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GENERALIZATION OF THE GIRSANOV THEOREM

Václav SKOKAN

Abstract. The purpose of this paper is to generalize the Girsanov theorem [1]. In particular, on a probability space $[\Omega,\mathfrak{A},P_0]$ a process $\xi(t)=$ $=w(t)+\int_{E_n} x\,\bar{p}(t,dx)-\int_0^t A(s)ds$ is considered, where $w(t)$, $(w(0)=0)$, is a Wiener process and $p(t,A)$ is an integer-valued random measure with a characteristic $\Pi(t,A)$, $\bar{p}(t,A)=p(t,A)-\Pi(t,A)$. It is proved that there exists a probability measure P_1 such that $P_1 \ll P_0$ and on the probability space $[\Omega,\mathfrak{A},P_1]$ it holds $\xi(t)=\tilde{w}(t)+\int_{E_n} x\,\bar{p}(t,dx)$, where $\tilde{w}(t)$ is a Wiener process on $[\Omega,\mathfrak{A},P_1]$ and $\int_{E_n} x\,\bar{p}(t,dx)$ is a locally square integrable martingale on $[\Omega,\mathfrak{A},P_1]$ with the same characteristic with respect to P_1 as well as to P_0. This result is then further generalized.

Key words: Integer-valued random measure, local martingale, locally square integrable martingale, locally square integrable martingale measure, natural process, characteristic of a local martingale (of an integer valued random measure).

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1. Introduction. The object of this paper is to generalize the Girsanov theorem [1]. This theorem was derived in connection with an absolutely continuous transformation of measures associated with Ito-processes. In particular, if the class of the Ito-processes is investigated on $[\Omega,\mathfrak{A},P_0]$ then from this theorem it follows that there exists a probability measure P_1 such that $P_1 \ll P_0$ and the class of the Ito-processes is the same on $[\Omega,\mathfrak{A},P_1]$ as on $[\Omega,\mathfrak{A},P_0]$.

Generalizations of this theorem were done in the direction to diffusion processes and to processes of diffusion type [2, 3, 4, 5].

This paper deals with the generalization in another direction, which we have not met in the accessible literature. The main results are contained in Theorem 1 and Theorem 2 of the paragraph 4. Theorem 1 is an analogue of the Girsanov theorem. Theorem 2 is a generalization of Theorem 1.

2. Notation and definitions. It is supposed that a probability space $[\Omega, \mathfrak{A}, P_0]$ with a nondecreasing system of σ-algebras $\mathcal{F} = [\mathcal{F}_t, t \in \langle 0;T \rangle]$, $(T < +\infty, \mathcal{F}_t \subset \mathfrak{A})$, is given, where the σ-algebras $\mathcal{F}_t$ include P_0-zero sets and the system $\mathcal{F}$ is continuous from the right. By the symbol E_r r-dimensional Euclidean space is denoted. $\mathcal{B}_r$ denotes the system of all Borel sets from E_r and $\mathcal{B}_r^0$ the system of all Borel sets from E_r, the closure of which does not include the origin.

Definition 1. The function $q(t,A)$ defined on $\langle 0;T \rangle \times \mathcal{B}_r^0 \times \Omega$ which assumes non-negative integer values is called an integer-valued random measure, if the following assumptions are fulfilled: 1) $q(0,A)=0$, 2) for fixed $t \in \langle 0;T \rangle$ and fixed $A \in \mathcal{B}_r^0$ $q(t,A)$ is a random variable, measurable with respect to $\mathcal{F}_t$, 3) for fixed A as a function of t, $q(t,A)$ is nondecreasing, 4) if $A_i \in \mathcal{B}_r^0$ $(i=1,2)$, $A_1 \cap A_2 = \emptyset$, then $q(t,A_1 \cup A_2) = q(t,A_1) + q(t,A_2)$, 5) if $\varepsilon > 0$, $S_\varepsilon = [x \in E_r : |x| < \varepsilon]$, then $M_0[q(T, E_r - S_\varepsilon)] < +\infty$ (M_0 denotes the mean value with respect to P_0), 6) if $[\tau_n]$ is an arbitrary sequence of Markov times on $\mathcal{F}$ such that $0 \leq \tau_n \leq \tau_{n+1} \leq T$, $\lim \tau_n = \tau \leq T$, then $M_0[q(\tau_n, A)] \to$ $\to M_0[q(\tau, A)$ for each $A \in \mathcal{B}_r^0$.

Remark 1. The condition 6) of Definition 1 is a necessary and sufficient condition for the continuity of the characteristic of the submartingale $q(t,A)$ [3, p. 41 , Theorem 13]. If $q(t,A)$ satisfies Definition 1, then for $q(t,A)$ the Doob-Meyer's theorem [3, p. 121, Theorem 8] holds true.

Remark 2. The integer-valued random measure can be slightly generalized if we replace the conditions 5), 6) in Definition 1 by the following condition: 5a) on $\mathcal{F}$ there exists a sequence of Markov times $[\tau_n]$, $0 \leq \tau_n \leq$ $\leq \tau_{n+1} \leq T$ and for a.e. $\omega \in \Omega$ there exists $n_0(\omega)$ such that for $n > n_0(\omega)$ $\tau_n(\omega) = T$ and the conditions 5), 6) hold for $q_n(t,A) = q(t \wedge \tau_n, A)$, $t \in \langle 0,T \rangle$, $A \in \mathcal{B}_r^0$ for each natural n.

