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REDUCIBILITIES OF SETS BASED ON CONSTRUCTIVE FUNCTIONS OF A REAL VARIABLE

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Abstract: Reducibilities of sets based on mappings corresponding to constructive functions of a real variable are introduced and studied. Connections between them and tt- and T-reducibilities and their relativizations to $\emptyset'$ are discussed.

Key words: Recursion theory, tt-reducibility, T-reducibility, constructive functions of a real variable, arithmetical real numbers, sets of reals of B-measure zero.

Classification: 03D30, 03F65

Binary expansions of reals give us a many-to-many correspondence between reals and sets of natural numbers (NNs). For any set A of NNs we denote by r_A the sum of the series $\sum_{x\in A} 2^{-x-1}$ and, for any real X, we denote by Set(X) the infinite set B of NNs for which $X-r_B$ is equal to an integer. Using reals to study sets of NNs we can restrict ourselves to reals from the closed unit interval $[0,1]$. A real X is said to be A-recursive if $\mathrm{Set}(X)\leq_T A$ holds. In constructive mathematics in the sense of Markov we study, among other things, constructive reals (i.e. codes of $\emptyset$-recursive reals) and everywhere (i.e. for any constructive real) defined constructive functions of a real variable (briefly:constructive functions). Let us remember that any constructive function is an algorithm transforming equal constructive reals into equal ones and it is constructive continuous (i.e., as we shall see later, $\emptyset$-continuous) at any constructive real [2].

For any constructive function F we denote by R[F] a classical function of a real variable being a maximal (as to domain) continuous (with respect to its domain) extension of F. Thus, R[F] is defined, in particular, at any $\emptyset$-recursive real and transforms it into a real being $\emptyset$-recursive, too.

A set A of NNs is said to be <u>f-reducible</u> to a set B of NNs <u>by</u> F (notation: $A\leq_f B$ via F) if F is a constructive function for which $R[F](r_B)$ is defined and $R[F](r_B)=r_A$ holds. Studying f-reducibility we can limit ourselves to constructive functions constant on both $(-\infty, 0]$ and $[1, +\infty)$, which will be named by us <u>c-functions</u>. A set A of NNs is said to be <u>f-reducible</u>, <u>Ø'-ucf-reducible</u>, Ø-ucf-<u>reducible</u>, mf-<u>reducible</u>, respectively, to a set B (notation: $A \leq_f B$, $A \leq_{\emptyset'\text{-ucf}} B$, $A \leq_{\emptyset\text{-ucf}} B$, $A \leq_{mf} B$, respectively) if there is a c-function F, a classically uniformly continuous (i.e., as we shall see later, Ø'-uniformly continuous) c-function F, a constructively uniformly continuous (i.e. Ø-uniformly continuous) c-function F, a monotone c-function F, respectively, such that $A \leq_f B$ via F holds.

Basic properties of these reducibilities and their relation to tt- and T-reducibility and their Ø'-relativizations are studied in this paper.

We use the notation and terminology of [11]. $\rightleftharpoons$ stands for "denotes" and $\doteq$ for graphical equality. For any function (or algorithm) $\mathcal{G}$ and for any objects P and R (of corresponding types), $!\,\mathcal{G}(P) \rightleftharpoons (\mathcal{G}(P)$ is defined) and $\mathcal{G}(P) \simeq R \rightleftharpoons (!\,\mathcal{G}(P) \,\&\, (\mathcal{G}(P) \doteq R))$. Note that $\simeq$ means: both sides are defined and equal, or both are undefined.

Objects of constructive mathematical analysis used in this paper are either words in the alphabet Ξ (containing, among other things, the symbols 0, |, - , / , $\square$, $\triangle$, ∇) or objects finitely codable by words in Ξ (e.g. algorithms are codable by NNs). Thus, 0, 0|, 0||,... are (codes of) NNs. Also integers, rational numbers (RtNs) and strings are words in Ξ of appropriate types. As an abbreviatory notation for these constructive objects, we employ the standard notation (e.g., for NNs, 0,1,2,...). It is not necessary for us to make any explicit difference between intuitive objects of these types and their constructive counterparts (used by us). The set of all words in Ξ is denoted by Ξ^* , the set of all NNs (or RtNs) by N (or by Q) and the set of all strings (of 0's and 1's) by St. In the sequel, the symbols U, V are variables for words in Ξ , s, t, u, v, w, x, y and z are variables for NNs, i and j for integers, a, b, c and d for RtNs, ρ , σ and τ for strings (of 0's and 1's), A, B and C for sets of NNs and X and Y for reals. A RtN a is said to be <u>binary rational</u> if $\exists iy\,(a=i.2^{-y})$ holds. Let Q^b denote the set of all binary rational numbers. Thus, Q^b, Q, N and St are subsets of Ξ^*.

Notions and notations for strings, sets of strings, partial recursive functions (PRFs), recursively enumerable (r.e.) sets of NNs and their relativizations were introduced in [11]. In particular, σ_x denotes the string with the number x, $\langle A \rangle = \{\sigma_z : z \in A\}$ and A[0,x] is the string of length (x+1) extended by A for any NN x and any set A of NNs. For any set A and any NNs z and s, we denote $\{x:\ !\varphi_z^{A[0,s]}(x) \,\&\, x < s\}$ by $W_z^{A,s}$ (for the notation see [11]).

