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EXTENSIONS OF NONEXPANSIVE MAPPINGS IN THE HILBERT BALL WITH THE HYPERBOLIC METRIC. PART I.

Tadeusz KUCZUMOW and Adam STACHURA

Abstract: In an open unit disc $\Delta \subset C$ we have the Poincaré metric ρ_1. If $T:X\to\Delta$ is a ρ_1-nonexpansive mapping of a subset X of Δ into Δ, then there exists a ρ_1-nonexpansive mapping $\tilde{T}: \Delta\to\Delta$ such that its restriction to X is identical with T.

If in a complex Hilbert space H we take an open unit ball B with the hyperbolic metric, then for dim $H\geq 2$ the above fact is not true. Similarly as in Δ^n for $n\geq 2$.

Key words: Hyperbolic metric, nonexpansive mappings, fixed points.

Classification: 47H10, 32H15

Let B denote an open unit ball of a complex Hilbert space H. B can be furnished with the invariant hyperbolic metric ρ_1 given by the formula

$$\rho_1(x,y)=\tanh^{-1}[(1-\quad(x,y))^{1/2}],$$

where

$$\sigma(x,y)=\frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-(x,y)|^2}.$$

B(x,r) denotes a closed ball in (B,ρ_1) centered at x and of radius r.

It has been recently shown ([2],[3],[5]) that several ideas from the theory of nonexpansive mappings in Banach spaces can be used to yield new results concerning holomorphic self-mappings of B which are ρ_1-nonexpansive. In particular, it is useful to observe that certain metrical properties of (B,ρ_1) are analogous to properties of Hilbert spaces. Therefore there is a natural question if the Kirszbraun-Valentine theorem ([6]) on the existence of nonexpansive extensions for nonexpansive mappings in an arbitrary Hilbert space is still true in (B,ρ_1). In this paper we give the answer to this question.

We will consider first the case dim H=1. Then B is equal to the unit disc Δ on the complex plane $\mathbf{C}$ and ρ_1 is the Poincaré metric. The key role in our considerations will be played by the following

Lemma 1. Let a_1,a_2,a_3,b_1,b_2,b_3 be points of Δ satisfying inequalities $\rho_1(b_k,b_j) \leq \rho_1(a_k,a_j)$ for $k,j=1,2,3$. Then there exist points c_1,c_2,c_3 in Δ such that

(i) the inequalities $\rho_1(c_k,c_j) \leq \rho_1(a_k,a_j)$ $(k,j=1,2,3,\ k\neq j)$ are satisfied and at least two of them are actually equalities;

(ii) if the balls $B(c_1,r_1)$, $B(c_2,r_2)$, $B(c_3,r_3)$ have a nonempty intersection, then

$$\bigcap_{k=1}^{3} B(b_k,r_k)\neq\emptyset.$$

Proof: Without loss of generality we may assume that $0=b_1$, $b_2\in\mathbf{R}$, $\mathrm{Im}(b_3)\leq 0$, $\rho_1(0,b_2)<\rho_1(a_1,a_2)$ and $\rho_1(0,b_3)<\rho_1(a_1,a_3)$. It is easy to observe that there exists a point $c_1'=i\alpha\in\Delta$ $(0<\alpha\in\mathbf{R})$ satisfying

$$\rho_1(c_1',b_2)=\rho_1(a_1,a_2),\quad \rho_1(c_1',b_3)\leq\rho_1(a_1,a_3)$$

or

$$\rho_1(c_1',b_2)\leq\rho_1(a_1,a_2),\quad \rho_1(c_1',b_3)=\rho_1(a_1,a_3).$$

Let us denote $c_2'=b_2$, $c_3'=b_3$. Obviously if $\bigcap_{k=1}^{3} B(c_k',r_k)\neq\emptyset$ then $\bigcap_{k=1}^{3} B(b_k,r_k)\neq\emptyset$ ([3]). Applying this construction at most twice we get the sought points c_1,c_2,c_3.