Also in this case we have for $q(t,A)$ with respect to $\mathcal{F}$ and to P_0 the decomposition

$$q(t,A) = \bar{q}(t,A) + \Pi(t,A) \tag{1}$$

but in this case $\bar{q}(t,A)$ does not have to be a martingale and $\Pi(t,A)$ does not have to be integrable, but in the set of processes $\Pi(t,A)$, for which $\Pi(t \wedge \sigma_n, A)$ are natural processes ($[\sigma_n]$ is a sequence of Markov times, which fully generates $\bar{q}(t,A)$), the preceding decomposition is unique.

The term Poisson integer-valued random measure is understood according to [3, p. 146, Theorem 13].

About all processes which are further considered, it is supposed that they are measurable with respect to $\mathcal{F}$ and their trajectories are continuous from the right and have a finite limit from the left.

3. Several auxiliary assertions. In this paragraph we want to derive several lemmas, to which we shall refer in the following

Lemma 1. Let two probability measures be given on $[\Omega, \mathcal{U}]$, where $P_1 \ll P_0$. If ρ is their Radon-Nikodym derivative ($dP_1 = \rho\, dP_0$), let the process $\rho(t) = M_0[\rho / \mathcal{F}_t]$, $t \in \langle 0;T\rangle$, be a non-negative square integrable martingale with continuous trajectories with respect to P_0. Let $q(t,A)$, $t \in \langle 0;T\rangle$, $A \in \mathcal{B}_r^0$ be an integer-valued random measure satisfying Definition 1 with respect to the measure P_0. Let us denote its characteristic with respect to P_0 as $\Pi(t,A)$, so that $\Pi(t,A)$ has continuous trajectories. Let $M_0[q^2(T,A)] < +\infty$, $M_0[\Pi^2(T,A)] < +\infty$ for each $A \in \mathcal{B}_r^0$ so that $\bar{q}(t,A)$ is an orthogonal square integrable martingale measure with respect to P_0.

Then $q(t,A)$ is the integer-valued random measure satisfying Definition 1 with the characteristic $\Pi(t,A)$ also with respect to P_1 and $\bar{q}(t,A) = q(t,A) - \Pi(t,A)$ is the orthogonal locally square integrable martingale measure with the characteristic $\Pi(t,A)$ also with respect to P_1. Especially, if $\Pi(t,A)$ is non-random, then $q(t,A)$ is the Poisson integer-valued random measure with respect to P_1, too.

Proof. First of all we should verify whether $q(t,A)$ satisfies Definition 1 also with respect to P_1. But that follows immediately from the assumptions of the lemma from the Hölder's inequality and from [7, p. 173, Consequence 3].

Hence and according to Remark 1, the decomposition

$$q(t,A) = \bar{q}^+(t,A) + \Pi^+(t,A) \tag{2}$$

holds for $q(t,A)$ on $[\Omega, \mathcal{U}, P_1]$, too.

Now we shall prove that $\Pi^+(t,A) = \Pi(t,A)$. As the decomposition (2) is unique, we shall have $\bar{q}^+(t,A) = \bar{q}(t,A)$, so that $\bar{q}(t,A)$ will be an orthogonal locally square integrable martingale measure also with respect to P_1.

As it follows from Remark 1, the characteristic $\Pi^+(t,A)$ is a natural continuous process also with respect to the measure P_1. But then according to [3, p. 58, Theorem 2] we have for $\Pi^+(t,A)$

$$\Pi^+(t,A) = \lim \sum_{k=1}^{m} M_1[q(t_k,A) - q(t_{k-1},A) / \mathcal{F}_{t_{k-1}}] \tag{3}$$

and the limit is considered for $|\sigma| \to 0$ in the sense of the convergence in L_1, $|\sigma| = \max \Delta t_k$ is a norm of a partition σ of the type $0 = t_0 < t_1 < \ldots < t_n = T$. Hence we must prove the relation

(4) $\lim M_1[\,|\sum_{k=1}^{m} M_1[\Delta q(t_k,A)/\mathcal{F}_{t_{k-1}}]-\Pi(t,A)|\,] = 0$

We easily obtain

$$M_1[\,|\sum_{k=1}^{m} M_1[\Delta q(t_k,A)/\mathcal{F}_{t_{k-1}}]-\Pi(t,A)|\,]\le M_0[\max_{1\le k\le m}|\Delta\varphi(t_k)|(q(t,A)+\Pi(t,A)]+$$

$$+M_0[\max_{1\le k\le m}|\Delta\varphi(t_k)|\Pi(t,A)]+M_0[\varphi(T)|\sum M_0[\Delta\Pi(t_k,A)/\mathcal{F}_{t_{k-1}}]-\Pi(t,A)|] =$$

$$= S_1+S_2+S_3$$

We shall prove only the relation $\lim S_1=0$. The proof of the relations $\lim S_i=0$ for $i=2,3$ is similar. Obviously,

(5) $P_0\text{-}\lim_{|\delta|\to 0}[\max_{1\le k\le m}|\Delta\varphi(t_k)|(q(t,A)+\Pi(t,A))] =0$

Hence to prove the relation $\lim S_1=0$, it is sufficient to show that the variables

(6) $$[\max_{1\le k\le m}|\Delta\varphi(t_k)|(q(t,A)+\Pi(t,A))]$$

are bounded for each partition δ by an integrable random variable. But for each partition δ it holds

(7) $[\max_{1\le k\le m}|\Delta\varphi(t_k)|(q(t,A)+\Pi(t,A))]\le 2\max_{0\le s\le t}\varphi(s)[q(t,A)+\Pi(t,A)]$

Taking the mean value of both sides in (7) and applying at first the Hölder and then the Doob inequality, we readily obtain that the variables (6) are bounded by an integrable variable and this together with (5) proves the relation $\lim S_1=0$.