For any set $\mathcal{S}$ of strings let $\mathcal{S}^E \rightleftharpoons \{A: \exists \tau\,(\tau \in \mathcal{S} \,\&\, (A \text{ extends } \tau))\}$. We remind that $\mu(\mathcal{S})$ denotes the Lebesgue measure of $\mathcal{S}^E$ (in the following we shall often write $\mu(\mathcal{G}^E)$ instead of $\mu(\mathcal{G}))$ and that $\mathcal{G}$ is said to be a proper covering if $\mathcal{G}$ is r.e., $\mathcal{G}^E$ contains all recursive sets but there is a set of NNs not contained in $\mathcal{G}^E$.

There exist recursive functions nol (non-overlapping) and hp (the halting problem) such that, for any NNs t and y,

(i) $\emptyset'(t)=\lim_{s\to+\infty} hp(s,t)$ and $hp(y,t)\leq hp(y+1,t)$ hold;

(ii) $\langle W_{nol(t)}\rangle$ is a set of mutually incomparable (with respect to $\subseteq$) strings; moreover, if even $\langle W_t\rangle$ has this property then $W_{nol(t)}=W_t$ holds;

(iii) $\langle W_{nol(t)}\rangle^E=\langle W_t\rangle^E$ is valid.

Remark 1. 1) There are recursive functions dom of two variables and val of three variables such that, for any NNs z,x and y,

$W_{dom(z,x)}=\{t:\ !\varphi_z^{\sigma_t}(x)\ \&\ \neg(\exists w)_{w<t}(\sigma_w\subseteq\sigma_t\ \&\ !\varphi_z^{\sigma_w}(x))\}$,

$W_{eval(z,x,y)}=\{t: t\in W_{dom(z,x)}\ \&\ (\varphi_z^{\sigma_t}(x)=y)\}$

(for the notation see [11]). Consequently, $W_{nol(dom(z,x))}=W_{dom(z,x)}$ and, for any set A of NNs, $\varphi_z^A(x)\simeq y\iff A\in\langle W_{eval(z,x,y)}\rangle^E$.

On the other hand, we can construct on the basis of an index of a recursive function f of two variables a NN z fulfilling

$\langle W_{eval(z,x,y)}\rangle^E\subseteq\langle W_{f(x,y)}\rangle^E$

and

$\bigcup_{t=0}^{+\infty}\langle W_{eval(z,x,t)}\rangle^E=\bigcup_{t=0}^{+\infty}\langle W_{f(x,t)}\rangle^E$ for any NNs x and y.

2) Using $\emptyset'$-recursively enumerable sets instead of r.e. ones in the definition of partial A-recursive functions [1, p. 132], we obtain, among other things, a generalization of T-reducibility: the concept of $\emptyset'$-T-reducibility. According to 1) and its relativization to $\emptyset'$ we have $A\leq_{\emptyset'-T}B\iff A\leq_T(B\oplus\emptyset')$ (there are recursive functions transforming indices of $\emptyset'$-T-reducibility into corresponding indices of T-reducibility and vice versa). Let us remark that the ($\emptyset'$-T)-jump of any set A is T-equivalent to $(A\oplus\emptyset')'$.

It will be useful for us to introduce also an $\emptyset'$-variant of tt-reducibility. A set A of NNs is said to be $\emptyset'$-tt-reducible to a set B of NNs (notation: $A\leq_{\emptyset'-tt}B$) if there exists an $\emptyset'$-recursive function f such that $A\leq_{\emptyset'-tt}B$ via f holds, i.e., for all NNs x, $x\in A\iff$ (tt-condition f(x) is satisfied by B) is valid.

Of course, $A\equiv_{\emptyset'-tt}(A\oplus\emptyset')$ and $A\leq_{\emptyset'-tt}B\implies A\leq_T(B\oplus\emptyset')$ hold.

Results from [10, Theorem 6] are reformulated in parts 1) and 2) of the following statement. We use the function e from [11].

Theorem 2. There are three recursive functions f and $\bar{g}$ of one variable and g of two variables and four $\emptyset'$-recursive functions $\bar{h}$ and $\bar{k}$ of two variables, $\bar{p}$ of three variables and $\bar{q}$ of one variable with the following properties.

1) For any NNs x and t, $\sum_{s=0}^{+\infty} sg(|g(s,x)-g(s+1,x)|) \leq f(x)$, $\mu(\langle W_{\lim_{y\to+\infty} g(y,x)}\rangle^E) \leq 2^{-x}$, $\langle W^{\emptyset'}_{\bar{g}(t)}\rangle^E = \bigcup_{v=t}^{+\infty} \langle W_{\lim_{y\to+\infty} g(y,v)}\rangle^E$ and $\langle W_{e(t+1)}\rangle^E \subseteq \langle W^{\emptyset'}_{\bar{g}(t)}\rangle^E$ hold and consequently, $\mu(\langle W^{\emptyset'}_{\bar{g}(t)}\rangle^E) \leq 2^{-t+1}$.

2) For any NNs z and t and any set A of NNs fulfilling $A \notin \langle W^{\emptyset'}_{\bar{g}(t)}\rangle^E$, we have

(i) $A \in \langle W_z\rangle^E \Longleftrightarrow A \in \langle W_z^{\bar{h}(t,z)}\rangle^E$;

(ii) if $A \notin \langle W_z\rangle^E$ (i.e. if $A \notin \langle W_z^{\bar{h}(t,z)}\rangle^E$) holds then

$\forall wy(\bar{k}(z,w+t) \leq y \Longrightarrow \mu(\langle W_z\rangle^E \cap \{A[0,y]\}^E) \leq 2^{-w}. \mu(\{A[0,y]\}^E))$
is valid; consequently,

(iii) either A is in $\langle W_z^{\bar{h}(t,z)}\rangle^E$ or it is a point of dispersion of $\langle W_z\rangle^E$ (with an $\emptyset'$-recursive function as a modulus).

