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Jiří Vanžura

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MAXIMAL IDEALS IN THE LIE ALGEBRA OF VECTOR FIELDS

Jiří VANŽURA

Abstract: We describe maximal ideals in the Lie algebra $\mathfrak{X}(V)$ of all C^∞-vector fields on a C^∞-manifold V. Further we show that the set Specm $\mathfrak{X}(V)$ of all maximal ideals in $\mathfrak{X}(V)$ endowed with the Stone topology is homeomorphic with the Stone-Čech compactification βV of V.

Key words: C^∞-manifold, Lie algebra of C^∞-vector fields, maximal ideal, Stone-Čech compactification.

Classification: 17B65

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1. Maximal ideals in the associative algebra C(V). Let V be a connected paracompact real C^∞-manifold, dim V=m, and let C=C(V) denote the commutative and associative algebra of all real C^∞-functions on V.

For $f \in C$ we define the zero-set Z(f) of f by

$$Z(f)=\{p \in V; f(p)=0\}.$$

Z(f) is a closed subset of V. We recall the well known fact that every closed subset of V is the zero-set of some function from C. We shall now consider an ideal $I \subset C$. (Ideal in C will always mean proper ideal.) But first we introduce

Definition 1. A nonempty family $\mathcal{F}$ of closed subsets of V is called <u>z-filter</u> on V provided that

(i) $\emptyset \notin \mathcal{F}$

(ii) $Z,Z' \in \mathcal{F} \Rightarrow Z \cap Z' \in \mathcal{F}$

(iii) $Z \in \mathcal{F}$, $Z \subset Z'$, Z' is a closed subset of $V \Rightarrow Z' \in \mathcal{F}$.

By a <u>z-ultrafilter</u> on V we shall mean a maximal z-filter, i.e. one not contained in any other z-filter.

In the same way as in [1] we can prove the following

Proposition 1: (i) If $I \subset C$ is an ideal, then the family

$$Z[I]=\{Z(f); f \in I\}$$

is a z-filter on V.

(ii) If $\mathcal{F}$ is a z-filter on V, then the family

$$Z^{\leftarrow}[\mathcal{F}] = \{f; Z(f) \in \mathcal{F}\}$$

is an ideal in C.

It is easy to see that for any z-filter $\mathcal{F}$, and for any ideal I there is

$$Z[Z^{\leftarrow}[\mathcal{F}]] = \mathcal{F} \quad \text{and} \quad Z^{\leftarrow}[Z[I]] \supset I.$$

The last inclusion may be proper. (Take V=**R**, and let $I=(x^2)$ be the principal ideal generated by the function x^2. Then $Z^{\leftarrow}[Z[I]] =(x)$. If an ideal I satisfies $Z^{\leftarrow}[Z[I]] =I$ we shall call it <u>z-ideal.</u> Obviously every maximal ideal is a z-ideal.

Following again [1] we get the next two propositions.

Proposition 2: (i) If $M \subset C$ is a maximal ideal, then $Z[M]$ is a z-ultrafilter on V.

(ii) If $\mathcal{A}$ is a z-ultrafilter on V, then $Z^{\leftarrow}[\mathcal{A}]$ is a maximal ideal in C.

(iii) The mapping $Z^{\leftarrow}$ is one-one from the set of all z-ultrafilters on V onto the set of all maximal ideals in C.

Proposition 3: (i) Let $M \subset C$ be a maximal ideal. If Z(f) meets every member of $Z[M]$, then $f \in M$.

(ii) Let $\mathcal{A}$ be a z-ultrafilter on V. If a closed set $Z \subset V$ meets every member of $\mathcal{A}$, then $Z \in \mathcal{A}$.

Let $I \subset C$ be an ideal. We shall call I <u>fixed ideal</u> if $\bigcap Z[I] \neq \emptyset$, and <u>free ideal</u> if $\bigcap Z[I] =\emptyset$. We shall now describe fixed maximal ideals in C. Let $M \subset C$ be a fixed maximal ideal. We denote $S= \bigcap Z[M]$. Obviously $M \subset \{f \in C; f|S=0\}$, where the latter set is a (proper) ideal in C. Hence $M= \{f \in C; f|S=0\}$. But because for any two closed subsets $S_1 \subsetneq S_2$ there is

$$\{f \in C; f|S_1=0\} \supsetneq \{f \in C; f3S_2=0\},$$

we can see that S contains just one point, i.e. $S=\{p\}$. Then $M= \{f \in C; f(p)=0\}$. Conversely, for any point $p \in M$ the set

$$M_p = \{f \in C; f(p)=0\}$$

is an ideal in C. Moreover, it is a maximal ideal, because the factor $C/M_p \cong$ $\cong$**R** is a field. We have thus proved the following

Theorem 1. The fixed maximal ideals in C are precisely the sets

$$M_p=\{f\in C; f(p)=0\},\ p\in V.$$

The ideals M_p are distinct for distinct p.

Before proceeding further we shall need

Proposition 4: If a manifold is compact, then every ideal $I\subset C$ is fixed.