It remains to prove that $q(t,A)$ is a Poisson integer-valued random measure also with respect to P_1 in the case, when $\Pi(t,A)$ is non-random. But in accordance with the preceding, the characteristic of $q(t,A)$ does not change when we pass from P_0 to P_1 and the fact that $q(t,A)$ is a Poisson integer-valued random measure also with respect to P_1 then follows from [3, p. 146, Theorem 8]. The lemma is thus completely proved.

The following lemma is an analogue of Lemma 1, but the assumptions which are set on $q(t,A)$ and $\varphi(t)$ are not so strong.

Lemma 2. Let the assumptions of Lemma 1 hold, only the process $\varphi(t)=M_0[\varphi/\mathcal{F}_t]$ is now assumed to be a non-negative martingale with continuous trajectories with respect to P_0, and $\bar{q}(t,A)$ is assumed to be an orthogonal locally square integrable martingale measure with respect to P_0.

Then the assertion of Lemma 1 also holds, but in this case $q(t,A)$ satisfies with respect to P_1 Definition 1 in which the assumptions 5), 6) are replaced by the assumption 5a).

Proof: As it follows from [3, p.121, Theorem 8], $q(t,A)$ can be written in the form

$$q(t,A)=\bar{q}(t,A)+\Pi(t,A) \tag{8}$$

with respect to P_0.

If $[\sigma_n]$ is a sequence of Markov times on $\mathcal{F}$, which fully generates $\bar{q}(t,A)$ with respect to P_0, we put

$$\sigma_n'=\inf\,[t:\rho(t)\geq n],\quad \sigma_n^*=\inf\,[t:\Pi(t,E_r)\geq n],\quad T=\inf\,\emptyset$$

$$\tau_n=\sigma_n\wedge\sigma_n'\wedge\sigma_n^* \tag{9}$$

Evidently $[\tau_n]$ is the sequence of Markov times on $\mathcal{F}$ with respect to P_0 as well as to P_1 and for $q(t\wedge\tau_n,A)$, $\rho(t\wedge\tau_n)$, the assumptions of Lemma 1 are fulfilled and so the decomposition

$$q(t\wedge\tau_n,A)=\bar{q}(t\wedge\tau_n,A)+\Pi(t\wedge\tau_n,A) \tag{10}$$

holds with respect to P_1, too. When we pass in (10) to the limit, we obtain P_0 and also P_1 a.s.

$$q(t,A)=\bar{q}(t,A)+\Pi(t,A) \tag{11}$$

We have still to prove that the decomposition (11) is unique in the set of the processes considered in Remark 2.

It will be so, if we show that $q(t,A)$ satisfies with respect to P_1 Definition 1, in which the conditions 5), 6) are replaced by the condition 5a).

As a sequence of Markov times from the condition 5a) it is considered the sequence $[\tau_n]$ given by (9). Then we have

$$M_1[q(T\wedge\tau_n,E_r-S_\varepsilon)]=M_0[\rho(T\wedge\tau_n)q(T\wedge\tau,E_r-S_\varepsilon)]\leq$$

$$\leq n\,M_0[q(T\wedge\tau_n,E_r-S_\varepsilon)]<+\infty$$

Now if $[\sigma_j]$ is an arbitrary sequence of Markov times on $\mathcal{F}$ such that $0\leq\sigma_j\leq\sigma_{j+1}\leq T$, $\lim\sigma_j=\sigma\leq T$, then we obviously have

$$P_0\text{-}\lim\rho(\sigma_j\wedge\tau_n)q(\sigma_j\wedge\tau_n,A)=\rho(\sigma\wedge\tau_n)q(\sigma\wedge\tau_n,A)$$

Further

$$\rho(\sigma_j\wedge\tau_n)q(\sigma_j\wedge\tau_n,A)\leq n\,q(T\wedge\tau_n,A) \tag{12}$$

and the variable on the right hand side of (12) is integrable with respect to P_1, so that

$$\lim_{j\to+\infty}M_1[q(\sigma_j\wedge\tau_n,A)]=\lim_{j\to+\infty}M_0[\rho(\sigma_j\wedge\tau_n)q(\sigma_j\wedge\tau_n,A)]=$$

$$=M_0[\rho(\sigma\wedge\tau_n)q(\sigma\wedge\tau_n,A)]=M_1[q(\sigma\wedge\tau_n,A)]$$

This proves the first part of the lemma.

It remains to prove that $q(t,A)$ is a Poisson integer-valued random measure also with respect to P_1, if $\Pi(t,A)$ is nonrandom. As the relation (11) holds also in this case, it is sufficient in accordance with [3, p. 146, Theorem 13] to show that $q(t,A)$ satisfies Definition 1 also with respect to P_1. Obviously, it is sufficient to prove the validity of the conditions 5), 6) of this definition.

If we take as a sequence of Markov times the sequence (9), the validity of the condition 5) follows immediately from the theorem about the convergence of a nonnegative monotonous sequence.

