_F(X)$ in $R^{|F|}$ is a compactification of X. This compactification is said to be generated by F and is denoted by e_FX. Of course, e_FX is the smallest compactification of X to which all functions from F are continuously extendable.

For a positive integer n, denote $B_n(X)=\{f\in C(X)$: there exists a compact set $K\subset X$ such that $|f(X\setminus K)|\leq n\}$. It is easily verified that $B_n(X)$ separates points from closed sets, and so belongs to $\mathcal{E}(X)$. For simplicity, denote $e_nX=e_FX$ where $F=B_n(X)$. It follows from [2; Theorem 3.3] and [5; Theorem 3.3´] (cf. also [3; Corollary 6.5, p. 67]) that if $|\beta X\setminus X|=n$, then $\beta X=e_nX$.

In [3; Remarks 6.9, p. 71] R.E. Chandler asked the question whether $|e_n X \setminus X|$ equals n for X having an n-point compactification (i.e. a compactification with the remainder of cardinality n). In this note, we shall show that $|e_n X \setminus X| = n$ if and only if X has exactly one (up to equivalence) n-point compactification which holds if and only if $B_n(X)$ is a subalgebra of $C^*(X)$ but $B_m(X)$ is not whenever $1 < m < n$. We shall also give a number of other necessary and sufficient conditions for X to have only one (up to equivalence) n-point compactification.

We shall use the following theorem proved in [2] by B.J. Ball and Shoji Yokura:

Theorem 0. For any subset F of $C^*(X)$ and any compactification αX of X, the following conditions are equivalent:

(i) $F \in \mathcal{E}(X)$ and $e_F X = \alpha X$;

(ii) $F \subset C_\alpha$ and F^α separates points of αX.

For notation and terminology not defined here, see [3] and [4].

Results. To begin with, let us observe that if X is a noncompact locally compact space and ωX is the one-point compactification of X, then $B_1(X) \subset C_\omega$ and $B_1(X)^\omega$ separates points of ωX. Th eorem 0 implies that $B_1(X)$ generatés ωX.

Lemma 1. If αX is a compactification of X for which $|\alpha X \setminus X|$ is finite, then $B_2(X) \cap C_\alpha$ generates αX.

Proof. Without any difficulties one can check that the set $B_1(X)^\alpha$ separates each pair of distinct points $y, z \in \alpha X$ such that $y \in X$.

Suppose that $y, z \in \alpha X \setminus X$ and $y \neq z$. As the set $\alpha X \setminus X$ is finite, there exist sets V, W open in αX and such that $y \in V$, $(\alpha X \setminus X) \setminus \{y\} \subset W$ and $(cl_{\alpha X} V) \cap (cl_{\alpha X} W) = \emptyset$. Take a function $f^\alpha \in C(\alpha X)$ such that $f^\alpha(cl_{\alpha X} V) = \{0\}$ and $f^\alpha(cl_{\alpha X} W) = \{1\}$ and put $f = f^\alpha|_X$. The set $K = X \setminus (V \cup W) = \alpha X \setminus (V \cup W)$ is compact and $f(X \setminus K) \subset \{0,1\}$, so $f \in B_2(X) \cap C_\alpha$. Clearly, $f^\alpha(y) \neq f^\alpha(z)$, hence $[B_2(X) \cap C_\alpha]^\alpha$ separates points of αX. It follows from Theorem 0 that $B_2(X) \cap C_\alpha$ generates αX.

Lemma 2. If αX is an n-point compactification of X where $n > 1$, then there exist functions $f_i \in B_2(X) \cap C_\alpha$ $(i=1,\dots,n)$ such that

$\sum_{i=1}^{n} f_i \in B_n(X) \setminus B_{n-1}(X)$.

Proof. Let $z_1,\dots,z_n$ be distinct points of $\alpha X\setminus X$. Take sets V_i, open in αX, such that $z_i\in V_i$ and $(cl_{\alpha X}V_i)\cap(cl_{\alpha X}V_j)=\emptyset$ for $i\neq j$ $(i,j=1,\dots,n)$. There exist functions $f_i^{\alpha}\in C(\alpha X)$ such that $f_i^{\alpha}(cl_{\alpha X}V_i)=\{i\}$ and $f_i^{\alpha}(\bigcup_{j\neq i} cl_{\alpha X}V_j)=\{0\}$ $(i=1,\dots,n)$. Denote $f_i=f_i^{\alpha}|_X$ $(i=1,\dots,n)$ and $f=\sum_{i=1}^{n} f_i$. Then $f_i\in B_2(X)\cap C_{\alpha}$ and $f^{\alpha}(cl_{\alpha X}V_i)=\{i\}$ for $i=1,\dots,n$. If there is a compact set $K\subset X$ such that $|f(X\setminus K)|\leq n-1$, then $V_i\cap X\subset K$ for some $i\in\{1,\dots,n\}$, which is impossible because $z_i\in cl_{\alpha X}(V_i\cap X)$. Hence $f\in B_n(X)\setminus B_{n-1}(X)$.

