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ON SUPERTIGHTNESS AND FUNCTION SPACES

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Abstract: For a Tychonoff space X, we denote by $C_p(X)$ the function space on X with the topology of pointwise convergence. M.O. Asanov showed that X^n has countable tightness for every natural number n if $C_p(X)$ is Lindelöf. In this note we shall strengthen Asanov's result. We show that X^n has countable supertightness for every natural number n if $C_p(X)$ is Lindelöf.

Key words: Function space, supertightness, Lindelöf space.

Classification: 54A25, 54C35, 54D20

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1. Introduction. In this paper by a space we shall always mean a Tychonoff space. **N** denotes the positive integers. Unexplained notions and terminology follow [2]. We begin with some definitions. We denote by $C_p(X)$ the function space on a space X with the topology of pointwise convergence. Basic open sets of $C_p(X)$ are of the form $[x_1,x_2,\dots,x_k;\ U_1,U_2,\dots,U_k]=$ $=\{f\in C_p(X):f(x_i)\in U_i\ \ i=1,2,\dots,k\}$, where $k\in \mathbf{N}$, $x_i\in X$ and each U_i is an open subset of the real-line.

A collection of subsets $\mathcal{F}$ of a space X is called a π-network for $x\in X$ provided that every neighborhood of x contains a member from $\mathcal{F}$. The supertightness st(x,X) of x in X is defined to be the least cardinal κ for which every π-network $\mathcal{F}$ for x consisting of finite subsets of X contains a subfamily $\mathcal{G}\subset\mathcal{F}$ of cardinality $\le\kappa$ which is a π-network for x. The supertightness st(X) of X is defined by $st(X)=\omega\cdot\sup\{st(x,X):x\in X\}$. The concept of supertightness was introduced in [3] to estimate the character of supercompact spaces. It is clear that $t(X)\le st(X)$ for a space X, where t(X) is the tightness of X.

This paper is motivated by the concept of supertightness and Asanov's result. Asanov showed in [1] that X^n has countable tightness for every $n\in \mathbf{N}$ if $C_p(X)$ is Lindelöf. In this note we shall strengthen Asanov's result. We show that X^n has countable supertightness for every $n \in \mathbf{N}$ if $C_p(X)$ is Lindelöf. In addition, we also show the equality $st(C_p(X))=t(C_p(X))$.

There is a supercompact space X such that $t(X^\omega)=\omega$ while $st(X)=2^\omega$ [3, Example 2.6].

2. Results. For a space X we set $l(X)=\min\{\kappa :$ every open cover of X has a subcover of cardinality $\leq \kappa\}$. For $x_1,\dots,x_n \in X$ and $f \in C_p(X)$ we define $(x_1+\dots+x_n)(f)=f(x_1)+\dots+f(x_n)$, then $x_1+\dots+x_n$ is a continuous function on $C_p(X)$.

Theorem 2.1. $l(C_p(X)) \geq st(X^n)$ holds for each $n \in \mathbf{N}$.

Proof. We set $l(C_p(X))=\kappa$. We fix $n \in \mathbf{N}$ and $(z_1,\dots,z_n) \in X^n$. Let $\mathcal{F}$ be a π-network of finite subsets of X^n for $(z_1,\dots,z_n)$. We must find a subfamily $\mathcal{G} \subset \mathcal{F}$ of cardinality $\leq \kappa$ which is a π-network for $(z_1,\dots,z_n)$. Let U_i be an open neighborhood of z_i such that $U_i \cap U_j=\emptyset$ if $z_i \neq z_j$, and $U_i=U_j$ if $z_i=z_j$. We may assume $F \subset U_1 \times \dots \times U_n$ for every $F \in \mathcal{F}$. For $A=\{z_1,\dots,z_n\}$ we set $C_p(X;A)=\{f \in C_p(X): f|A=0\}$. Since $C_p(X;A)$ is closed in $C_p(X)$, $l(C_p(X;A)) \leq \kappa$. We claim that

$$C_p(X;A) \subset \bigcup_{F \in \mathcal{F}} \left(\bigcap \{(x_1+\dots+x_n)^{\leftarrow}(-1,1):(x_1,\dots,x_n) \in F\}\right).$$

In fact, for $f \in C_p(X;A)$ $f^{\leftarrow}(-1/n,1/n) \times \dots \times f^{\leftarrow}(-1/n,1/n)$ is a neighborhood of $(z_1,\dots,z_n)$, hence there is an $F \in \mathcal{F}$ such that $F \subset f^{\leftarrow}(-1/n,1/n) \times \dots \times f^{\leftarrow}(-1/n,1/n)$. This means that for each $(x_1,\dots,x_n) \in F$, $|(x_1+\dots+x_n)(f)| \leq |f(x_1)|+\dots+|f(x_n)| < 1$. Thus $f \in \bigcap\{(x_1+\dots+x_n)^{\leftarrow}(-1,1):(x_1,\dots,x_n) \in F\}$.

