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A NEW WAY TO FIND COMPACT ZERO-DIMENSIONAL FIRST COUNTABLE PREIMAGES OF FIRST COUNTABLE COMPACT SPACES

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Abstract: Any compact first countable space X possesses a base B such that the family $P_B = \{Fr(U): U \in B\}$ has the order less than $\mathfrak{c}$ at every $x \in X$. Therefore CH implies X has a peripherally point-countable base. We prove also that every first countable compact space with a peripherally point-countable base is a continuous image of a zero-dimensional first countable compact space, giving thus a new easier way to prove A.V. Ivanov's theorem [1].

Key words: First countable compact space, order, peripherally point-countable base.

Classification: 54A25, 54C35

It is not yet known within ZFC if for any first countable compact space X there exists a first countable zero-dimensional compact Y and a continuous onto mapping $f: Y \longrightarrow X$. A.V. Ivanov proved using inverse spectra technique that such a Y exists in case $w(X) = \omega_1$ [1]. Hence it follows from CH that the answer to the above question (which is actually V.I. Ponomarev's problem [2]) is positive. In this paper we extend the result of A.V. Ivanov over the classes of Corson and linearly ordered first countable compact spaces. Thereby some new properties of a first countable compact X come into consideration and seem to be interesting in themselves. For example, if CH is assumed, then any X as above has a base B such that the family $P_B = \{Fr(U): U \in B\}$ is point-countable (we will call such a base peripherally point-countable).

It is relevant to mention hereby A.S. Mishchenko's theorem [3]: any compact space having a point-countable base is metrizable and B.E. Shapirovskii's result [4]: if the tightness of a compact X is countable, then X has a point-countable π-base. Unfortunately, the author did not succeed to clear up whether it is true in ZFC that any first countable compact X has a peripherally point-countable base.

Our notations and terminology are standard. All spaces under consideration are Tychonoff (and in fact compact). For a space X by $T(X)$ is denoted its topology and $T^*(X)=T(X)\setminus\{\emptyset\}$. The boundary $Fr(A)$ of a set $A\subset X$ is the set $\overline{A}\cap\overline{X\setminus A}$. If $X=\prod\{X_\alpha:\alpha\in\tau\}$ is a Tychonoff product of the spaces X_α and $A\subset B\subset\tau$, then $p^B_A:\prod\{X_\alpha:\alpha\in B\}\to\prod\{X_\alpha:\alpha\in A\}$ is the natural projection and $p_A=p^\tau_A$. Functions are treated as their graphs so that $f=\cup\{f_\alpha:\alpha\in\tau\}$ means f is the common extension of f_α's. For $x\in X$, the cardinal number $\chi(x,X)$ is the weight of X at x, and $\chi(X)=\sup\{\chi(x,X):x\in X\}$. By the order $ord(\gamma,x)$ of a family γ of subsets of X at the point x is meant the power of the set $\{U\in\gamma:x\in U\}$. The expression $x_n\to x$ means the sequence $\{x_n:n\in\omega\}$ converges to x. The space **R** is the real line with its natural topology and $D=\{0,1\}$ - the discrete two-point space.

1. Theorem. Any compact X with $\chi(X)=\omega$ has a base B such that $ord(P_B,x)<\mathfrak{c}$ for all $x\in X$ (recall that $P_B:\{Fr(U):U\in B\}$).

Proof. It is possible by the well known A.V. Arhangel'skii's theorem [5] to faithfully index all points of X by the ordinals from $\mathfrak{c}$: $X=\{x_\alpha:\alpha\in\mathfrak{c}\}$. Of course $|X|<\mathfrak{c}$ implies $|X|=\omega$ and the theorem is trivial in this case,so we assume from now on that $|X|=\mathfrak{c}$.

Fix a family $F=\{f_\alpha:\alpha<\mathfrak{c}\}$ of real-valued continuous functions on X satisfying the following conditions:

(1) $f_\alpha(X)\subset I=[0,1]$;

(2) $\{x_\alpha\}=f^{-1}_\alpha(0)$

for all $\alpha<\mathfrak{c}$.