Proof: Let us take a finite number of functions $f_1,\dots,f_k\in I$. Then $\bigcap_{i=1}^{k} Z(f_i)=Z(\sum_{i=1}^{k} f_i^2)\neq\emptyset$ (otherwise $\sum_{i=0}^{k} f_i^2$ would be an invertible element). This shows that the family $Z[I]$ of closed subsets has the finite intersection property (i.e. every finite subfamily has a nonempty intersection). But V is compact, which implies $\bigcap Z[I]\neq\emptyset$.

As an immediate consequence of Th. 1 and Prop. 4 we get

Theorem 2: If a C^∞-manifold V is compact, then the correspondence $p\mapsto M_p$ is one-one from V onto the set of all maximal ideals in C.

This theorem describes the maximal ideals in C=C(V) for V compact. We shall now focus our attention to the case when V is only paracompact. Every paracompact topological space is completely regular, so that we may use results from Chapter 6 of [1]. Let βV denote the Stone-Čech compactification of the manifold V (considered as a topological space). Let $\mathcal{F}$ be a z-filter on V. We shall say that $\mathcal{F}$ converges to the limit $p\in\beta V$ if every neighborhood (in βV) of p contains a member of $\mathcal{F}$. Let us recall that every z-ultrafilter $\mathcal{A}$ on V has a unique limit $p\in\beta V$, and that p is a unique point such that $p\in\bigcap_{Z\in\mathcal{A}} cl_{\beta V}Z$, where $cl_{\beta V}$ denotes the closure in βV. Moreover every point $p\in\beta V$ is a limit of a unique z-ultrafilter $\mathcal{A}$ on V. In this way we get one-one mapping from βV onto the set of all z-ultrafilters on V. The unique z-ultrafilter having the limit $p\in\beta V$ we shall denote by $\mathcal{A}^p$. There is

$$\mathcal{A}^p=\{Z; Z\subset X \text{ is closed in } X,\ p\in cl_{\beta V}Z\}.$$

If $p\in V$ there is even a simpler description:

$$\mathcal{A}^p=\{Z; Z\subset X \text{ is closed in } X,\ p\in Z\}.$$

Theorem 3: Let V be a paracompact C^∞-manifold. The maximal ideals in C are precisely the sets

$$M^p=\{f\in C,\ p\in cl_{\beta V}Z(f)\},\ p\in\beta V.$$

The ideals M^p are distinct for distinct p. If $p\in V$, then $M^p=M_p$.

Proof: Let $M \subset C$ be a maximal ideal. Then according to Prop. 2 $Z[M]$ is a z-ultrafilter on V. Therefore there exists a unique $p \in \beta V$ such that $Z[M] = \mathcal{A}^p$. Now we have

$$M = Z^{\leftarrow}[Z[M]] = Z^{\leftarrow}[\mathcal{A}^p] = \{f \in C;\ Z(f) \in \mathcal{A}^p\} = \{f \in C;\ p \in cl_{\beta V} Z(f)\} = M^p.$$

Conversely, if $p \in \beta V$ is any point, then

$$M^p = \{f \in C;\ p \in cl_{\beta V} Z(f)\} = \{f \in C,\ Z(f) \in \mathcal{A}^p\} = Z^{\leftarrow}[\mathcal{A}^p],$$

which shows that M^p is a maximal ideal in C. The rest of the proof is obvious.

Furthermore we get easily

Theorem 4: Let V be a paracompact C^{∞}-manifold. Then the correspondence $p \longmapsto M^p$ is one-one from βV onto the set of all maximal ideals in C. This correspondence maps $V \subset \beta V$ onto the set of all fixed maximal ideals in C.

As usual we denote by Specm C the set of all maximal ideals in C. (It is called maximal spectrum of C.) We provide Specm C with the Stone topology. Namely, we take the family of all sets $\{M \in \text{Specm } C;\ f \in M\}$, $f \in C$ as a base for the closed sets. Along the same lines as in [1] we get

Theorem 5: The correspondence $p \longmapsto M^p$ is a homeomorphism from βV onto Specm C.

2. Maximal ideals in the Lie algebra $\mathfrak{X}(V)$. Let us denote by $\mathfrak{X} = \mathfrak{X}(V)$ the Lie algebra of all C^{∞}-vector fields on V. We recall the well known fact that $\mathfrak{X}$ can be naturally identified with the Lie algebra of all derivations on the algebra C.