Lemma 3. If $n>1$ and $f\in B_n(X)\setminus B_{n-1}(X)$, then there exists an n-point compactification αX of X such that $f\in C_{\alpha}$.

Proof. Suppose that K is a compact subset of X such that $|f(X\setminus K)|=n$. Let $f(X\setminus K)=\{a_1,\dots,a_n\}$ and, for $i=1,\dots,n$, let us put $G_i=f^{-1}(a_i)\setminus K$. It is easily seen that the sets G_i are open in X, pairwise disjoint and $X\setminus\bigcup_{i=1}^{n} G_i=K$. If $K\cup G_i$ is compact for some i, then, since $|f[X\setminus(K\cup G_i)]|\leq n-1$, we have that $f\in B_{n-1}(X)$ - a contradiction. Hence all the sets $K\cup G_i$ are not compact. The proof of Magill's theorem (cf. [6; the proof of Theorem 2.1] or [3; the proof of Theorem 6.8, p. 70]) implies that there exists an n-point compactification αX of X such that if $\alpha X\setminus X=\{z_1,\dots,z_n\}$, then the set $G_i\cup\{z_i\}$ is a neighbourhood of z_i in αX $(i=1,\dots,n)$. Let us define $f^{\alpha}(z)=f(z)$ for $z\in X$ and $f^{\alpha}(z_i)=a_i$ for $i=1,\dots,n$. The function f^{α} is a continuous extension of f to αX, so $f\in C_{\alpha}$.

Let us recall the notion of β-families (cf. [3; Definition 5.15,p.52]).

Definition. Let αX be a compactification of X and let $h:\beta X\to\alpha X$ be a continuous mapping such that $h\circ\beta=\alpha$. The set $\{h^{-1}(z):z\in\alpha X\setminus X\}$ is denoted by $\mathcal{F}(\alpha X)$ and is called the β-family of αX.

Lemma 4. If αX and γX are nonequivalent n-point compactifications of X, then neither $\alpha X\leq\gamma X$ nor $\gamma X\leq\alpha X$.

Proof. Suppose that $\alpha X\leq\gamma X$, $\mathcal{F}(\alpha X)=\{A_1,\dots,A_n\}$ and $\mathcal{F}(\gamma X)=\{E_1,\dots,E_n\}$. For $j=1,\dots,n$, denote $N_j=\{i\in\{1,\dots,n\}:E_i\subset A_j\}$. Then the sets N_j are pairwise disjoint and, moreover, Lemma 5.16 of [3; p. 52] yields that $\bigcup_{j=1}^{n} N_j=\{1,\dots,n\}$. As $\bigcup_{i=1}^{n} E_i=\bigcup_{i=1}^{n} A_i=\beta X\setminus X$, then $|N_j|=1$ for each

$j\in\{1,\dots,n\}$. This implies that $\mathcal{F}(\alpha X)=\mathcal{F}(\gamma X)$. By virtue of [3; Corollary 5.17, p. 53], we have that $\alpha X=\gamma X$ - a contradiction.

Our next lemma is a consequence of Lemmas 6.12 and 6.13 of [3; p. 72].

Lemma 5. Suppose that X has an n-point compactification for some $n>1$. Then all n-point compactifications of X are equivalent if and only if X has no m-point compactification where $m>n$.

Theorem 1. For every locally compact space X and any positive integer $n>1$, the compactifications e_2X and e_nX of X are equivalent.

Proof. Let us fix a positive integer $n>1$. Since $B_2(X)\subset B_n(X)$, according to Theorem 2.10 of [3; p. 14], it suffices to show that $B_n(X)\subset C_{e_2}$.

Suppose that $f\in B_n(X)\setminus B_2(X)$ and let p be the smallest positive integer for which $f\in B_p(X)$. It follows from Lemma 3 that X has a p-point compactification αX such that $f\in C_\alpha$. By virtue of Lemma 1, the set $B_2(X)\cap C_\alpha$ generates αX. Using Theorem 2.10 of [3], we obtain that $\alpha X\leq e_2X$; thus, $C_\alpha\subset C_{e_2}$ and $f\in C_{e_2}$. Consequently, $B_n(X)\subset C_{e_2}$.

Theorem 2. For every locally compact space X and any positive integer $n>1$, the following conditions are equivalent:

(i) X has exactly one (up to equivalence) n-point compactification;

(ii) $B_m(X)=B_n(X)\neq B_{n-1}(X)$ for each $m\geq n$;

(iii) $B_{n+1}(X)=B_n(X)\neq B_{n-1}(X)$;

(iv) $|e_2X\setminus X|=n$.