Lemma 2. If a_1,a_2,a_3,b_1,b_2,b_3 are points of Δ such that $\rho_1(b_k,b_j)\leq$ $\leq\rho_1(a_k,a_j)$ $(k,j=1,2,3)$ and $\bigcap_{k=1}^{3} B(a_k,r_k)\neq\emptyset$ for some $r_1,r_2,\ r_3>0$, then $\bigcap_{k=1}^{3} B(b_k,r_k)\neq\emptyset$.

Proof: By Lemma 1 we may assume that $a_1=b_1=0$, $0<a_2=b_2\in\mathbf{R}$, $a_3=re^{i\alpha}$, $re^{i\beta}$, where $0<r<1$. We have

$$(1-a_2^2)(1-r^2)(1+a_2^2r^2-2a_2r\cos\alpha)^{-1}=\sigma(a_2,a_3)\leq\sigma(b_2,b_3)=$$
$$=(1-b_2^2)(1-r^2)(1+b_2^2r^2-2b_2r\cos\beta)^{-1}$$

and therefore $0\leq\beta\leq\alpha\leq\pi$. In $\bigcap_{k=1}^{3} B(a_k,r_k)$ we may find a point $Re^{it\alpha}$, where $0\leq t\leq 1$ ([3]). Hence the point $Re^{it\beta}$ lies in $\bigcap_{k=1}^{3} B(b_kr_k)$.

Now we may prove the following

Theorem 1. Let $\{B(x_\alpha,r_\alpha)\}_{\alpha\in I}$, $\{B(x'_\alpha,r_\alpha)\}_{\alpha\in I}$ be two families of balls in the disc Δ. If $\rho_1(x'_\alpha,x'_\beta)\le\rho_1(x_\alpha,x_\beta)$ for all $\alpha,\beta\in I$ and the intersection $\bigcap_{\alpha\in I} B(x_\alpha,r_\alpha)$ is nonempty then so is the intersection $\bigcap_{\alpha\in I} B(x'_\alpha,r_\alpha)$.

Proof: To prove this theorem it is sufficient to apply the Helly's Theorem ([4]) and Lemma 2.

The usual procedure based on the Kuratowski-Zorn Lemma gives the theorem on the existence of nonexpansive extension.

Theorem 2. Let $T:X\to\Delta$ be a ρ_1-nonexpansive mapping of a subset X of Δ into Δ. There exists a ρ_1-nonexpansive mapping $\tilde{T}:\Delta\to\Delta$ such that its restriction to X is identical with T.

Lemma 2, Theorem 1 and Theorem 2 fail to be true without assumption that dim H=1 as shown by the following example. In $\mathbf{C}^2$ we take points $a_1=(\alpha,0)$, $a_2=(i\alpha,0)$, $a_3=(-i\alpha,0)$, $b_1=(\alpha',0)$, $b_2=(0,\alpha)$, $b_3=(0,-\alpha)$, where $\alpha,\alpha'\in(0,1)$ and $\alpha'=\alpha[(1+\alpha^2)(1+\alpha^4)^{-1}]^{1/2}$.

For $r=\tanh^{-1}\alpha$ we have $0\in\bigcap_{k=1}^{3} B(a_k,r_k)$ and $\bigcap_{k=1}^{3} B(b_k,r_k)=\emptyset$.

Now let us consider the domain $B^n=B\times\ldots\times B$. The hyperbolic metric ρ_n on this domain is defined by

$$\rho_n((x_1,\ldots,x_n),\ (y_1,\ldots,y_n))=\max_{1\le k\le n}\rho_1(x_k,y_k)$$

for $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n)\in B^n$ ([1]). If we take H=**C**, n=2 and $B\times B=\Delta\times\Delta$ then for the points $a_1=(0,0)$, $a_2=(\alpha,0)$, $a_3=(0,\alpha)$, $b_1=(0,0)$, $b_2=(\alpha,0)$,

$$b_3=\left(\frac{\alpha(\alpha^2+1)}{2}+i\frac{\alpha[4-(\alpha^2+1)^2]^{1/2}}{2},0\right)$$

and $r=\frac{1}{2}\tanh^{-1}\alpha$ $(0<\alpha<1)$ we obtain $\bigcap_{k=1}^{3} B(a_k,r_k)\neq\emptyset$ and $\bigcap_{k=1}^{3} B(b_k,r_k)=\emptyset$.

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