We shall now consider an ideal $I \subset C$. Following [2] we define for any $n \in \mathbf{N}^* = \mathbf{N} \cup \{0\}$

$$I(n) = \{f \in I;\ Y_k(Y_{k-1}(\ldots(Y_1 f)\ldots)) \in I \text{ for any } Y_1, \ldots, Y_k \in \mathfrak{X} \text{ and } k = 0, \ldots, n\}.$$

Obviously there is $I = I(0) \supset I(1) \supset \ldots$. It can be easily checked that for any $n \in \mathbf{N}^*$ $I(n)$ is an ideal in C. For any $p \in V$, $f \in C$, and $n \in \mathbf{N}^* \cup \{\infty\}$ we denote by $j^n_p(f)$ the n-jet of the function f at the point p. Further we define the n-jet zero-set $Z_n(f)$ of f by

$$Z_n(f) = \{p \in V;\ j^n_p(f) = 0\}.$$

Obviously $Z_0(f) = Z(f)$. $Z_n(f)$ is a closed subset of V. But it can be shown that every closed subset of V is the n-jet zero-set of some function from C. (We recall that $n \in \mathbf{N}^* \cup \{\infty\}$ is arbitrary.) Let $\mathcal{F}$ be a z-filter on V. Then for any $n \in \mathbf{N}^* \cup \{\infty\}$ we define

$$Z_n^{\leftarrow}[\mathcal{F}] = \{f \in C;\ Z_n(f) \in \mathcal{F}\}.$$

It can be easily seen that $Z_n^{\leftarrow}[\mathcal{F}]$ is an ideal in C. We are now going to prove the following proposition.

Proposition 5: Let $\mathcal{F}$ be a z-filter on V. Then for any $n \in \mathbf{N}^*$ there is

$$(Z^{\leftarrow}[\mathcal{F}])(n) = Z_n^{\leftarrow}[\mathcal{F}].$$

Before starting the proof of Prop. 5 we shall introduce on V certain special functions and special vector fields which will be needed several times in the sequel. Because dim V=m, we can find (see [3]) m+1 families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_m$ of open subsets in V

$$\mathcal{U}_i = U_{i\alpha};\ \alpha \in \Sigma_i,\ 0 \leq i \leq m\}$$

with the following properties

(i) $\bigcup_{i=0}^{m} \bigcup_{\alpha \in \Sigma_i} U_{i\alpha} = V$

(ii) For any $0 \leq i \leq m$, and any $\alpha, \beta \in \Sigma_i$, $\alpha \neq \beta$ there is $U_{i\alpha} \cap U_{i\beta} = \emptyset$.

(iii) Each $U_{i\alpha}$ is a domain of a chart $(x_1^{(i\alpha)}, \ldots, x_m^{(i\alpha)})$.

Furthermore we can find open subsets $V_{i\alpha}$, $0 \leq i \leq m$, $\alpha \in \Sigma_i$ such that

(iv) $\mathrm{cl}_V V_{i\alpha} \subset U_{i\alpha}$

(v) $\bigcup_{i=0}^{m} \bigcup_{\alpha \in \Sigma_i} V_{i\alpha} = V$.

Now it can be easily seen that there exist functions $f_{ij} \in C$ and vector fields $X_{ij} \in \mathcal{X}$, $0 \leq i \leq m$, $1 \leq j \leq m$ such that

for any $\alpha \in \Sigma_i$ and any $p \in V_{i\alpha}$ there is

$$f_{ij}(p) = x_j^{(i\alpha)}(p),\quad X_{ij}(p) = \frac{\partial}{\partial x_j^{(i\alpha)}}(p).$$

Proof of Prop. 5: Let $f \in Z_n^{\leftarrow}[\mathcal{F}]$, i.e. $Z_n(f) \in \mathcal{F}$. For any $Y_1, \ldots, Y_k \in \mathcal{X}$, $0 \leq k \leq n$ we have

$$\{p \in V;\ (Y_k(Y_{k-1}(\ldots(Y_1 f)\ldots)))(p) = 0\} \supset Z_n(f),$$

and thus $\{p \in V;\ (Y_k(Y_{k-1}(\ldots(Y_1 f)\ldots)))(p) = 0\} \in \mathcal{F}$. From this follows $Y_k(Y_{k-1}(\ldots(Y_1 f)\ldots)) \in Z^{\leftarrow}[\mathcal{F}]$. We have proved that $Z_n^{\leftarrow}[\mathcal{F}] \subset (Z^{\leftarrow}[\mathcal{F}])(n)$.

Conversely let $f \in (Z^{\leftarrow}[\mathcal{F}])(n)$. We denote by $\mathcal{X}_0$ the finite subset of $\mathcal{X}$ consisting of the vector fields X_{ij}, $0 \leq i \leq m$, $1 \leq j \leq m$. If $p \in V$ and $g \in C$, then $j_p^n(g)=0$ if and only if

$$(Y_k(Y_{k-1}(\dots(Y_1 g)\dots)))(p)=0$$

for any $Y_1,\dots,Y_k \in \mathfrak{X}_0$, $0 \leq k \leq n$. Obviously

$$Z_n(g)=\bigcap_{k=0}^{n}\ \bigcap_{Y_1,\dots,Y_k \in \mathfrak{X}_0} \{p \in V;\ (Y_k(Y_{k-1}(\dots(Y_1 g)\dots)))(p)=0\}.$$

Using this formula we find easily that for $f \in (Z^{\leftarrow}[\mathcal{F}])(n)$ there is $Z_n(f) \in \mathcal{F}$. Thus we have proved that $(Z^{\leftarrow}[\mathcal{F}])(n) \subset Z_n^{\leftarrow}[\mathcal{F}]$.