Now let $[\sigma_n]$ be an arbitrary sequence of Markov times such that $0 \leq \sigma_n \leq \leq \sigma_{n+1} \leq T$, $\lim \sigma_n = \sigma \leq T$. We want to show that

$$\lim M_1[q(\sigma_n,A)] = M_1[q(\sigma,A)]. \tag{13}$$

From the validity of the condition 6) with respect to P_0 it follows

$$P_1\text{-}\lim q(\sigma_n,A)=q(\sigma,A) \tag{14}$$

Further

$$q(\sigma_n,A) \leq q(T,A),\ P_1\text{- a.s.}$$

and as the condition 5) holds with respect to P_1 too, $q(T,A)$ is an integrable random variable with respect to P_1, which together with (14) proves (13). The fact that $q(t,A)$ is a Poisson integer-valued measure now follows from [3, p. 146, Theorem 13]. The lemma is thus proved.

Remark 3. Lemma 2 can be slightly generalized. We can require that $q(t,A)$ satisfies with respect to P_0 only Definition 1 in which the conditions 5), 6) are replaced by the assumption 5a). The assertion of Lemma 2 will remain valid except that part where the Poisson integer-valued measure is spoken about. According to Remark 2 the decomposition

$$q(t,A)=\bar{q}(t,A)+\Pi(t,A)$$

holds for $q(t,A)$ with respect to P_0 also in this case, only $\Pi(t,A)$ does not have to be integrable. But the integration of $\Pi(t,A)$ with respect to P_0 is not used in the proof of Lemma 2, so that it holds also in this case.

Lemma 3. Let $P_0, P_1, \varphi(t)$ and $q(t,A)$ satisfy the assumptions of Lemma 2, only $q(t,A)$ (in accordance with Remark 3) can also satisfy Definition 1 in which the conditions 5), 6) are replaced by 5a). Let $c(t,x)$ be a random function, defined on $\langle 0;T\rangle \times E_r \times \Omega$ with values in E_r, which is $\mathfrak{B}_1^+ \times \mathfrak{B}_r \times \mathfrak{A}$ measurable ($\mathfrak{B}_1^+$ is a system of Borel sets of $\langle 0;T\rangle$), for fixed x being measurable with respect to $\mathcal{F}$ and such that

$$(15) \qquad \int_0^T \int_{E_r} |c(s,x)|^2 \Pi(ds,dx) < +\infty \qquad P_0\text{-a.s.}$$

Let a random process $\eta(t)$ be given by the relation

$$(16) \qquad \eta(t) = \int_0^T \int_{E_r} c(s,x)\bar{q}(ds,dx)$$

so that $\eta(t)$ is a locally square integrable martingale with respect to P_0.

Then $\eta(t)$ is a locally square integrable martingale also with respect to P_1. If $r(t,A)$, $t \in \langle 0,T \rangle$, $A \in \mathcal{B}_r^0$, denotes the integer-valued random measure associated with the process $\eta(t)$, then $r(t,A)$ satisfies Definition 1 in which the conditions 5), 6) are replaced by 5a) and

$$(17) \qquad r(t,A) = \int_0^t \int_{E_r} \chi_A(c(s,x))q(ds,dx)$$

with respect to P_0 as well as to P_1. If $\Psi(t,A)$ is a characteristic of $r(t,A)$, then

$$(18) \qquad \Psi(t,A) = \int_0^t \int_{E_r} \chi_A(c(s,x))\Pi(ds,dx)$$

with respect to P_0 as well as to P_1. When $\Pi(t,A)$ and $c(t,x)$ are non-random, then $r(t,A)$ is a Poisson integer-valued random measure with respect to P_0 as well as to P_1.

Proof: According to Lemma 2, Remark 3 and with respect to (15), $\eta(t)$ is immediately obtained to be a locally square integrable martingale with respect to P_0 as well as to P_1.

Let $\xi(t)$ be a process given by

$$\xi(t) = \int_0^t \int_{E_r} x\, q(ds,dx)$$

From the definition of the integer-valued random measure it is then obtained for $r(t,A)$

$$r(t,A) = \sum_{0 \le s \le t} \chi_A(\delta\eta(s)) = \sum_{0 \le s \le t} \chi_A\Big[\int_{s-0}^{s} \int_{E_r} c(u,x)\bar{q}(du,dx)\Big] =$$

$$= \sum_{\substack{0 \le s \le t \\ \delta\xi(s) \ne 0}} \chi_A(c(s,\delta\xi(s))) = \int_0^t \int_{E_r} \chi_A(c(s,x))q(ds,dx)$$

This relation proves (17) with respect to P_0 as well as to P_1.

Now $r(t,A)$ is verified to satisfy Definition 1 in which the conditions 5), 6) are replaced by 5a) as well as the validity of (18) with respect to P_0. Then the validity of both with respect to P_1 will follow from Lemma 2 and Remark 3.