3) For any NNs t, z and x and any set A of NNs,

a) $(A \notin \langle W^{\emptyset'}_{\bar{g}(t)}\rangle^E) \& !\varphi_z^A(x) \Longrightarrow \varphi_z^A(x) \leq \bar{p}(t,z,x)$ holds;

b) if

$$A \notin \bigcap_{v=0}^{+\infty} \langle W^{\emptyset'}_{\bar{g}(v)}\rangle^E \tag{1}$$

is fulfilled, then A is a NAP-set (thus, a non-recursive non-semigeneric set [11]) and

(i) $!\varphi_z^A(y) \Longrightarrow \varphi_z^A(y) \leq \bar{q}(y)$ is valid for any sufficiently large NN y;

(ii) $A' \equiv_{\emptyset'\text{-}tt} A$ and, thus, $A' \equiv_T (A \oplus \emptyset')$ hold;

(iii) for any sets B and C of NNs, we have $C \leq_T B \leq_T A \Longrightarrow (C \leq_{\emptyset'\text{-}tt} B) \&$ $\&$(B is not $\emptyset'$-hyperimmune).

Let us remark that, as we have seen, for any set A fulfilling (1), any partial A-recursive function is almost everywhere majorized by the $\emptyset'$-recursive function $\bar{q}$ and that (1) is valid for almost any set A, where "almost any" is effective with respect to $\emptyset'$ (cf. [9], the corresponding notion will be also introduced below).

Proof of Theorem 2. It is sufficient to show that part 3) is a consequence of the preceding two parts of the statement. We use Remark 1 and construct $\emptyset'$-recursive functions $\bar{p}$ and $\bar{q}$ such that, for any NNs t,z and x, $\bar{p}(t,z,x)$ is the maximal element of the set

$$\{0\} \cup \{\varphi_z^{\sigma_y}(x) : y \in W_{dom(z,x)}^{\bar{h}(t,dom(z,x))}\} \text{ and } \bar{q}(x) = \max_{v,w \leq x} \bar{p}(v,w,x).$$

To finish the proof we notice that if $A \notin \langle W_{\bar{g}(t)}^{\emptyset'} \rangle^{E}$ then

$$x \in A' \Longleftrightarrow \exists\, y(y \in W_{dom(x,x)}^{\bar{h}(t,dom(x,x))} \,\&\, \sigma_y \subseteq A[0, lh(\sigma_y)]).$$

By a slightly modified $\emptyset'$-relativization of the proof of [6, Theorem 3.12] we can prove the following statement.

Theorem 3. For any set A of NNs the following are equivalent.

(a) Any A-recursive function is majorized by some $\emptyset'$-recursive function.

(b) $deg_T(A)$ (the T-degree of A) is $\emptyset'$-hyperimmune-free.

(c) $deg_T(A)$ has a non-empty intersection with only one $\emptyset'$-tt-degree.

(d) $deg_T(A)$ has non-empty intersections with only finitely many $\emptyset'$-tt-degrees.

We introduce some constructive concepts now (cf. [7]).

Let us fix two Markov algorithms: wd establishing a one-to-one correspondence between N and Ξ^*, and en being the algorithm inverse to wd. For any set B of NNs and any NN x, the correspondence $\lambda U(\underline{wd}(\varphi_x^B(\underline{en}(U))))$ will be denoted by $[\![\varphi_x^B]\!]$ and called the B-algorithm with B-index x.

Let M_0 and M_1 be subsets of Ξ^*. A B-algorithm F is said to be a) of the type $(M_0 \to M_1)$ if $\forall U(U \in M_0 \Rightarrow\, !\, F(U) \,\&\, F(U) \in M_1)$;

b) a B-sequence of elements of M_1 if it is of the type $(N \to M_1)$;

c) a B-sequence of C-algorithms if it is a B-sequence of C-indices of corresponding algorithms.

In the following we shall present B-sequences of words (or algorithms) by their "members" using notation $\{\ldots\}_x^B$.

Let F be a sequence of RtNs. A sequence of NNs (or a function) G is called a modulus of fundamentality of F if $\forall xy(|F(G(x))-F(G(x)+y)| \leq 2^{-x})$ holds. F is called B-fundamental if there is a B-sequence of NNs (or, equivalently, a B-recursive function) being a modulus of fundamentality of F. If the function $\lambda x(x)$ is a modulus of fundamentality of F then F is said to be canonically fundamental. Let us notice that any B-sequence of RtNs is fundamental (in the classical sense) if and only if it is B'-fundamental and that a real X is B-recursive if and only if there exists a canonically fundamental (or a B-fundamental) B-sequence of RtNs converging to X.

We meet a situation similar to the one just presented, among other cases, in the cases of convergence, continuity, uniform continuity and measurability. Also we can introduce here (in parallel with classical concepts) B-concepts, where the availability of a B-recursive function being a corresponding modulus, is required. Speaking about constructive (or, equivalently, effective) fundamentality, constructive continuity etc. we always mean $\emptyset$-fundamentality, $\emptyset$-continuity etc.

Let x be a NN. $\emptyset^{(x)}$-constructive real numbers ($\emptyset^{(x)}$-CRNs) are words in Ξ being either rational numbers or codes of $\emptyset^{(x)}$-recursive reals such that any $\emptyset^{(x)}$-CRN is an $\emptyset^{(x+1)}$-CRN. $D^{[x]}$ denotes the set of all $\emptyset^{(x)}$-CRNs ([x] stands for $\emptyset^{(x)}$). Elements of $\mathcal{A}$, where $\mathcal{A} \rightleftharpoons \bigcup_{y=0}^{+\infty} D^{[y]}$, are called <u>arithmetical real numbers</u> (ARNs). On $\mathcal{A}$, the relations of equality and of order are defined in an obvious way, basic algebraic operations can be realized, thanks to the s-m-n theorem, by $\emptyset$-algorithms, i.e. effectively [7]. Let S and T be variables for $\emptyset$-CRNs.