We can find a subfamily $\mathcal{G} \subset \mathcal{F}$ of cardinality $\leq \kappa$ such that

$$C_p(X;A) \subset \bigcup_{F \in \mathcal{G}} \left(\bigcap\{(x_1+\dots+x_n)^{\leftarrow}(-1,1):(x_1,\dots,x_n) \in F\}\right).$$

We claim that $\mathcal{G}$ is a π-network for $(z_1,\dots,z_n)$. Let V_i be an open neighborhood of z_i. Without loss of generality we may assume that $V_i \subset U_i$, and $V_i=V_j$ if $z_i=z_j$. Let f be a nonnegative continuous function on X such that $f|A=0$ and $f|X-V_1 \cup \dots \cup V_n=1$. Then there is an $F \in \mathcal{F}$ such that $f \in \bigcap\{(x_1+\dots+x_n)^{\leftarrow}(-1,1):(x_1,\dots,x_n) \in F\}$. This means that for each $(x_1,\dots,x_n) \in F$ $f(x_1)+\dots+f(x_n) < 1$. Since f is non-negative, we have $\{x_1,\dots,x_n\} \subset V_1 \cup \dots \cup V_n$. Thus $(x_1,\dots,x_n) \in V_{i_1} \times \dots \times V_{i_n} \cap U_1 \times \dots \times U_n$, where $V_{i_j} \in \{V_1,\dots,V_n\}$. It is not difficult to see that $V_{i_j}=V_j$ because of $V_{i_j} \cap U_j$ is not empty for $j=1,\dots,n$. Consequently we have $F \subset V_1 \times \dots \times V_n$. The proof is complete.

Corollary 2.2. If $C_p(X)$ is Lindelöf, then X^n has countable supertightness for every $n \in \mathbf{N}$.

Theorem 2.3. For a space X $st(C_p(X))=t(C_p(X))$ holds.

Proof. We set $t(C_p(X))=\kappa$. Let $\mathcal{F}$ be a π-network of finite subsets of $C_p(X)$ for $f \in C_p(X)$. Since $C_p(X)$ is homogeneous, we may assume f is the constant function 0. For each $F \in \mathcal{F}$ and $n \in \mathbf{N}$ we set $U_n(F)=\bigcap_{f \in F} f^{\leftarrow}(-1/n,1/n)$. We set $A_n=\{h \in C_p(X): h|X-U_n(F)=1$ for some $F \in \mathcal{F}\}$ for each $n \in \mathbf{N}$. Then we have $f \in \bigcap_n \overline{A}_n$. In fact, let $G=[x_1,\dots,x_k;\ W_1,\dots,W_k]$ be any basic neighborhood of f. We take an $\varepsilon > 0$ such that $\varepsilon > 1/n$ and $W=(-\varepsilon,\varepsilon) \subset \bigcap\{W_i: i=1,\dots,k\}$. Since $[x_1,\dots,x_k;\ W,\dots,W]$ is a neighborhood of f, there is an $F \in \mathcal{F}$ such that $F \subset [x_1,\dots,x_k;\ W,\dots,W]$. This means $\{x_1,\dots,x_k\} \subset U_n(F)$. We take a continuous function g such that $g|(x_1,\dots,x_k)=0$ and $g|X-U_n(F)=1$. Obviously $g \in G \cap A_n$.

From $t(C_p(X))=\kappa$, we can find a subset $B_n \subset A_n$ of cardinality $\le \kappa$ such that $f \in \bigcap_n \overline{B}_n$. For each $g \in B_n$ we select $F(g) \in \mathcal{F}$ such that $g|X-U_n(F(g))=1$. We set $\mathcal{F}_n=\{F(g): g \in B_n\}$ for each $n \in \mathbf{N}$ and $\mathcal{G}=\bigcup_n \mathcal{F}_n$. The cardinality of $\mathcal{G}$ is less than or equal to κ. We claim $\mathcal{G}$ is a π-network for f. Let $G=[x_1,\dots,x_k;\ W,\dots,W]$ be a neighborhood of f, where $W=(-1/n,1/n)$. Then there is a $g \in B_n \cap G$. This means $\{x_1,\dots,x_k\} \subset U_n(F(g))$. Thus we have $F(g) \subset G$ and we have proved that $st(C_p(X)) \le \kappa$. The other inequality is trivial. The proof is complete.

References

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