Again following [2], for any ideal $I \subset C$, and any $n \in \mathbf{N}^*$ we define

$$\mathcal{L}_I^n = \{X \in \mathfrak{X}\ ;\ Xf \in I(n) \text{ for every } f \in C\}.$$

Furthermore we define

$$\mathcal{L}_I^\infty = \bigcap_{n=0}^{\infty} \mathcal{L}_I^n.$$

It can be proved (see [2]) that $\mathcal{L}_I^n$ for any $n \in \mathbf{N}^*$, and consequently $\mathcal{L}_I^\infty$, is an ideal in the Lie algebra $\mathfrak{X}$. As usual for any $p \in V$, $X \in \mathfrak{X}$, and $n \in \mathbf{N}^* \cup \{\infty\}$ we denote by $j_p^n(X)$ the n-jet of the vector field X at the point p. Similarly as for functions we define

$$\mathcal{Z}_n(X)=\{p \in V;\ j_p^n(X)=0\}.$$

Proposition 6: Let $\mathcal{F}$ be a z-filter on V. Then for $I=Z^{\leftarrow}[\mathcal{F}]$ and any $n \in \mathbf{N}^*$ there is

$$\mathcal{L}_I^n = \{X \in \mathfrak{X}\ ;\ \mathcal{Z}_n(X) \in \mathcal{F}\}.$$

Moreover

$$\mathcal{L}_I^\infty = \{X \in \mathfrak{X}\ ;\ \mathcal{Z}_n(X) \in \mathcal{F} \text{ for every } n \in \mathbf{N}^*\}.$$

Proof: Let $X \in \mathfrak{X}$ be such that $\mathcal{Z}_n(X) \in \mathcal{F}$. Then for any $f \in C$ we have $Z_n(Xf) \supset \mathcal{Z}_n(X)$, which shows that $Z_n(Xf) \in \mathcal{F}$. By virtue of Prop. 5 there is $Xf \in I(n)$, and thus $X \in \mathcal{L}_I^n$. We have proved that $\{X \in \mathfrak{X}\ ;\ \mathcal{Z}_n(X) \in \mathcal{F}\} \subset \mathcal{L}_I^n$.

Conversely, let $X \in \mathcal{L}_I^n$. Obviously there is $\mathcal{Z}_n(X)= \bigcap_{i=0}^{m} \bigcap_{j=1}^{m} Z_n(Xf_{ij})$. But $X \in \mathcal{L}_I^n$, which means that $Xf_{ij} \in I(n)$ for any $0 \leq i \leq m$, $1 \leq j \leq m$. By virtue of Prop. 5 it follows that $Z_n(Xf_{ij}) \in \mathcal{F}$ for any $0 \leq i \leq m$, $1 \leq j \leq m$. Using the above formula we can see that $\mathcal{Z}_n(X) \in \mathcal{F}$. We have proved that $\mathcal{L}_I^n \subset \{X \in \mathfrak{X}\ ;\ \mathcal{Z}_n(X) \in \mathcal{F}\}$. The assertion concerning $\mathcal{L}_I^\infty$ is now obvious.

We are now going to describe maximal ideals in the Lie algebra $\mathfrak{X}$. (Ideal in $\mathfrak{X}$ will always mean proper ideal.) First we shall state a fundamental result by Grabowski (see [2]):

(G) Let $\mathcal{L} \subset \mathfrak{X}$ be an ideal. Then there exists an ideal $I_0 \subset C$ such that for each prime ideal I containing I_0 there is $\mathcal{L} \subset \mathcal{L}_I^\infty$.

Keeping the above notation let us assume that $\mathcal{L} = \mathfrak{M} \subset \mathfrak{X}$ is a maximal ideal. We take I=M, where $M \subset C$ is a maximal ideal containing the ideal I_0. We denote $\mathcal{A} = Z[M]$. It is an easy consequence of Prop. 6 that $\mathcal{L}_M^\infty \subset \mathfrak{X}$ is a (proper) ideal. Thus for the maximal ideal $\mathfrak{M} \subset \mathcal{L}_M^\infty$ we have $\mathfrak{M} = \mathcal{L}_M^\infty$. This means that there is

$$\mathfrak{M} = \{X \in \mathfrak{X} \; ; \; \mathcal{Z}_n(X) \in \mathcal{A} \text{ for every } n \in \mathbf{N}^*\},$$

where $\mathcal{A}$ is a z-ultrafilter on V. Now it is natural to introduce

Definition 2: Let $\mathcal{F}$ be a z-filter on V. We define

$$\mathcal{Z}^{\leftarrow}[\mathcal{F}] = \{X \in \mathfrak{X} \; ; \; \mathcal{Z}_n(X) \in \mathcal{F} \text{ for every } n \in \mathbf{N}^*\}.$$

It is easy to see that $\mathcal{Z}^{\leftarrow}[\mathcal{F}]$ is an ideal in the Lie algebra $\mathfrak{X}$. Using this notation we can formulate the above result as

Theorem 6: Let $\mathfrak{M} \subset \mathfrak{X}$ be a maximal ideal. Then there exists a z-ultrafilter $\mathcal{A}$ on V such that $\mathfrak{M} = \mathcal{Z}^{\leftarrow}[\mathcal{A}]$.