Suppose we have a family $S=\{S_\alpha:\alpha<\mathfrak{c}\}$ where $S_\alpha=\{r^n_\alpha:n\in\omega\}$ is a decreasing sequence of positive elements of I converging to zero. It is straightforward to verify that

$$B_S=\{f^{-1}([0,r^n_\alpha)):n\in\omega,\alpha<\mathfrak{c}\}$$

is a base of X. To find a base promised in the theorem we will construct an appropriate S by recursion along $\alpha<\mathfrak{c}$.

Assume that the sequences $S_\alpha=\{r^n_\alpha:n\in\omega\}$ have been constructed for all $\alpha<\beta<\mathfrak{c}$. As $|\{f_\beta(x_\alpha):\alpha<\beta\}|<\mathfrak{c}$ there exists a decreasing sequence $\{r^n_\beta:n\in\omega\}\subset(0,1]\setminus\{f_\beta(x_\alpha):\alpha<\beta\}$ converging to 0. Let $S_\beta=\{r^n_\beta:n\in\omega\}$.

The family $S=\{S_\alpha:\alpha<\mathfrak{c}\}$ being at hand let us prove that the base $B=B_S$ is as required. Since $Fr(f^{-1}_\alpha([0,r^n_\alpha)))\subset f^{-1}_\alpha(r^n_\alpha)$, it suffices to prove that $ord(\gamma,x)<\mathfrak{c}$ for $\gamma=\{f^{-1}_\alpha(r^n_\alpha):\alpha<\mathfrak{c},n\in\omega\}$ and any point $x\in X$.

Indeed, there is an $\alpha<\mathfrak{c}$ with $x_\alpha=x$. For every $\beta>\alpha$ and $n\in\omega$ it is impossible that $f_\beta(x)=r^n_\beta$, so $\{E:E\in\gamma$ and $x\in E\}\subset\{f^{-1}_\beta(r^n_\beta):n\in\omega,\beta\leq\alpha\}$ and this finishes our proof.

2. Corollary. (CH). Any first countable compact X has a peripherally point-countable base.

The peripherally point-countable base (abbr.: PPC-base) seems to be an interesting notion in itself. It is hereditary and looks like a dimensional property since all zero-dimensional spaces have a PPC-base. Any compact space is a continuous image of a zero-dimensional compact space. Therefore our following example shows that the PPC-base property is not invariant with respect to perfect mappings.

3. Example. The space $X=I^{\omega_1}$ has no PPC-base.

Proof. Let B be a base in X, $A\subset\omega_1$ - a countable set and $z\in I^A$. Then there is a $U\in B$, a countable $A_1\supset A$, $A_1\subset\omega_1$ and $z_1\in I^{A_1}$ such that $p^{A_1}_A(z_1)=z$ and $p^{-1}_{A_1}(z_1)\subset Fr(U)$. To prove this, pick any $U\in B$ with $U\cap p^{-1}_A(z)\neq\emptyset\neq p^{-1}_A(z)\setminus\overline{U}$. There is a countable $A_1\supset A$ for which $p^{-1}_{A_1}p_{A_1}(\overline{U})=\overline{U}$ holds. The set $p_{A_1}(U)$ is open in I^{A_1} and $\emptyset\neq p_{A_1}(U)\cap(p^{A_1}_A)^{-1}(z)\neq(p^{A_1}_A)^{-1}(z)$ for if $p_{A_1}(U)\supset(p^{A_1}_A)^{-1}(z)$, then $\overline{U}=p^{-1}_{A_1}p_{A_1}(\overline{U})\supset p^{-1}_{A_1}(p^{A_1}_A)^{-1}(z)=p^{-1}_A(z)$.

The space $(p^{A_1}_A)^{-1}(z)$ being connected, there is a point

$$z_1\in Fr(p_{A_1}(U)\cap(p^{A_1}_A)^{-1}(z))\subset Fr(p_{A_1}(U)).$$

Thus $z_1\in\overline{p_{A_1}(U)}\setminus p_{A_1}(U)=p_{A_1}(\overline{U})\setminus p_{A_1}(U)$ so that $p^{-1}_{A_1}(z_1)\subset Fr(U)$. Of course $p^{A_1}_A(z_1)=z$.