A rational segment (or interval) is a word of the form $a \triangle b$ (or $a \triangledown b$), where a and be are RtNs and $a < b$ holds. For any words of the form $U \curlyvee V$, where $\curlyvee$ is either $\triangle$ or $\triangledown$ and U and V are ARNs, and any real X, $X \in U \triangle V \rightleftharpoons$ $\rightleftharpoons$ ($U \leq X \leq V$ holds), $X \in U \triangledown V \rightleftharpoons$ ($U < X < V$ holds), $(U \triangle V)^{0} \rightleftharpoons U \triangledown V$, $|U \curlyvee V| \rightleftharpoons$ $\rightleftharpoons (V-U)$, $E_l(U \curlyvee V) \rightleftharpoons U$, $E_r(U \curlyvee V) \rightleftharpoons V$. Let us suppose to have a fixed enumeration of rational segments given by an $\emptyset$-algorithm $\mathcal{L}$ and two $\emptyset$-algorithms $\mathcal{L}^{0}$ and Seg such that, for any NN x and any string τ , $\mathcal{L}^{0}(x) \rightleftharpoons (\mathcal{L}(x))^{0}$ and Seg(τ) is the rational segment $c \triangle (c+2^{-lh(\tau)})$, where

$$c \rightleftharpoons \sum_{x < lh(\tau)} \tau(x).2^{-x-1}.$$

For any set A of NNs, we define $[A]_c \rightleftharpoons \{X: \exists x(x \in A \,\&\, X \in \mathcal{L}(x))\}$ and $[A] \rightleftharpoons \{X: \exists x(x \in A \,\&\, X \in \mathcal{L}^{0}(x))\}$. Sometimes, we use the word $U \curlyvee V$, where U and V are ARNs and $\curlyvee$ is either $\triangle$ or $\triangledown$, instead of $\{X: X \in U \curlyvee V\}$. Let $\overline{\text{In}}$ be a recursive function such that, for any NN z, $W_{\overline{\text{In}}(z)}$ is the set of all NNs y for which the segment $\mathcal{L}(y)$ is a union of a finite subset of the set $\{\text{Seg}(\sigma_x): x \in W_z\}$ of rational segments.

A <u>class</u> $\mathfrak{C}$ of sets of NNs is said to be <u>of</u> B-<u>measure zero</u> if there exist two B-recursive functions f of one variable and g of two variables such that, for any NN x, $\forall yzv(g(x,y) \leq z \Rightarrow |\mu(\langle W_{f(x)}^{B,z+v} \rangle^{E}) - \mu(\langle W_{f(x)}^{B,z} \rangle^{E})| \leq 2^{-y})$, $\mu(\langle W_{f(x)}^{B} \rangle^{E}) \leq 2^{-x}$ and $\mathfrak{C} \subseteq \langle W_{f(x)}^{B} \rangle^{E}$ hold.

Let us remark that the class of all weakly 1-generic sets [5] is of $\emptyset$-measure zero, the class of all sets being either semigeneric or recursive [11] is of $\emptyset'$-measure zero and, for any non-recursive set A, the class $\{B: A \leq_T B\}$ is of $(A \oplus \emptyset')$-measure zero.

Theorem 4. For any class $\mathcal{E}$ of B-measure zero and for any string τ and any set C of NNs, there is a set A of NNs such that $A \in (\{\tau\}^E \setminus \mathcal{E})$, $A \leq_T (B \oplus C)$ and $C \leq_T (B \oplus A)$ hold (in fact, we have $C \equiv_{B-tt} A$).

Proof. Using standard methods of measure theory, we construct a B-algorithm T of the type (St $\rightarrow$ St) (determining a B-recursive tree) such that (i) T of the empty string is τ, (ii) for any string σ, the segments $Seg(T(\sigma * 0))$ and $Seg(T(\sigma * 1))$ are disjoint, contained in $(Seg(T(\sigma)))^0$, they have the same length, the first of them is situated to the left of the other and none of them contains any of RtNs of the type $i.2^{-lh(\sigma)-1}$ and (iii) no set of NNs corresponding to an infinite path of the tree T is in the class $\mathcal{E}$. Then, for a given set C of NNs, we put $A = \{x : T(C[0,x])(x) = 1\}$.

Corollary 5. For any string τ and any set B of NNs, $\emptyset' \leq_T B$, there is a set A such that (1) is valid, $\tau \subseteq A$, $A \equiv_{\emptyset'-tt} B$ and $A' \equiv_T B$ hold. Thus, A is a NAP-set, $\neg(\emptyset' \leq_T A)$ and $(\emptyset' <_T B \Rightarrow \neg(A \leq_T \emptyset'))$.

Proof. It is easy to show that the class $\bigcap_{v=0}^{+\infty} \langle W^{\emptyset'}_{\bar{g}(v)} \rangle^E$ of sets is of $\emptyset'$-measure zero. Then we use Theorems 4 and 2.