Proof. Assume (i). Applying Lemma 2, we deduce that $B_n(X)\neq B_{n-1}(X)$. If $B_m(X)\neq B_n(X)$ for some $m>n$, then there exists a positive integer $p>n$ such that $B_p(X)\setminus B_{p-1}(X)\neq\emptyset$. Thus, by Lemma 3, X has a p-point compactification. This, together with Lemma 5, contradicts (i). Hence (i) $\Rightarrow$ (ii).

Assume (iii). According to Lemma 3, there exists a compactification αX of X such that $|\alpha X\setminus X|=n$. Let us take a function $f\in B_2(X)$ and suppose that $f\notin C_\alpha$. As $B_1(X)\subset C_\alpha$, by virtue of Lemma 3, X has a 2-point compactification γX such that $f\in C_\gamma$. Denote $\mathcal{F}(\alpha X)=\{A_1,\dots,A_n\}$ and $\mathcal{F}(\gamma X)=\{E_1,E_2\}$. Since $C_\gamma\setminus C_\alpha\neq\emptyset$, the inequality $\gamma X\leq\alpha X$ does not hold. It follows from Lemma 5.16 of [3; p. 52] that there is an $i\in\{1,\dots,n\}$ such that $A_i\cap E_1\neq\emptyset$ and $A_i\cap E_2\neq\emptyset$. Then there exists a compactification δX of X for which $\mathcal{F}(\delta X)=\{A_1,\dots,A_{i-1},A_i\cap E_1,A_i\cap E_2,A_{i+1},\dots,A_n\}$. Clearly, $|\delta X\setminus X|=n+1$ and, by using Lemma 2, we obtain that $B_{n+1}(X) \neq B_n(X)$ - a contradiction. Hence, $f \in C_\alpha$ and $B_2(X) \subset C_\alpha$. It follows from Lemma 1 that $e_2X = \alpha X$, so (iii) $\Rightarrow$ (iv). It remains to show that (iv) $\Rightarrow$ (i).

Assume (iv) and let αX be an arbitrary n-point compactification of X. By Lemma 1, the set $B_2(X) \cap C_\alpha$ generates αX. This, along with Theorem 2.10 of [3; p. 14], yields that $\alpha X \leq e_2X$. Lemma 4 implies that $\alpha X = e_2X$; hence (iv) $\Rightarrow$ (i).

Corollary 1. For every locally compact space X and any positive integer n, the following conditions are equivalent:

(i) X has exactly one (up to equivalence) n-point compactification;

(ii) $|e_nX \setminus X| = n$.

Corollary 2. For every locally compact space X, the following conditions are equivalent:

(i) X does not have any 2-point compactification;

(ii) $B_n(X) = B_1(X)$ for each positive integer n;

(iii) $|e_2X \setminus X| \leq 1$.

Corollary 3. For every locally compact space X, the following conditions are equivalent:

(i) X has an n-point compactification for any positive integer n;

(ii) $B_{n+1}(X) \neq B_n(X)$ for any positive integer n;

(iii) $|e_2X \setminus X| \geq \aleph_0$.

Example. Let X be an infinite discrete space. It is easily seen that if y, z are distinct points of βX, then there exist sets V and W, open in βX, such that $y \in V$, $z \in W$, $V \cap W = \emptyset$ and $V \cup W = \beta X$. This implies that $B_2(X)^\beta$ separates points of βX; thus, by Theorem 0, $e_2X = \beta X$.

In connection with the above example one may suspect that $e_2X = \beta X$ whenever $|e_2X \setminus X|$ is infinite. That this is false is shown by the following

Theorem 3. For every cardinal number $m \neq 0$, there exists a locally compact space X such that $|e_2X \setminus X| = m$ and $e_2X \neq \beta X$.

Proof. Let Y be the discrete spce of cardinality $m \neq 0$. By virtue of [8; Proposition 4.17, p. 36], there exists a locally compact space X such that $\beta X \setminus X$ is homeomorphic to $[0;1] \times \omega Y$. For simplicity, assume that $\beta X \setminus X = [0;1] \times \omega Y$. Let us observe that if z_0, z_1 are distinct points of ωY, then one can find a function $f \in B_2(X)$ such that $f^\beta([0;1] \times \{z_i\}) = \{i\}$ for i=0,1.

Since $B_2(X)^{\beta}$ does not separate points of $[0;1]\times\{z\}$ where $z\in\omega Y$, it follows from Theorem 0 that $\mathcal{F}(e_2X)=\{[0;1]\times\{z\}: z\in\omega Y\}$. Hence $|e_2X\setminus X|=\mathfrak{m}$ and, moreover, $e_2X\neq\beta X$.