Let $\{\tau_n\}$ be the sequence of Markov times given by (9) and

$$\sigma_n^+=\inf\{t: \int_0^t\int_{E_n}|c(s,x)|^2\Pi(ds,dx)\geq n\},\quad \tau_n^*=\tau_n\wedge\sigma_n^+,\quad \eta_n(t)=\eta(t\wedge\tau_n^*)$$

and let $r_n(t,A)$ be the integer-valued random measure of $\eta_n(t)$, $\Upsilon_n(t,A)$ its characteristic with respect to P_0. As before it is obtained

$$r_n(t,A)=\int_0^{t\wedge\tau_n^*}\int_{E_n}\chi_A(c(s,x))q(ds,dx)\qquad \Upsilon_n(t,A)=\int_0^{t\wedge\tau_n^*}\int_{E_n}\chi_A(c(s,x))\Pi(ds,dx)$$

According to [3,pp. 117-121] $r_n(t,A)$ satisfies with respect to P_0 Definition 1, so that $r(t,A)$ satisfies with respect to P_0 Definition 1 in which 5), 6) are replaced by 5a).

Obviously $r_n(t,A)=\bar{r}_n(t,A)+\Upsilon_n(t,A)$, where

$$\bar{r}_n(t,A)=\int_0^{t\wedge\tau_n^*}\int_{E_n}\chi_A(c(s,x))\bar{q}(ds,dx)$$

Taking the limit on both sides in the relation for $r_n(t,A)$ it is obtained P_0-a.d.

(19) $$r(t,A)=\bar{r}(t,A)+\Upsilon(t,A)$$

where

$$\Upsilon(t,A)=\int_0^t\int_{E_n}\chi_A(c(s,x))\Pi(ds,dx),\quad \bar{r}(t,A)=\int_0^t\int_{E_n}\chi_A(c(s,x))\bar{q}(ds,dx)$$

According to Remark 2, the decomposition (19) is unique and $\bar{q}(t,A)$ is an orthogonal locally square integrable martingale measure with respect to P_0. This proves (18) with respect to P_0. According to Lemma 2 and Remark 3 the relation (19) holds with respect to P_1 too, and this implies the validity of (18) also with respect to P_1.

If $\Pi(t,A)$ is nonrandom, $q(t,A)$ must satisfy Definition 1 and according to Lemma 2, $q(t,A)$ is a Poisson integer-valued random measure with respect to P_0 as well as to P_1. When $c(t,x)$ is nonrandom too, then the condition (15) can be written as

$$M_0\{\int_0^T\int_{E_n}|c(s,x)|^2\Pi(ds,dx)\}<+\infty$$

and the process (16) is a square integrable martingale. As it follows from [3, pp.117-121], $r(t,A)$ satisfies Definition 1 with respect to P_0. $\Upsilon(t,A)$ is obviously nonrandom, too, and from Lemma 2 it is obtained that $r(t,A)$ is a Poisson integer-valued measure with respect to P_0 as well as to P_1. The lemma is proved.

4. The generalization of the Girsanov theorem. The following theorem is an analogue of the Girsanov theorem.

Theorem 1. Let $A(t)=[A_1(t),\dots,A_r(t)]$ be a random function $\mathcal{F}_t$ and progressively measurable and such that

$$\int_0^T |A(t)|^2 dt < +\infty \qquad P_0\text{-a.s.}$$

Let a process $Z \equiv [Z(t), t \in \langle 0;T\rangle]$ be given by the relation

$$Z(t)=\exp\left[\int_0^t A(s)dw^*(s)-\frac{1}{2}\int_0^t |A(s)|^2 ds\right]$$

where $w(t)=[w_1(r),\dots,w_k(t)]$, $w(0)=0$, is a Wiener process (($*$) denotes the transposition) with respect to P_0. Let $M_0[Z(T)]=1$ hold, so that $Z(t)$ is a martingale with respect to P_0. Let the measure P_1 be defined on $[\Omega,\mathcal{U}]$ by the relation

$$P_1(A)=\int_A Z(T)d\,P_0,\ A \in \mathcal{U} \tag{20}$$

Let $p(t,A)$ be an integer-valued random measure satisfying Definition 1' or Definition 1 in which 5), 6) are replaced by 5a), so that its characteristic $\Pi(t,A)$ is a continuous non-decreasing process, $\Pi(0,A)=0$ and let

$$\int_0^T \int_{E_n} |x|^2 \Pi(ds,dx) < +\infty \qquad P_0\text{-a.s.} \tag{21}$$

Let a random process $\eta(t)$ be given by

$$\eta(t)=\int_0^t \int_{E_n} x\,\overline{p}(ds,dx) \tag{22}$$

where $\overline{p}(t,A)=p(t,A)-\Pi(t,A)$, $t \in \langle 0,T\rangle$, $A \in \mathcal{B}_r^0$ so that $\eta(t)$ is a locally square integrable martingale with respect to P_0.

Then on the probability space $[\Omega,\mathcal{U},P_1]$

$$w(t)+\eta(t)-\int_0^t A(s)ds=\tilde{w}(t)+\tilde{\eta}(t) \tag{23}$$

where $\tilde{w}(t)$ is a Wiener process with respect to P_1 and $\tilde{\eta}(t)$ is a locally square integrable martingale with respect to P_1, for which

$$\tilde{\eta}(t)=\int_0^t \int_{E_n} x\,\overline{p}(ds,dx), \quad P_1\text{-a.s.} \tag{24}$$

Proof: Let us denote

$$\xi(t)=w(t)+\eta(t)-\int_0^t A(s)ds$$

Applying to $Z(t)\,\xi_k(t)$ $(k=1,\dots,r)$ the generalized Ito formula [3, p. 143, Theorem 11] we obtain

$$d[Z(t)\xi_k(t)]=Z(t)\xi_k(t)\sum_{i=1}^{n} A_i(t)dw_i(t)+Z(t)dw_k(t)+Z(t)\int_{E_n} u_k\overline{p}(dt,du)$$