We return to constructive functions now. Concepts introduced by us make it possible to give definitions. An $\emptyset$-algorithm F is said to be a <u>constructive function</u> if it is of the type $(D^{[0]} \rightarrow D^{[0]})$ and $\forall ST(S=T \Rightarrow F(S)=F(T))$ holds. A constructive function is called a <u>c-function</u> if $\forall S((S \leq 0 \Rightarrow F(S) = F(0)) \& (1 \leq S \Rightarrow F(S)=F(1)))$ is valid.

Remark 6. 1) By Cejtin [2], for any c-function F, there is a recursive function $cont_F$ such that, for any NN v, the set $\langle W_{cont_F(v)} \rangle$ of strings is a proper covering and $\mu(\langle W_{cont_F(y)} \rangle^E) \leq 2^{-v}$ and $\forall yab(y \in W_{cont_F(v)} \& (a,b \in Seg(\sigma_y)) \Rightarrow |F(b)-F(a)| \leq 2^{-v})$ hold. Consequently, any c-function F is $\emptyset$-continuous at any $\emptyset$-CRN and R[F] is defined at least on $\bigcap_{v=0}^{+\infty} \langle W_{cont_F(v)} \rangle^E$, in particular, it has a B-limit at r_B for any semigeneric set B (cf. [11]). However, a c-function can be unbounded on $0 \Delta 1$.

2) For any c-function F, (F is (classically) uniformly continuous) $\Leftrightarrow$ (F is $\emptyset'$-uniformly continuous) $\Leftrightarrow$ (R[F] is defined at any $\emptyset''$-recursive real) $\Leftrightarrow$ (R[F] is everywhere defined) [7]. There are uniformly continuous c-functions being not $\emptyset$-uniformly continuous [4]. On the other hand, any monotone c-function is $\emptyset$-uniformly continuous [3].

3) The predicates $S < T$ of variables S and T, $S \in \mathcal{L}^0(x)$ and $S \notin \mathcal{L}(x)$ of variables S and x are recursively enumerable. For any $\emptyset$-uniformly continuous (resp. $\emptyset'$-uniformly continuous) c-function F the predicates $R[F](\mathcal{L}(x)) \subseteq \mathcal{L}^0(y)$ and $R[F](\mathcal{L}(x)) \cap \mathcal{L}(y) = \emptyset$ of variables x and y are $\emptyset$-recursively (resp. $\emptyset'$-recursively enumerable.

Remark 7. On the basis of Remark 6 and [11, Lemma 10] it is easy to prove the following.

1) Let B be f-reducible to A. Then we have

(i) if B is a non-recursive and non-semigeneric set, then A also has these properties;

(ii) if A is 1-generic [12] then B is either recursive or 1-generic;

(iii) if A is hyperimmune then B is contained in a class of sets of $\emptyset$-measure zero.

2) Weakly 1-generic sets [5] can be mf-reducible to semigeneric sets only.

We shall have some trouble with binary rational reals in our considerations. Let us notice that the real r_A is not equal to a binary rational number if and only if A is a bi-infinite set. In the sequel, for any NN t, $\mathfrak{B}(t)$ (or, as the case may be, $\mathfrak{B}^{\emptyset'}(t)$) denotes: $\langle W_t \rangle$ (or $\langle W^{\emptyset'} \rangle$) is a set of mutually incomparable strings which, for any non-empty finite set A, covers either A or $Set(r_A)$.

A set A of NNs is said to be

a) a <u>strongly bi-infinite set</u>, an SBI<u>-set</u>, (resp. and $\emptyset'$-strongly bi-infinite set, an $\emptyset'$-SBI-set) if it is a bi-infinite set and there is a NN t fulfilling $\mathfrak{B}(t)$ (resp. $\mathfrak{B}^{\emptyset'}(t)$) and $A \notin \langle W_t \rangle^E$ (resp. $A \notin \langle W_t^{\emptyset'} \rangle^E$);

b) a <u>weakly bi-infinite set</u>, a WBI<u>-set</u>, if it is a bi-infinite set, but it is not an SBI-set.

Remark 8. 1) Any bi-infinite recursive set is an SBI-set. Any SBI-set is an $\emptyset'$-SBI-set. Any weakly 1-generic set is, naturally, a WBI-set. Any WBI-set is by [11, Remark 15 and Corollary 12] a bi-hyperimmune and, consequently, semigeneric set. So, according to [5], WBI T-degrees are just the hyperimmune ones. The class of all WBI-sets is both a $\Pi_2^{0,\emptyset'}$ class and a Π_3^0 class of $\emptyset$-measure zero which is not a Σ_3^0 class.

2) Let t_0 be a NN such that $W_{t_0} = \{x : \exists y(lh(\sigma_x) = 2y+2 \,\&\, (\forall z)_{z<y}(\sigma_x(2z) =$

$= \sigma_x(2z+1)) \& \sigma_x(2y) \neq \sigma_x(2y+1))\}$. Then $\mathfrak{B}(t_0)$ holds and $A \oplus A \notin \langle W_{t_0} \rangle^E$ is valid for any set A of NNs. Thus, for any bi-infinite set B (in particular, for any WBI-set), $B \oplus B$ is an SBI-set. Consequently, there are bi-hyperimmune SBI-sets.

3) We can construct bi-infinite $\emptyset'$-recursive sets A_0 and B_0 such that $A_0 \equiv_{tt} B_0$ holds and $A_0 \oplus (B_0 \oplus B_0)$ is a WBI-set which according to [11, Theorems 4 and 5] cannot be weakly 1-generic. By [11, Theorem 9], 1) and 2) $(B_0 \oplus B_0)$ is a semigeneric set not being a WBI-set.

We begin with $\emptyset$-ucf-reducibility. Let us remark that according to the part 2) of Remark 6 $A \leq_{mf} B \Rightarrow A \leq_{\emptyset\text{-ucf}} B$ holds for any sets A and B of NNs.

