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ANY ORTHOMODULAR POSET IS A PASTING OF BOOLEAN ALGEBRAS

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Abstract: We prove that every orthomodular poset can be constructed from Boolean algebras using the technique developed by Dichtl [1].

Key words: Orthomodular poset, pasting of Boolean algebras.

Classification: Primary 06C15
Secondary 81B10

"Pasting" Boolean algebras suggested by Greechie [2] has been used to construct a lot of interesting examples of orthomodular posets (abbr. OMP). This technique was generalized by Dichtl [1]. He introduced the notion of a pasted family Ψ of Boolean algebras and formulated a necessary and sufficient condition in order $L=\bigcup\Psi$ (with the orthocomplementation $'$ and the partial ordering $\leq$ inherited from the members of Ψ) be an OMP (Theorem 2).We call OMPs constructed by this way Dichtl OMPs. Dichtl proved that if an OMP is a lattice, then it is a Dichtl OMP. We show that any OMP belongs to the class of Dichtl OMPs. This result enables us to combine the Dichtl construction with other techniques, e.g. with products of OMPs.

We preserve the definitions and the symbols of [1] and [3]. Let us only recall two notions. The elements $a,b\in L$ are called compatible in L (in symbols: $a\overset{L}{\longleftrightarrow}b$) if there are $a_1,b_1,c\in L$ such that $a_1\leq b_1'$, $a_1\leq c'$, $b_1\leq c'$ and $a=a_1\vee c$, $b=b_1\vee c$. An n-cycle $((B_i,m_i))_{i=0}^{n-1}$ is a set of n not necessarily distinct members $B_i\in\Psi$ and n not necessarily distinct elements $m_i\in B_i\cap B_{i+1}$, such that $[0,m_i]_{B_i}$ equals $[0,m_i]_{B_{i+1}}$ and that $m_{i-1}\leq_{B_i} m_i'$ (indices mod n).

1. Definition ([1], p. 381). A family Ψ of Boolean algebras is pasted if for every pair $C_0,C_2\in\Psi$, the following conditions hold true:

(i) C_0 is not properly contained in C_2,

(ii) $C_0\cap C_2$ is a subalgebra of C_0 and of C_2, on which the operations of

C_o and C_2 coincide,

(iii) for every element $m \in C_o \cap C_2$ there exists a 4-cycle $((B_i, m_i))_{i=o}^{3}$ such that $B_o = C_o$, $B_2 = C_2$, $m_o = m = m_2$ and $m_1 = m' = m_3$ (i.e. an astroid for m).

2. Theorem ([1], Theorem 9). The pasting L of a pasted family Ψ is an OMP if and only if for every 3-cycle $((B_i, m_i))_{i=o}^{2}$ in Ψ there is $B \in \Psi$ containing $\bigcup_{i=o}^{3} [0, m_i]_{B_i}$.

3. Theorem. The system of all blocks of an OMP is pasted.

Proof. Let L be an OMP and let Ψ be the family of all blocks in L. It suffices to prove (iii) of Definition 1. Let $C_o, C_2 \in \Psi$ and $m \in C_o \cap C_2$, $m \neq 0,1$. For any $m_o \lneqq_{C_o} m$ and $m_2 \lneqq_{C_2} m'$ we have $m_o \lneqq_L m_2'$. Hence $m_o \overset{L}{\longleftrightarrow} m_2$ and there is a Boolean algebra B in L isomorphic to $[0,m]_{C_o} \times [0,m']_{C_2}$. Now B is contained in some $B_1 \in \Psi$. In fact, $B = B_1$. Indeed, let $a \in B_1$, $a \notin B$, and let us write $a = (a \wedge_{B_1} m) \vee_{B_1} (a \wedge_{B_1} m')$. Then $(a \wedge_{B_1} m) \overset{L}{\longleftrightarrow} [0,m]_{C_o}$ and $(a \wedge_{B_1} m) \overset{L}{\longleftrightarrow} b$ for any $b \in C_o$. Assuming $a \wedge_{B_1} m \neq 0$ we have got a contradiction with the maximality of C_o. In the case $a \wedge_{B_1} m = 0$ we use $a \wedge_{B_1} m'$ and the Boolean algebra C_2. Analogously there is $B_3 \in \Psi$ isomorphic to $[0,m]_{C_2} \times [0,m']_{C_o}$. Put $B_o = C_o$ and $B_2 = C_2$. We have constructed the 4-cycle (B_o, B_1, B_2, B_3) satisfying (iii) and hence L is the pasting of the pasted family Ψ.

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