Hence and according to [8, p. 102, Theorem 1], $\xi(t)$ is a locally square integrable martingale with respect to P_1. But then according to [3, p. 127, Consequence] the decomposition

$$\xi(t)=\xi_c(t)+\xi_d(t)=\xi_c(t)+\int_{E_n} u\,\bar{q}(t,du)$$

holds for $\xi(t)$. Here $\bar{q}(t,A)=q(t,A)-\Psi(t,A)$ and $q(t,A)$ is an integer-valued measure associated with the process $\xi(t)$, $\Psi(t,A)$ is its characteristic with respect to P_1 and $\xi_c(t)$ is a local martingale with respect to P_1 with the continuous trajectories. Obviously $q(t,A)$ is defined only by the process $\eta(t)$ so that according to Lemma 3 it holds for $q(t,A)$

$$q(t,A)=\int_0^t\int_{E_n}\chi_A(u)p(ds,du)=p(t,A) \tag{25}$$

and for $\Psi(t,A)$

$$\Psi(t,A)=\int_0^t\int_{E_n}\chi_A(u)\Pi(ds,du)=\Pi(t,A) \tag{26}$$

Hence we obtain for $\xi_d(t)$ on $[\Omega,\mathfrak{A},P_1]$ the relation

$$\xi_d(t)=\tilde{\eta}(t)=\int_{E_n} u\,\bar{p}(t,du) \tag{27}$$

From the decomposition

$$\xi(t)=\xi_c(t)+\xi_d(t)=\xi_c(t)+\int_0^t\int_{E_n} u\,\bar{p}(ds,du)$$

it follows that

$$\xi_c(t)=w(t)-\int_0^t A(s)ds$$

and using the Girsanov theorem [3, p. 330, Theorem 11] we obtain for the process $\xi_c(t)$ on $[\Omega,\mathfrak{A},P_1]$

$$\xi_c(t)=w(t)-\int_0^t A(s)ds=\tilde{w}(t) \tag{28}$$

where $\tilde{w}(t)$ is a Wiener process with respect to P_1. (27) and (28) prove for $\xi(t)$ on $[\Omega,\mathfrak{A},P_1]$ the relation

$$\xi(t)=\xi_c(t)+\xi_d(t)=\tilde{w}(t)+\tilde{\eta}(t)$$

where $\tilde{w}(t)$ is a Wiener process with respect to P_1 and $\tilde{\eta}(t)$ is a locally square integrable martingale with respect to P_1 for which (24) holds. The theorem is proved.

Consequence 1. Let in Theorem 1 $p(t,A)$ be a Poisson integer-valued random measure with respect to P_0, so that $\eta(t)$ is a centered Poisson process with respect to P_0. Then also the process $\tilde{\eta}(t)$ is a centered Poisson process with respect to P_1 which is homogeneous, if $\eta(t)$ is homogeneous on

$[\Omega, \mathfrak{A}, P_0]$. The process

$$\xi(t)= \int_0^t A(s)ds+w(t)+\eta(t)$$

is on $[\Omega, \mathfrak{A}, P_1]$ a process with independent increments and random vectors

$$\int_0^t A(s)ds+w(t); \ \eta(t)$$

are for each $t\in\langle 0;T\rangle$ mutually independent on $[\Omega, \mathfrak{A}, P_1]$.

Proof: The fact that $\tilde{\eta}(t)$ is a centered Poisson process with respect to P_1 whenever $p(t,A)$ is a Poisson integer-valued random measure with respect to P_0, follows from the relations (25),(26),(27).

Further, if $w(t)$ is a Wiener process with respect to P_0, then the process $(-w(t))$ is a Wiener process with respect to P_0, too, and according to the Girsanov theorem [3, p. 330, Theorem 11] we have on $[\Omega, \mathfrak{A}, P_1]$

$$-w(t)= \int_0^t A(s)ds=w^+(t)$$

where $w^+(t)$ is a Wiener process with respect to P_1 and thus the process

$$\tilde{w}(t)= -w^+(t)=w(t)+ \int_0^t A(s)ds$$

is a Wiener process on $[\Omega, \mathfrak{A}, P_1]$, too. But then from Theorem 1 it follows on $[\Omega, \mathfrak{A}, P_1]$ for the process

$$\xi(t)= \int_0^t A(s)ds+w(t)+\eta(t)$$

the relation

$$\xi(t)=\tilde{w}(t)+\tilde{\eta}(t)$$

where $\tilde{w}(t)$ is a Wiener process with respect to P_1 and $\tilde{\eta}(t)$ is a centered Poisson process with respect to P_1.

Applying now the generalized Ito formula [3, p. 143, Theorem 11] to the function

$$X(t)=\exp\{i[\sum_{k=1}^{n} \lambda_k x_k+\sum_{k=1}^{n} z_k y_k]$$

where $x_k=\tilde{w}_k(t)$, $y_k=\tilde{\eta}_k(t)$; λ_k, z_k are arbitrary r.n., we directly see that the probability distribution of the random vector $\xi(t)-\xi(s)$, considered on $[\Omega, \mathfrak{A}, P_1]$, does not depend on the σ-algebra $\mathcal{F}_s$. But this means that the process $\xi(t)$, considered on $[\Omega, \mathfrak{A}, P_1]$, is a process with independent increments. Applying once more the generalized Ito formula to the function $X(t)$ and denoting

$$I(t)=M_1[X(t)/\mathcal{F}_0]$$

we obtain

(29) $$M_1[I(t)] =\exp\{-\tfrac{1}{2}\sum\lambda_k^2 t\}\exp\{\int_{E_n}[\exp(i\sum z_k u_k)-1- i\sum z_k u_k]\Pi(t,du)\}$$

The relation (29) proves that the vectors $\int_0^t A(s)ds+w(t)$; $\eta(t)$ considered on $[\Omega,\mathfrak{A},P_1]$ are for each $t\in\langle 0,T\rangle$ mutually independent. The proof is finished.