Theorem 9. 1) For any $\emptyset$-uniformly continuous c-function F and any NN t fulfilling $0 \leq F \leq 1 \& \mathfrak{B}(t)$, we can construct a recursive function f such that, for any sets A and B of NNs, where $B \notin \langle W_t \rangle^E$ and B is bi-infinite, we have

(2) $(B \leq_{\emptyset\text{-ucf}} A$ via $F) \Leftrightarrow (B \leq_{tt} A$ via $f)$.

2) For any recursive function f and any NN t fulfilling $\mathfrak{B}(t)$, we can construct an $\emptyset$-uniformly continuous c-function F such that $0 \leq F \leq 1$ and (2) hold for any sets A and B of NNs, where $A \notin \langle W_t \rangle^E$ and B is bi-infinite.

Proof. Let t be a NN, $\mathfrak{B}(t)$. We construct an increasing recursive function g such that $\forall x i (0 < i < 2^{x+1} \Rightarrow \exists y (y \in W_t^{g(x)} \& i.2^{-x-1} \in \mathrm{Seg}(\sigma_y)))$.

1) Let F be an $\emptyset$-uniformly continuous c-function, $0 \leq F \leq 1$. There are an increasing recursive function h and an $\emptyset$-algorithm $\underline{H}$ of the type $(St \rightarrow St)$ such that $0 < h(0)$ and, for any NN x and any strings σ and τ fulfilling $lh(\tau) = h(x)$, we have $(\sigma \in \langle W_t^{g(x)} \rangle \Rightarrow \forall ab(a,b \in \mathrm{Seg}(\tau) \Rightarrow |F(a)-F(b)| < 2^{-lh(\sigma)-x-2}))$, $lh(\underline{H}(\tau)) = x+1$, $R[F](\mathrm{Seg}(\tau)) \subseteq (\mathrm{Seg}(\underline{H}(\tau)) \cup \bigcup_{y \in W_t^{g(x)}} \mathrm{Seg}(\sigma_y))$, $((\tau$ extends $\sigma) \Rightarrow (\underline{H}(\tau)$ extends $\underline{H}(\sigma))$ and $(|F(E_1(\mathrm{Seg}(\tau)))- -E_1(\mathrm{Seg}(\underline{H}(\tau)))| < 2^{-x} \vee \underline{H}(\tau)(x)=0)$ (see Remark 6). We construct a recursive function f such that, for any NN x, f(x) is the (code number of) tt-condition of norm h(x) with an associated set $\{0,1,\ldots,h(x)-1\}$ and h(x)-ary Boolean function transforming any string τ of length h(x) into $\underline{H}(\tau)(x)$. Obviously, f has all the required properties.

2) Let f be a recursive function. We shall construct an $\emptyset$-sequence $\{F_x\}_x^{\emptyset}$ of polygonal c-functions and a recursive function h such that $0 \leq F_x \leq 2^{-x-1}$ and F_x fulfils the Lipschitz condition with h(x) for any NN x and a c-function F to which the canonically uniformly fundamental $\emptyset$-sequence

$\{\sum_{v=0}^{r} F_v\}_y^{\emptyset}$ of c-function converges, has all the required properties.

Let x be a NN and p_x the greatest of the elements of associated sets of tt-conditions f(0),f(1),...,f(x). We denote by $\{c_{x,i}\}_{i=0}^{q_x}$ the increasing finite sequence formed by RtNs: 0, 1 and all end points of $Seg(\sigma_z)$ for $z \in W_t^{g(p_x)}$. The c-function F_x will be linear on the rational segment $c_{x,j} \Delta c_{x,j+1}$ for any NN $j < q_x$. Let i be a NN, $0 \leq i \leq q_x$, and let $v_{x,i} \rightleftharpoons \mu y(lh(\sigma_y)=p_x+1 \& c_{x,i} \in Seg(\sigma_y) \& (c_{x,i} < E_r(Seg(\sigma_y)) \vee i=q_x \vee \exists z(z \in W_t^{g(p_x)} \& E_l(Seg(\sigma_z))=c_{x,i})))$. We find a string σ fulfilling $lh(\sigma)=x+1$ and $\sigma \leq_{tt} \sigma_{v_{x,i}}$ via f (see [11, p. 73]) and put $F_x(c_{x,i})= \sigma(x).2^{-x-1}$. It is easy to find a NN being the Lipschitz constant for F_x.

These results cannot be improved in a general case.

Example 10. Let $g \rightleftharpoons \lambda x(\mu y(x<2y+2))$. Then, for any set A,

(a) $A \oplus A \leq_m A$ via g holds;

(b) if A is bi-infinite and $A \oplus A \leq_f A$ is valid then A is necessarily an SBI-set.

Remark 11. Let F be a non-decreasing c-function fulfilling F(0)=0, F(1)= =1 and $\forall a(a \in Q^b \& 0<a<1 \Longrightarrow$ (Set(F(a)) is bi-infinite)). Then, for any set A and any bi-infinite set B of NNs such that $B \leq_f A$ via F holds and $B \leq_{tt} A$ (or $B \leq_{\emptyset'-tt} A$) is valid, the set B must be an SBI-set (or, an $\emptyset'$-SBI-set, respectively).