Our next goal will be to prove that any ideal of the above form is in fact a maximal ideal. But first we shall establish the existence of maximal ideals in $\mathfrak{X}$. Here we have at least two possibilities how to proceed. We have chosen that one which fits better into our setting.

Proposition 7: Let $\mathcal{L} \subset \mathfrak{X}$ be an ideal. Then for any $X \in \mathcal{L}$ we have

$$\mathcal{Z}_n(X) \neq \emptyset \text{ for any } n \in \mathbf{N}^*.$$

Proof: Let $I_0 \subset C$ be the ideal described in (G). We take any maximal ideal $M \subset C$ such that $M \supset I_0$. According to (G) there is $\mathcal{L} \subset \mathcal{L}_M^\infty$. Denoting $\mathcal{A} = Z^{\leftarrow}[M]$ we have by virtue of Prop. 6 $\mathcal{L} \subset \mathcal{Z}^{\leftarrow}[\mathcal{A}]$. Thus for any $X \in \mathcal{L}$ and any $n \in \mathbf{N}^*$ we have $\mathcal{Z}_n(X) \in \mathcal{A}$, which implies $\mathcal{Z}_n(X) \neq \emptyset$.

Theorem 7: Let $\mathcal{L} \subset \mathfrak{X}$ be an ideal. Then there exists a maximal ideal $\mathfrak{M} \subset \mathfrak{X}$ such that $\mathcal{L} \subset \mathfrak{M}$.

Proof: First we shall prove that there exists a vector field $Y \in \mathfrak{X}$ such that $\mathcal{Z}_1(Y) = \emptyset$. For this purpose let us take a Morse function $f \in C$ (i.e. a function with nondegenerate critical points), and let us choose an auxiliary riemannian metric g on V. We define a vector field $Y \in \mathfrak{X}$ by the equation

$$g(\cdot, Y) = df.$$

It can be immediately seen that $\mathcal{Z}_1(Y)=\emptyset$.

By virtue of the previous proposition the vector field Y cannot belong to any (proper) ideal. Let us consider now a family $\{\mathcal{L}_\sigma;\ \sigma\in\Sigma\}$ of (proper) ideals in $\mathfrak{X}$, each of which contains the ideal $\mathcal{L}$, and let us assume that this family is totally ordered with respect to the inclusion. The union $\bigcup_{\sigma\in\Sigma}\mathcal{L}_\sigma$ is obviously an ideal in $\mathfrak{X}$ (possibly improper). But because for any $\sigma\in\Sigma$ there is $Y\notin\mathcal{L}_\sigma$, we have $Y\notin\bigcup_{\sigma\in\Sigma}\mathcal{L}_\sigma$, which shows that $\bigcup_{\sigma\in\Sigma}\mathcal{L}_\sigma$ is a proper ideal. Thus by virtue of the Zorn's lemma there exists a maximal ideal $\mathfrak{M}\subset\mathfrak{X}$ such that $\mathcal{L}\subset\mathfrak{M}$.

Let us consider an ideal $\mathcal{L}\subset\mathfrak{X}$, and let $\mathfrak{M}\subset\mathfrak{X}$ be a maximal ideal such that $\mathcal{L}\subset\mathfrak{M}$. Let $\mathcal{A}$ be a z-ultrafilter on V with the property $\mathfrak{M}=$ $=\mathcal{Z}^{\leftarrow}[\mathcal{A}]$, which exists by virtue of Th. 6. The family

$$\{\mathcal{Z}_n(X);\ X\in\mathcal{L},\ n\in\mathbf{N}^*\}$$

of closed sets in V is a subfamily of the z-ultrafilter $\mathcal{A}$, and therefore has the finite intersection property. Consequently it generates a z-filter on V.

Definition 3: Let $\mathcal{L}\subset\mathfrak{X}$ be an ideal. The z-filter on V generated by the family $\{\mathcal{Z}_n(X),\ X\in\mathcal{L},\ n\in\mathbf{N}^*\}$ we shall denote by $\mathcal{Z}[\mathcal{L}]$. We shall call $\mathcal{L}$ <u>fixed ideal</u> if $\bigcap\mathcal{Z}[\mathcal{L}]\neq\emptyset$, and <u>free ideal</u> if $\bigcap\mathcal{Z}[\mathcal{L}]=\emptyset$.

Let $\mathcal{L}\subset\mathfrak{X}$ be a fixed ideal. It is easy to see that $p\in\bigcap\mathcal{Z}[\mathcal{L}]$ if and only if $j^\infty_p(X)=0$ for every $X\in\mathcal{L}$.

