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AN INTEGRAL FORMULA FOR CLOSED SURFACES
AND A GENERALIZATION OF Hp-THEOREM

Giovanni ROTONDARO

Abstract: Let H , r , p be the mean curvature, distance and support functions for an immersion $f:M \rightarrow \mathbb{R}^3$ of a closed orientable surface, with area element dS . We prove the integral formula $\int_M (p^2 - Hpr^2)/r^4 dS = 0$, and deduce that, if $Hp=1$, then M is embedded as a standard sphere.

Key words: Closed surface, support function, mean curvature, sphere.

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Let $f:M \rightarrow \mathbb{R}^3$ be a C^∞ immersion of a closed orientable C^∞ surface into Euclidean three-space. Let n denote the unit normal field of f , dS the area element, H the mean curvature, K the Gauss curvature, $p = -f \cdot n$ the support function and $r = |f|$ the distance function. The well-known Hp -theorem ([1],[2]) asserts that if $Hp=1$ and $K > 0$ then M is embedded as a standard sphere. In this note we prove an integral formula (Theorem 1) and deduce (Theorem 2) a generalization of Hp -theorem which avoids the stringent hypothesis of convexity.

Theorem 1. In the above situation, if the origin of coordinates does not lie in $f(M)$, then

$$\int_M \frac{p^2 - Hpr^2}{r^4} ds = 0.$$

Proof. Consider the conformal diffeomorphism

$$i: x \in \mathbb{R}^3 \rightarrow \frac{c^2}{(r(x))^2} x \in \mathbb{R}^3$$

where $c > 0$ is a fixed real number. Immerse M in \mathbb{R}^3 via $f^* = i \circ f$ and denote by n^* , H^* , ... the differential-geometric entities associated with f^* . Then, by

a routine calculation, we have (*)

$$\begin{aligned} df^* \cdot df^* &= \frac{c^4}{r^4} df \cdot df & dS^* &= \frac{c^4}{r^4} ds \\ - df^* \cdot dn^* &= \frac{c^2}{r^2} df \cdot dn + \frac{2pc^2}{r^4} df \cdot df. \end{aligned}$$

Hence

$$K^* dS^* = KdS + 4 \frac{p^2 - Hpr^2}{r^4} ds.$$

On integration, we have

$$\int_M \frac{p^2 - Hpr^2}{r^4} ds = \frac{1}{4} \int_M (KdS - K^*dS^*) = 0$$

by the Gauss-Bonnet theorem.

Theorem 2. If $Hp=1$, then M is embedded as a standard sphere.

Proof. Applying our formula, we have

$$\int_M \frac{p^2 - r^2}{r^4} ds = 0,$$

which implies $p^2 = r^2$. Changing orientation, if necessary, this gives $f = -pn$. Then, denoting by subscripts partial derivatives with respect to some local coordinates, $f_i = -p_i n - pn_i$ ($i=1,2$), which implies $p_1 = p_2 = 0$. Therefore p is a constant, and so $|f|$.

(*) The reader can consult [3,p.110], paying attention to some misprints.

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