By a relativization of the proof of the part 1 of the preceding theorem we get the following statement:

Theorem 12. For any $\emptyset'$-uniformly continuous c-function F and any NN t fulfilling $0 \leq F \leq 1 \& \mathcal{B}^{\emptyset'}(t)$, we can construct an $\emptyset'$-recursive function f such that, for any sets A and B of NNs, where $B \notin \langle W_t^{\emptyset'} \rangle^E$ and B is bi-infinite, $(B \leq_{\emptyset'-ucf} A$ via F$) \Longleftrightarrow (B \leq_{\emptyset'-tt} A$ via f) holds.

In the case of monotone c-functions we can get more.

Theorem 13. 1) Let F be a c-function and f a recursive function such that $(B \leq_{tt} A$ via f$) \Longrightarrow (B \leq_f A$ via F) holds for any sets A and B of NNs. Then $(B \leq_f A$ via F$) \Longrightarrow (B \leq_{tt} A$ via f) is valid for any set A and any bi-infinite set B of NNs.

2) Let F be a non-decreasing c-function, F(0)=0&F(1)=1. Then the following two conditions are equivalent:

(a) There is a recursive function f such that, for any sets A and B of NNs, $(B \leq_{tt} A$ via f$) \Longrightarrow (B \leq_{mf} A$ via F) holds.

(b) There is an $\emptyset$-algorithm $\underline{E}$ of the type $(Q^b \longrightarrow Q^b)$ such that $\forall a(a \in Q^b \,\&\, 0 < a < 1 \Longrightarrow F(\underline{E}(a))=a)$ is valid.

Theorem 14. Let F be a non-decreasing c-function and $\underline{E}$ an $\emptyset$-algorithm of the type $(Q^b \longrightarrow Q^b)$ for which $F(0)=0 \,\&\, F(1)=1 \,\&\, \forall a(a \in Q^b \,\&\, 0 < a < 1 \Longrightarrow$ $\Longrightarrow F(a)=\underline{E}(a))$ holds. Then there is a recursive function f such that $(B \leq_{mf} A$ via $F) \Longleftrightarrow (A \leq_{tt} B$ via $f)$ holds for any set A and any bi-infinite set B of NNs.

Theorem 15. For any $\emptyset$-uniformly continuous c-function F we can construct NNs z_0 and z_1 such that, for any sets A and B of NNs, we have

(a) (B bi-infinite) $\Longrightarrow ((B \leq_T A$ via $z_0) \Longleftrightarrow (B \leq_{\emptyset\text{-}ucf} A$ via F));

(b) if F is non-decreasing and $0 \leq F \leq 1$ holds then $\emptyset <_T B \Longrightarrow$ $\Longrightarrow ((B \leq_{mf} A$ via $F) \Longleftrightarrow (A \leq_T B$ via $z_1))$.

Theorem 16. For any c-function $\overline{F}$ we can construct NNs v_0 and v_1 such that $(B \leq_f A$ via $\overline{F}) \Longleftrightarrow (B \leq_T (A \oplus \emptyset')$via $v_0)$ and (A is either recursive or semigeneric) $\Longrightarrow ((B \leq_f A$ via $\overline{F}) \Longleftrightarrow (B \leq_T A$ via $v_1))$ hold for any set A and any bi-infinite set B of NNs.

Proofs of Theorems 15 and 16. We use Remark 6, the s-m-n theorem and its relativization and construct everywhere defined $\emptyset$-algorithms $\underline{L}$ and $\underline{R}$ and recursive functions g_0, g_1, h_0 and h_1 of two variables such that $\underline{L}(xyi) \doteq$ $\doteq (2i+y).2^{-x-1}$, $\underline{R}(xyi) \doteq (2i+y+1).2^{-x-1}$,

$$W_{g_0(x,y)} = \{t : y \leq 1 \,\&\, \exists j(R[F](Seg(\sigma_t)) \subseteq \underline{L}(xyj) \triangledown \underline{R}(xyj))\},$$

$$W_{g_1(x,y)} = \{t : y \leq 1 \,\&\, \exists j(Seg(\sigma_t) \subseteq F(\underline{L}(xyj)) \triangledown F(\underline{R}(xyj)))\},$$

$$W^{\emptyset'}_{h_0(x,y)} = \{t : y \leq 1 \,\&\, \exists j \neg \exists a(a \in Seg(\sigma_t) \,\&\, (\overline{F}(a) < \underline{L}(xyj) \vee \underline{R}(xyj) < \overline{F}(a)))\},$$

$$W_{h_1(x,y)} = \{t : y \leq 1 \,\&\, \exists jva(t \in W_{cont_F(v)} \,\&\, a \in Seg(\sigma_t) \,\&\, \underline{L}(xyj)+2^{-v+1} < \overline{F}(a) < \underline{R}(xyj)-2^{-v+1})\}$$

for any NNs x and y and any integer i. For the completion of the proof it is sufficient to use Remark 1.

Remark 17. 1) Using Theorems 13 and 14, we can prove by a construction of corresponding non-decreasing c-functions the following:

(a) Any $\emptyset'$-recursive 1-generic set is tt-equivalent to a weakly 1-generic set being not 1-generic.

(b) Any weakly 1-generic set is tt-equivalent to a WBI-set being not weakly 1-generic.

As we already know, for any WBI-set A, the set $A \oplus A$ being tt-equivalent to A is a bi-hyperimmune SBI-set and, thus, by [11, Corollary 12], a semigeneric set.

2) By Remark 11 there exists an increasing c-function F such that $F(0)=0$, $F(1)=1$ and, for any sets A and B fulfilling $B \leq_{mf} A$ via F we have: $B \equiv_{mf} A$, $B \equiv_{T} A$ (Theorem 15), if one of the sets A and B is 1-generic (resp. weakly 1-generic) then the other one is also 1-generic (resp. weakly 1-generic) and, in addition, B and A are tt-incomparable WBI-sets (Remark 8).