We recall that any family of closed sets with the finite intersection property in a compact topological space has a nonempty intersection. From this follows easily

Proposition 8: If a manifold V is compact, then every ideal $\mathcal{L}\subset\mathfrak{X}$ is fixed.

Proposition 9: For any z-filter $\mathcal{F}$ on V, and any ideal $\mathcal{L}\subset\mathfrak{X}$ there is

$$\mathcal{Z}[\mathcal{Z}^{\leftarrow}[\mathcal{F}]]=\mathcal{F} \text{ and } \mathcal{Z}^{\leftarrow}[\mathcal{Z}[\mathcal{L}]]\supset\mathcal{L}.$$

Proof: The inclusions $\mathcal{L}\subset\mathcal{Z}^{\leftarrow}[\mathcal{Z}[\mathcal{L}]]$ and $\mathcal{Z}[\mathcal{Z}^{\leftarrow}[\mathcal{F}]]\subset\mathcal{F}$ are obvious. It remains to prove the inclusion $\mathcal{F}\subset\mathcal{Z}[\mathcal{Z}^{\leftarrow}[\mathcal{F}]]$. Let $Z\in\mathcal{F}$ be a closed set. There exists a function $f\in C$ such that $Z_0(f)=Z_\infty(f)=Z$. Obviously for any $0\leq i\leq m$ and any $n\in\mathbf{N}^*$ there is $\mathcal{Z}_n(fX_{i1})\supset Z$, and thus $\mathcal{Z}_n(fX_{i1})\in\mathcal{F}$. This shows that $fX_{i1}\in\mathcal{Z}^{\leftarrow}[\mathcal{F}]$ for every $0\leq i\leq m$. Moreover there is

$\bigcap_{i=0}^{m} \mathcal{Z}_0(fX_{i1})=Z$, which proves that $Z \in \mathcal{Z}[\mathcal{Z}^{\leftarrow}[\mathcal{F}']]$.

Theorem 8: Let $\mathcal{A}$ be a z-ultrafilter on V. Then $\mathfrak{M} = \mathcal{Z}^{\leftarrow}[\mathcal{A}]$ is a maximal ideal in $\mathfrak{X}$.

Proof: We know already that $\mathfrak{M}$ is a (proper) ideal. According to Th. 7 there exists a maximal ideal $\mathfrak{M}' \subset \mathfrak{X}$ such that $\mathfrak{M} \subset \mathfrak{M}'$. Furthermore by virtue of Th. 6 there is a z-ultrafilter $\mathcal{A}'$ such that $\mathfrak{M}' = \mathcal{Z}^{\leftarrow}[\mathcal{A}']$. We have therefore $\mathcal{Z}^{\leftarrow}[\mathcal{A}] \subset \mathcal{Z}^{\leftarrow}[\mathcal{A}']$, which implies $\mathcal{Z}[\mathcal{Z}^{\leftarrow}[\mathcal{A}]] \subset \mathcal{Z}[\mathcal{Z}^{\leftarrow}[\mathcal{A}']]$. From this, using Prop. 9, we obtain $\mathcal{A} \subset \mathcal{A}'$. But $\mathcal{A}$ is a z-ultrafilter, and therefore $\mathcal{A} = \mathcal{A}'$. Now we get $\mathfrak{M} = \mathfrak{M}'$, which proves that $\mathfrak{M} = \mathcal{Z}^{\leftarrow}[\mathcal{A}]$ is a maximal ideal.

We recall that for any $p \in \beta V$ we have denoted by $\mathcal{A}^p$ the unique z-ultrafilter on V having the limit p.

Theorem 9: Let V be a paracompact C^{∞}-manifold. Then the correspondence $p \longmapsto \mathcal{Z}^{\leftarrow}[\mathcal{A}^p]$ is one-one from βV onto the set of all maximal ideals in $\mathfrak{X}$. This correspondence maps $V \subset \beta V$ onto the set of all fixed maximal ideals in $\mathfrak{X}$. If $p \in V$, then

$$\mathcal{Z}^{\leftarrow}[\mathcal{A}^p] = \{X \in \mathfrak{X} \; ; \; j_p^{\infty}(X)=0\}.$$

Proof: Using Th. 6 we can easily see that the mapping $p \longmapsto \mathcal{Z}^{\leftarrow}[\mathcal{A}^p]$ is surjective. Let us consider two points $p,q \in \beta V$ such that $\mathcal{Z}^{\leftarrow}[\mathcal{A}^p] = \mathcal{Z}^{\leftarrow}[\mathcal{A}^q]$. By virtue of the first formula in Prop. 9 we get $\mathcal{A}^p = \mathcal{A}^q$, and consequently p=q. This proves that the mapping $p \longmapsto \mathcal{Z}^{\leftarrow}[\mathcal{A}^p]$ is injective.