Theorem 2. Let $B(t)=[b_{jk}(t)]_{j,k=1}^{r}$ be progressively and $\mathcal{F}_t$ measurable random matrix process, defined on $[\Omega,\mathfrak{A},P_0]$ and such that

$$\int_0^T |B(s)|^2 ds < +\infty \qquad P_0\text{-a.s.}$$

and $B(t)$ is a regular matrix for each t and P_0-a.s. For the random function $A(t)=[A_1(t),\dots,A_r(t)]$ let the assumptions of Theorem 1 be fulfilled and let the inequality

$$\int_0^T |B^{-1}(s)A^*(s)|^2 ds < +\infty \qquad P_0\text{-a.s.}$$

hold (($*$) denotes the transposition). Let $M_0[Z(T)]=1$ hold, where $Z(t)$, $t\in\langle 0;T\rangle$, is given by the relation

$$Z(t)=\exp\left\{\int_0^t [B^{-1}(s)A^*(s)]^* \, dw^*(s)-\frac{1}{2}\int_0^t |B^{-1}(s)A^*(s)|^2 ds\right\}$$

Here $w(t)=[w_1(t),\dots,w_r(t)]$ is a Wiener process with respect to P_0. Let us define on $[\Omega,\mathfrak{A}]$ the measure P_1 by the relation

$$P_1(A)=\int_A Z(T)d\,P_0,\quad A\in\mathfrak{A} \tag{30}$$

Let $q(t,A)$ be an integer-valued random measure and $c(t,x)$ a random function for which the assumptions of Lemma 3 hold and the inequality

$$\int_0^T\int_{E_\kappa} |B^{-1}(s)c^*(s,x)|^2\,\Pi(ds,dx) < +\infty \qquad P_0\text{-a.s.} \tag{31}$$

holds, too. Let a random process $\eta(t)$ be given by (16).

Then on the probability space $[\Omega,\mathfrak{A},P_1]$

$$\int_0^t [B(s)dw^*(s)]^* + \eta(t)-\int_0^t A(s)ds=\int_0^t [B(s)d\tilde{w}^*(s)]^* + \tilde{\eta}(t) \tag{32}$$

where $\tilde{w}(t)=[\tilde{w}_1(t),\dots,\tilde{w}_r(t)]$ is a Wiener process with respect to P_1 and $\tilde{\eta}(t)$ is a locally square integrable martingale with respect to P_1 which is given by (16) and $q(t,A)$ is an integer-valued random measure with the same characteristic $\Pi(t,A)$ with respect to P_1 as to P_0, so that $\overline{q}(t,A)$ is the orthogonal locally square integrable martingale measure with the same characteristic with respect to P_1 as well as to P_0, too.

Proof: Let us denote

$$P(t)=\int_0^t [B(s)dw^*(s)]^* + \eta(t)-\int_0^t A(s)ds$$

Obviously, $P(t)$ has a stochastic differential and

$$dP(t)=[B(t)dw^*(t)]^* + \int_{E_r} c(t,u)\bar{q}(dt,du)-A(t)dt$$

$B(t)$ is for each $t\in\langle 0;T\rangle$ and P_0-a.s. regular, so that P_0-a.s.

$$\int_0^t [B^{-1}(s)dP^*(s)]^* = w(t)+\int_0^t \int_{E_r} [B^{-1}(s)c^*(s,x)]^* \bar{q}(ds,dx)-\int_0^t [B^{-1}(s)A^*(s)]^* ds$$

Let us put

$$\xi(t)=w(t)+\int_0^t \int_{E_r} [B^{-1}(s)c^*(s,x)]^* q(ds,dx)-\int_0^t [B^{-1}(s)A^*(s)]^* ds$$

In the same way as in the proof of Theorem 1 it can be verified that $\xi(t)$ is a locally square integrable martingale with respect to P_1, so that according to [3, p. 127, Consequence] the decomposition

$$\xi(t)=\xi_c(t)+\xi_d(t)=\xi_c(t)+\int_{E_r} u\bar{v}(t,du)$$

holds for $\xi(t)$ on $[\Omega,\mathcal{U},P_1]$. Here $\bar{v}(t,A)=v(t,A)-\varphi(t,A)$ and $v(t,A)$ is an integer-valued random measure associated with the process $\xi(t)$; $\varphi(t,A)$ is its characteristic with respect to P_1 and $\xi_c(t)$ is a local martingale with respect to P_1 with continuous trajectories. By means of Lemmas 2 and 3, in the same way as in the proof of Theorem 1 it can be derived that it holds on $[\Omega,\mathcal{U},P_1]$

$$\xi_d(t)=\int_0^t \int_{E_r} x\,\bar{v}(ds,dx)=\int_0^t \int_{E_r} [B^{-1}(s)c^*(s,x)]^* \bar{q}(ds,dx)$$