Example 18. There are two $\emptyset'$-recursive sets being recursively isomorphic (i.e. 1-equivalent) but f-incomparable (according to Theorem 9 these sets must be WBI-sets).

Theorem 19. For any proper covering $\langle W_t \rangle$ and any NN z we can construct a c-function F such that $0 \leq F \leq 1$ and, for any sets A and B of NNs, where $A \notin \langle W_t \rangle^E$ and B is bi-infinite, we have

$(B \leq_T (A \oplus \emptyset')$ via z$) \Longleftrightarrow (B \leq_f A$ via F$)$.

Theorem 20. For any proper covering $\langle W_t \rangle$ and any $\emptyset'$- recursive function f, we can construct an $\emptyset'$-uniformly continuous c-function F such that $0 \leq F \leq 1$ and, for any sets A and B of NNs, where $A \notin \langle W_t \rangle^E$ and B is bi-infinite, we have $(B \leq_{\emptyset'\text{-}tt} A$ via f$) \Longleftrightarrow (B \leq_{\emptyset'\text{-}ucf} A$ via F$)$.

Proof of Theorem 20. Let $\bar{k}$ be a recursive function whose range is the set $W_{nol(t)}$ and let $a_{y,0} \rightleftharpoons E_l(Seg(\delta_{\bar{k}(y)}))$ and $a_{y,1} \rightleftharpoons E_r(Seg(\delta_{\bar{k}(y)}))$ for any NN y. The RtN $a_{y,i}$, where $0 \leq i \leq 1$, is said to be vacant if $0 < a_{y,i} < 1 \,\&\, \neg(\exists v)_{v<y}(a_{y,i} = a_{v,1-i})$ holds.

We shall construct an $\emptyset$-sequence of ($\emptyset$-indices of) c-functions $\{F_x\}_x^{\emptyset}$ such that, for any NNs x,y and i, $0 \leq i \leq 1$, F_x is linear on the segment $a_{y,0} \Delta a_{y,1}$ and $F_x(a_{y,i})$ is either 0 or 2^{-x-1} and, thus, $0 \leq F_x \leq 2^{-x-1}$ holds.

Let $\bar{p}$ be a recursive function of two variables such that $f(x) = \lim_{y \to +\infty} \bar{p}(y,x)$ for any NN x.

We put $F_x(0)=F_x(1)=0$ for any NN x. The construction proceeds in stages.

Stage s. If neither $a_{s,0}$ nor $a_{s,1}$ is vacant, we go to the stage s+1. In the other case we denote by τ_s the longest string from the set $\{\tau : lh(\tau) \leq s \,\&\, (\tau \leq_{tt} \delta_{\bar{k}(s)}$ via $\lambda x\, \bar{p}(s,x))\}$ of strings (containing at least the empty string, see [11, p. 73]). For any NNs x and j, $0 \leq j \leq 1 \,\&\, (a_{s,j}$ is vacant), we put $F_x(a_{s,j}) = \tau_s(x).2^{-x-1}$, if $x < lh(\tau_s)$, and $F_x(a_{s,j})=0$, if

$lh(\tau_s) \leq x$. We go on to the stage s+1.

Let us remark that, for any NN x and sets A and B fulfilling $A \notin \langle W_t \rangle^E$ and $B \leq_{\emptyset'\text{-}tt} A$ via f, the function $R[F_x]$ is not only everywhere defined and, thus, by Remark 6, $\emptyset'$-uniformly continuous but even polygonal and $R[F_x](r_A) = B(x).2^{-x-1}$ is valid.

Let us denote by F a c-function being a limit of canonically uniformly fundamental $\emptyset$-sequence $\{\sum_{y=0}^{x} F_y\}_x^{\emptyset}$ of $\emptyset'$-uniformly continuous c-functions. Obviously, F has all the properties described in the theorem.

To prove Theorem 19 it is sufficient to modify the previous proof using the function hp.

Corollary 21. For any non-recursive non-semigeneric set A and any $\emptyset'$-recursive set B we have $B \leq_{\emptyset'\text{-}ucf} A$. Consequently, there are $\emptyset'$-recursive (and thus, $\emptyset'$-tt-equivalent) non-semigeneric $\emptyset'$-ucf-equivalent sets being T-incomparable and, thus, $\emptyset$-ucf-incomparable.

Proof. It is sufficient to use Theorems 15 and 20, the Friedberg-Muchnik theorem [1, § 10.2] and [11, Theorem 13].

We shall restate Theorem 33 from [11] here.

Lemma 22. For any NN t such that $\mu(\langle W_t \rangle) < 1$ and for any set A of NNs we can construct a set B of NNs such that $A \leq_T B$, $B \notin \langle W_t \rangle^E$ and $B \equiv_{\emptyset'\text{-}tt} A$ hold and, thus, $B \leq_T (A \oplus \emptyset')$ is valid.

Using this lemma we can get the following example.

Example 23. For any covering $\langle W_t \rangle$ such that $\mu(\langle W_t \rangle) < 1$ we can construct sets A and B of NNs not contained in $\langle W_t \rangle^E$ which are T-equivalent to $\emptyset''$ and $\emptyset'$-tt-incomparable. Consequently, according to Theorems 12 and 19 and Remark 8, A and B are f-equivalent and $\emptyset'$-ucf-incomparable.

Example 24. For any NAP-set A there is an NAP-set B such that A and B are tt-equivalent and mf-incomparable. Thus, A and B are SBI-sets [11, p. 74] which are by Theorem 9 $\emptyset$-ucf-equivalent.

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