The equality $\mathcal{Z}^{\leftarrow}[\mathcal{A}^p] = \{X \in \mathfrak{X} \; ; \; j_p^{\infty}(X)=0\}$ for $p \in V$ is obvious. (We recall that for $p \in V$ $\mathcal{A}^p$ is the family of all closed subsets of V containing the point p.) This shows that the mapping $p \longmapsto \mathcal{Z}^{\leftarrow}[\mathcal{A}^p]$ maps V into the set of all fixed maximal ideals in $\mathfrak{X}$. Conversely, let $\mathfrak{M} = \mathcal{Z}^{\leftarrow}[\mathcal{A}]$ be a fixed ideal. Then $S = \bigcap \mathcal{Z}[\mathcal{A}]$ is a nonempty set. Obviously

$$\mathfrak{M} \subset \{X \in \mathfrak{X} \; ; \; j_q^{\infty}(X)=0 \text{ for every } q \in S\},$$

where the latter set is a proper ideal in $\mathfrak{X}$. Now the maximality of $\mathfrak{M}$ implies

$$\mathfrak{M} = \{X \in \mathfrak{X} \; ; \; j_q^{\infty}(X)=0 \text{ for every } q \in S\}.$$

But because for any two closed subsets $S_1, S_2 \subset V$ satisfying $S_1 \subsetneq S_2$ there is

$$\{X \in \mathfrak{X} \; ; \; j_q^{\infty}(X)=0 \text{ for every } q \in S_1\} \supsetneq \{X \in \mathfrak{X}; \; j_q^{\infty}(X)=0 \text{ for every } q \in S_2\},$$

we can see that S contains just one point, i.e. $S = \{p\}$. Then $\mathfrak{M} = \{X \in \mathfrak{X}; \; j_p^{\infty}(X)=0\}$, which finishes the proof.

We denote now by Specm $\mathfrak{X}$ the set of all maximal ideals in the Lie algebra $\mathfrak{X}$ endowed with the Stone topology. Let us recall that this topology has the family $\{\mathfrak{m} \in \text{Specm } \mathfrak{X} ;\ X \in \mathfrak{m}\}$, $X \in \mathfrak{X}$ as a base for the closed sets.

Theorem 10: Let V be a paracompact C^{∞}-manifold. Then the correspondence $p \longmapsto \mathcal{Z}^{\leftarrow}[\mathcal{A}^{p}]$ is a homeomorphism from βV onto Specm $\mathfrak{X}$.

Proof: If $Z \subset V$ is a closed subset, we denote $\hat{Z} = \{p \in \beta V;\ Z \in \mathcal{A}^{p}\}$. We recall that the system of all sets of the form $\hat{Z}$, where $Z \subset V$ is an arbitrary closed set, represents a base for the closed sets in the Stone-Čech compactification βV.

We denote the mapping $p \longmapsto \mathcal{Z}^{\leftarrow}[\mathcal{A}^{p}]$ by ι. First we shall prove that ι is continuous. Let $X \in \mathfrak{X}$ be arbitrary, and let us denote $A_X = \{\mathfrak{m} \in \text{Specm } \mathfrak{X} ;\ X \in \mathfrak{m}\}$. Obviously it suffices to prove that $\iota^{-1}(A_X)$ is closed in βV. But for $p \in \beta V$ there is $\iota(p) = \mathcal{Z}^{\leftarrow}[\mathcal{A}^{p}]$, and $\iota(p) \in A_X$ if and only if $X \in \mathcal{Z}^{\leftarrow}[\mathcal{A}^{p}]$. This means that $\mathcal{Z}_n(X) \in \mathcal{A}^{p}$ for every $n \in \mathbf{N}^{*}$. We can now see that

$$\iota^{-1}(A_X) = \bigcap_{n=0}^{\infty} \{p \in \beta V;\ \mathcal{Z}_n(X) \in \mathcal{A}^{p}\} = \bigcap_{n=0}^{\infty} \widehat{\mathcal{Z}_n(X)}$$

is a closed subset in βV.

Next we prove that ι is a closed mapping. Here it suffices to prove that for any closed set $Z \subset V$ $\iota(\hat{Z})$ is a closed set in Specm $\mathfrak{X}$. Let $\mathfrak{m} \in \text{Specm } \mathfrak{X}$. We can see that $\mathfrak{m} \in \iota(\hat{Z})$ if and only if $Z \in \mathcal{Z}[\mathfrak{m}]$. Similarly as in the proof of Prop. 9 let us take a function $f \in C$ such that $Z_0(f) = Z_{\infty}(f) = Z$. We shall prove that

$$Z \in \mathcal{Z}[\mathfrak{m}] \Longleftrightarrow fX_{i1} \in \mathfrak{m} \text{ for } 0 \leq i \leq m.$$