where $\bar{q}(t,A)$ is an orthogonal locally square integrable martingale measure with the same characteristic $\Pi(t,A)$ with respect to P_1 as well as to P_0 and at the same time $\Pi(t,A)$ is the characteristic of $q(t,A)$ with respect to P_1 as well as to P_0. From the decomposition

$$\xi(t)=\xi_c(t)+\xi_d(t)=\xi_c(t)+\int_0^t \int_{E_r} [B^{-1}(s)c^*(s,x)]^* \bar{q}(ds,dx)$$

it follows that

$$\xi_c(t)=w(t)-\int_0^t [B^{-1}(s)A^*(s)]^* ds$$

is a local martingale with respect to P_1 with continuous trajectories. As the function $[B^{-1}(s)A^*(s)]^*$ satisfies the assumptions of Theorem 1, it is obtained in accordance with this theorem and with the Girsanov theorem that

$$\xi_c(t)=w(t)-\int_0^t [B^{-1}(s)A^*(s)]^* ds=\tilde{w}(t)$$

where $\tilde{w}(t)$ is a Wiener process with respect to P_1. But then

$$\int_0^t [B^{-1}(s)dP^*(s)]^* = \xi(t)=\tilde{w}(t)+\int_0^t \int_{E_r} [B^{-1}(s)c^*(s,x)]^* q(ds,dx)$$

Obviously, $\xi(t)$ has on $[\Omega, \mathcal{U}, P_1]$ a stochastic differential, so that

$$d\,\xi(t) = [d\,P(t)]\,B^{*-1}(t) = d\tilde{w}(t) + \int_{E_r} c(t,x)B^{*-1}(t)\bar{q}(dt,dx)$$

and so on $[\Omega, \mathcal{U}, P_1]$ it really holds

$$P(t) = \int_0^t [B(s)d\,\tilde{w}^*(s)]^* + \int_0^t \int_{E_r} c(s,x)\bar{q}(ds,dx)$$

The theorem is proved.

Consequence 2. Let $\xi(t)$ be a random process, defined on $[\Omega, \mathcal{U}, P_0]$ with values in E_r, measurable with respect to $\mathcal{F}$ and such that

$$\xi(t) = \int_0^t A(s)ds + \int_0^t [B(s)dw^*(s)]^* + \int_0^t \int_{E_r} c(s,x)\bar{q}(ds,dx)$$

where:

a) $w(t) = [w_1(t), \dots, w_r(t)]$ is a Wiener process with respect to P_0

b) $q(t,A)$, $t \in \langle 0;T \rangle$, $A \in \mathcal{B}_r^0$ is a Poisson integer-valued random measure with respect to P_0

c) $B(t) = [b_{jk}(t)]$ $(j,k=1,\dots,r)$ is a nonrandom matrix function which is regular for each t and such that

$$\int_0^T |B(s)|^2 ds < +\infty$$

d) $c(t,x) = [c_1(t,x), \dots, c_r(t,x)]$ is a nonrandom function such that

$$\int_0^T \int_{E_r} |c(s,x)|^2 \Pi(ds,dx) < +\infty$$

e) $A(t) = [A_1(t), \dots, A_r(t)]$ is a random function satisfying the assumptions of Theorem 1.

Then there exists a probability measure P_1 such that the process $\xi(t)$ is on $[\Omega, \mathcal{U}, P_1]$ a process with independent increments with respect to $\mathcal{F}$ and the vectors

$$\int_0^t A(s)ds + \int_0^t [B(s)dw^*(s)]^*; \quad \int_0^t \int_{E_r} c(s,x)\bar{q}(ds,dx)$$

are for each $t \in \langle 0;T \rangle$ mutually independent on $[\Omega, \mathcal{U}, P_1]$.

Proof: The proof is similar to the proof of Consequence 1, so that we shall not repeat it.

References

[1] GIRSANOV I.V.: O preobrazovanii odnogo klassa sluchaĭnykh processov s pomoshchyu absolyutno nepreryvnoĭ zameny mery, Teoriya veroyatnosteĭ i ee primeneniya 5(1960), 314-330.

[2] ERSHOV M.P.: Ob absolyutnoĭ nepreryvnosti mer otvechayushchikh processam diffuzionnogo tipa, Teoriya veroyatnosteĭ i ee primeneniya 17 17(1972), 173-179.

[3] GIKHMAN I.I., SKOROKHOD A.V.: Teoriya sluchaĭnykh processov, tom III, Izd.-vo Nauka , 1975,str. 345, 350.

[4] LIPCER R.SH., SHIRYAEV A.N.: Ob absolyutnoĭ nepreryvnosti mer sootvetstvuyushchikh processam diffuzionnogo tipa otnositelno vinerovskoĭ, Izv. AN SSSR, ser. matem. 36(1972), 847-889.

[5] SKOROKHOD A.V.: O differenciruemosti mer sootvetstvuyushchikh sluchaĭnym processam, Teoriya veroyatnosteĭ i ee primeneniya 5(1960), 45-53.

[6] GIKHMAN I.I., SKOROKHOD A.V.: Stokhasticheskie differencialnye uravneniya, Nauka, Dumka, 1968.

[7] LOEVE M.: Teoriya veroyatnosteĭ, Izd. in. literatury, Moskva 1962.

[8] MANDL P.: Stochastic integrals and their applications (in Czech), ČSAV, Praha 1975.

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