COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,2 (1988)

AN INTEGRAL FORMULA FOR CLOSED SURFACES AND A GENERALIZATION OF Hp-THEOREM

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Abstract: Let H, r, p be the mean curvature, distance and support functions for an immersion $f:M \to R^3$ of a closed orientable surface, with area element dS. We prove the integral formula $\int_M (p^2-Hpr^2)/r^4\, dS=0$, and deduce that, if Hp=1, then M is embedded as a standard sphere.

Key words: Closed surface, support function, mean curvature, sphere.

Classification: Primary: 53A05
Secondary: 53C45

Let $f:M \to R^3$ be a C^∞ immersion of a closed orientable C^∞ surface into Euclidean three-space. Let n denote the unit normal field of f, dS the area element, H the mean curvature, K the Gauss curvature, $p= -f\cdot n$ the support function and $r=|f|$ the distance function. The well-known Hp-theorem ([1],[2]) asserts that if Hp=1 and $K>0$ then M is embedded as a standard sphere. In this note we prove an integral formula (Theorem 1) and deduce (Theorem 2) a generalization of Hp-theorem which avoids the stringent hypothesis of convexity.

Theorem 1. In the above situation, if the origin of coordinates does not lie in $f(M)$, then

$$\int_M \frac{p^2-Hpr^2}{r^4}\, ds=0.$$

Proof. Consider the conformal diffeomorphism

$$i:x\in R^3 \to \frac{c^2}{(r(x))^2}\, x\in R^3$$

where $c>0$ is a fixed real number. Immerse M in R^3 via $f^*=i\circ f$ and denote by n^*, H^*,... the differential-geometric entities associated with f^*. Then, by a routine calculation, we have (*)

$$df^{*} \cdot df^{*} = \frac{c^4}{r^4}\, df \cdot df \qquad\qquad dS^{*} = \frac{c^4}{r^4}\, ds$$

$$- df^{*} \cdot dn^{*} = \frac{c^2}{r^2}\, df \cdot dn + \frac{2pc^2}{r^4}\, df \cdot df.$$

Hence

$$K^{*}\, dS^{*} = KdS + 4\, \frac{p^2 - Hpr^2}{r^4}\, ds.$$

On integration, we have

$$\int_M \frac{p^2 - Hpr^2}{r^4}\, ds = \frac{1}{4} \int_M (KdS - K^{*}dS^{*}) = 0$$

by the Gauss-Bonnet theorem.

Theorem 2. If $Hp=1$, then M is embedded as a standard sphere.

Proof. Applying our formula, we have

$$\int_M \frac{p^2 - r^2}{r^4}\, ds = 0,$$

which implies $p^2 = r^2$. Changing orientation, if necessary, this gives $f = -pn$. Then, denoting by subscripts partial derivatives with respect to some local coordinates, $f_i = -p_i n - p n_i$ $(i=1,2)$, which implies $p_1 = p_2 = 0$. Therefore p is a constant, and so $|f|$.

(*) The reader can consult [3,p.110],paying attention to some misprints.

References

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(Oblatum 4.1. 1988)