If $Z \in \mathcal{Z}[\mathfrak{m}]$, then $\mathcal{Z}_n(fX_{i1}) \supset Z$ for every $n \in \mathbf{N}^{*}$. This implies $\mathcal{Z}_n(fX_{i1}) \in \mathcal{Z}[\mathfrak{m}]$ for every $n \in \mathbf{N}^{*}$, and consequently $fX_{i1} \in \mathfrak{m}$. (Notice that by virtue of Th. 6 and Prop. 9 there is $\mathcal{Z}^{\leftarrow}[\mathcal{Z}[\mathfrak{m}]] = \mathcal{Z}^{\leftarrow}[\mathcal{Z}[\mathcal{Z}^{\leftarrow}[\mathcal{A}]]] = \mathcal{Z}^{\leftarrow}[\mathcal{A}] = \mathfrak{m}$.) Conversely let $fX_{i1} \in \mathfrak{m}$ for $0 \leq i \leq m$. Then $\mathcal{Z}_0(fX_{i1}) \in \mathcal{Z}[\mathfrak{m}]$. We can see that $Z = \bigcap_{i=0}^{m} \mathcal{Z}_0(fX_{i1}) \in \mathcal{Z}[\mathfrak{m}]$. Now it is obvious that

$$\iota(\hat{Z}) = \bigcap_{i=0}^{m} \{\mathfrak{m} \in \text{Specm } \mathfrak{X} ;\ fX_{i1} \in \mathfrak{m}\}$$

is a closed set in Specm $\mathfrak{X}$. This finishes the proof.

Let us assume now that the manifold V is not compact. We denote by $\mathfrak{X}_c$ the subset of $\mathfrak{X}$ consisting of all vector fields with compact support. $\mathfrak{X}_c$ is obviously a free ideal in $\mathfrak{X}$.

Theorem 11: Let V be a paracompact C^{∞}-manifold which is not compact. Then the intersection of all free maximal ideals in $\mathfrak{X}$ coincides with $\mathfrak{X}_c$.

Before starting the proof of Th. 11 we recall some facts. A z-filter $\mathcal{F}$ on V is called prime z-filter if it has the following property: if $Z', Z'' \subset V$ are two closed sets such that $Z' \cup Z'' \in \mathcal{F}$, then either $Z' \in \mathcal{F}$ or $Z'' \in \mathcal{F}$. Every z-ultrafilter is a prime z-filter (see [1]). We call a z-filter free or fixed according as the intersection of its members is empty or nonempty. Obviously an ideal $\mathcal{L} \subset \mathfrak{X}$ is free (fixed) if and only if the z-filter $\mathcal{Z}^{\leftarrow}[\mathcal{L}]$ is free (fixed). A closed set $Z \subset V$ is compact if and only if it belongs to no free z-filter (see [1]).

Proof of Th. 11: Let $X \in \mathfrak{X}_c$, and let $\mathfrak{M} \subset \mathfrak{X}$ be a free maximal ideal. Further let $\mathcal{A} = \mathcal{Z}^{\leftarrow}[\mathcal{F}]$. For any $n \in \mathbf{N}^*$ we denote

$$\mathrm{supp}_n X = \mathrm{cl}_V \{p \in V;\ j_p^n(X) \neq 0\},$$

where cl_V denotes the closure in V. Obviously

$$\mathrm{supp}_n X \cup \mathcal{Z}_n(X) = V \in \mathcal{A}.$$

$\mathcal{A}$ is a z-ultrafilter, and therefore a prime z-filter. Consequently either $\mathrm{supp}_n X \in \mathcal{A}$ or $\mathcal{Z}_n(X) \in \mathcal{A}$. But $\mathrm{supp}_n X \subset \mathrm{supp}\ X$, and this implies that $\mathrm{supp}_n X$ is compact. Therefore $\mathrm{supp}_n X \notin \mathcal{A}$ because $\mathcal{A}$ is a free z-filter. Thus we have $\mathcal{Z}_n(X) \in \mathcal{A}$ for every $n \in \mathbf{N}^*$, which means that $X \in \mathfrak{M}$. We have proved that $\mathfrak{X}_c$ is contained in the intersection of all free maximal ideals.

Conversely let us assume that $X \in \mathfrak{X}$ belongs to all free maximal ideals. Then $\mathcal{Z}_0(X)$ belongs to all free z-ultrafilters. If $\mathcal{Z}_0(X) = V$, then $X=0$ and $X \in \mathfrak{X}_c$. Thus let us assume that $\mathcal{Z}_0(X) \subsetneq V$. It suffices to prove that supp X is compact. Let us suppose that this is not the case. Then it is not difficult to see that there exists a closed noncompact set $Z \subset V - \mathcal{Z}_0(X)$. (To see this it suffices for example to embed V into a euclidean space.) Because Z is not compact, there exists a free z-ultrafilter $\mathcal{A}$ such that $Z \in \mathcal{A}$. We get therefore $\emptyset = Z \cap \mathcal{Z}_0(X) \in \mathcal{A}$, which is a contradiction. This contradiction shows that supp X is compact. We have proved that the intersection of all free maximal ideals is contained in $\mathfrak{X}_c$.

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Mathematical Institute of ČSAV, Mendlovo nám. 1, 603 00 Brno, Czechoslovakia

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