Theorem 4. For every cardinal number $\mathfrak{m}$, there exists a locally compact space X such that $|e_2X\setminus X|=\mathfrak{m}$ and $e_2X=\beta X$.

Proof. Let Y be the discrete space of cardinality $\mathfrak{m}$. If X is a locally compact space such that $\beta X\setminus X$ is homeomorphic to ωY (cf. [8; Proposition 4.17, p. 36]), then $B_2(X)^{\beta}$ separates points of βX; hence, by Theorem 0, $e_2X=\beta X$.

Let F be a nonempty subset of $C^*(X)$. For a positive integer n, denote $M^n(F)=\{h\circ\mathop{\Delta}\limits_{i=1}^{n} f_i : h\in C(R^n) \text{ and } f_i\in F \text{ for } i=1,\dots,n\}$ and $M^{\infty}(F)=\bigcup\limits_{n=1}^{\infty} M^n(F)$. The sets $M^n(F)$ and $M^{\infty}(F)$ were first considered by B.J. Ball and Shoji Yokura in [1]. As shown in [7], $M^{\infty}(F)$ is a subalgebra of $C^*(X)$ containing F and all constant functions. Denote by $\mathcal{A}(F)$ the smallest subalgebra of $C^*(X)$ which contains F and all constant functions, and let $\overline{\mathcal{A}(F)}$ be the closure of $\mathcal{A}(F)$ in $C^*(X)$ with the topology of uniform convergence. Proposition 1.10 of [7] says that $M^{\infty}(F)\subset\overline{\mathcal{A}(F)}$.

Without any difficulties we can check that $B_1(X)=M^{\infty}(B_1(X))$, so $B_1(X)$ is a subalgebra of $C^*(X)$. We are now going to generalize this result to sets $B_n(X)$ such that $|e_nX\setminus X|=n$.

Theorem 5. For every locally compact space X and any positive integer n, the following conditions are equivalent:

(i) $|e_nX\setminus X|=n$;

(ii) $M^{\infty}(B_n(X))=B_n(X)$, and if $1<m<n$, then $M^{\infty}(B_m(X))\neq B_m(X)$;

(iii) $B_n(X)$ is a subalgebra of $C^*(X)$, and if $1<m<n$, then $B_m(X)$ is not an algebra.

Proof. Assume (i). It is easily seen that $M^{\infty}(B_n(X))\subset\bigcup\limits_{m=1}^{\infty} B_m(X)$; thus, by virtue of Theorem 2, $B_n(X)\subset M^{\infty}(B_n(X))\subset\bigcup\limits_{m=1}^{n} B_m(X)=B_n(X)$; so that $M^{\infty}(B_n(X))=B_n(X)$ and, moreover, $B_n(X)$ is a subalgebra of $C^*(X)$. Suppose that $1<m<n$. Lemma 2 yields the existence of functions $f_i\in B_2(X)$ such that $\sum\limits_{i=1}^{n} f_i\notin B_m(X)$. Hence we have proved that (i) implies both (ii) and (iii).

Assume either (ii) or (iii), and suppose that (i) does not hold. Then $n>1$ and $B_n(X) \neq B_{n-1}(X)$. It follows from Theorem 2 that $B_m(X) \neq B_{m-1}(X)$ for some $m>n$. By Lemma 3, X has an m-point compactification. Lemma 2 implies that there exist functions $g_i \in B_n(X)$ such that $\sum_{i=1}^{m} g_i \notin B_{m-1}(X)$. As $B_n(X) \subset B_{m-1}(X)$, we have a contradiction. This completes the proof.

Remarks. Assume that αX is the unique (up to equivalence) n-point compactification of X. By our theorems, $B_n(X)$ is an algebra which generates αX. Applying Theorem 3.1 of [2], we deduce that $B_n(X)$ is a uniformly dense subset of C_α. In this way, we obtain a new proof of Theorem 3.1 of [5].

If $n>1$, then, by Theorem 2, $B_2(X)$ generates αX. It follows from Theorem 2.3 of [7] that C_α consists of all functions of the form $h \circ \mathop{\Delta}_{i=1}^{\infty} f_i$, where $h \in C(R^{\aleph_0})$ and $f_i \in B_2(X)$ for $i=1,2,\ldots$. Of course, by Theorem 2.3 of [7], a function $f \in C^*(X)$ is continuously extendable to the one-point compactification of X if and only if $f=h \circ \mathop{\Delta}_{i=1}^{\infty} f_i$ for some $h \in C(R^{\aleph_0})$ and $f_i \in B_1(X)$ $(i=1,2,\ldots)$.

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