Now it is not difficult to construct a transfinite sequence $\{\langle x_\alpha,A_\alpha,U_{t_\alpha}\rangle:$ $:\alpha<\omega_1\}$ with the following properties:

(3) $A_\alpha\subset\omega_1$, $|A_\alpha|=\omega$;

(4) $A_\alpha\subset A_\beta$ if $\alpha<\beta<\omega_1$;

(5) $x_\alpha\in I^{A_\alpha}$, $U_{t_\alpha}\in B$ and $p^{-1}_{A_\alpha}(x_\alpha)\subset Fr(U_{t_\alpha})$;

(6) $p^{A_\beta}_{A_\alpha}(x_\beta)=x_\alpha$ for $\alpha<\beta$;

(7) $U_{t_\beta}\cap Fr(U_{t_\alpha})\neq\emptyset$ for all $\alpha<\beta<\omega_1$.

Once this is done, let $A=\bigcup\{A_\alpha:\alpha<\omega_1\}$ and $x=\bigcup\{x_\alpha:\alpha<\omega_1\}$. Then $U_{t_\alpha}\neq U_{t_\beta}$ for different α,β : $x\in I^A$ and any $y\in p^{-1}_A(x)$ belongs to the set $\bigcap\{Fr(U_{t_\alpha}):\alpha<\omega_1\}$ which shows that B is not peripherally point-countable.

4. Main technical result. Given a first countable compact X and a base B in X, one can produce a zero-dimensional compact Y and a continuous onto mapping $f:Y\to X$ such that $\chi(y,Y)\le \mathrm{ord}(p_B,f(y))$ for any $y\in Y$.

Proof. Let $q_U(0)=U$, $q_U(1)=X\setminus\overline{U}$ for all $U\in B$. Define Y to be the subset of D^B consisting of those points $y=\langle y_U:U\in B\rangle$ for which the family $\{q_U(y_U): :U\in B\}$ has the finite intersection property. It is straightforward that Y is closed in D^B. For $y=\langle y_U:U\in B\rangle\in Y$ let $f(y)=x$, where $\{x\}=\bigcap\{\overline{q_U(y_U)}:U\in B\}$. To prove the consistency of our definition we must check that

$$|\bigcap\{\overline{q_U(y_U)}:U\in B\}|\le 1.$$

Take any $z\neq x$. There is a $U\in B$ with $x\in U\subset\overline{U}\not\ni z$. Therefore $q_U(y_U)\neq X\setminus\overline{U}$ and $z\notin\overline{q_U(y_U)}$ which is what we needed. That f is continuous and onto is routine. To verify the inequality $\chi(y,Y)\le \mathrm{ord}(P_B,f(y))$ let $C=\{U\in B:f(y)\in \mathrm{Fr}(U)\}$. Prove that p_C is one-to-one on $f^{-1}f(y)$. Pick $y_1,y_2\in f^{-1}f(y)$, $y_1=\langle y_U^1:U\in B\rangle$, $y_2=\langle y_U^2:\underline{U\in B}\rangle$. If $p_Cy_1=p_Cy_2$ then for any $U\in B\setminus C$ either $f(y)\in U$ or $f(y)\notin\overline{U}$. We have $\overline{q_U(y_U^i)}\ni f(y)$ for $U\in B$ and $i=1,2$. The set $q_U(y_U^i)$ contains f(y) iff $\overline{q_U(y_C^i)}\ni f(y)$ for $U\in B\setminus C$, so there is a single possibility to choose a set W out of the couple $\{U,X\setminus\overline{U}\}$ with $f(y)\in\overline{W}$. Hence $y_U^1=y_U^2$ for $U\in B\setminus C$ and $y_1=y_2$. Therefore $w(f^{-1}(f(y)))\le|C|$ and our proof is complete.

Let us list some consequences of 4.

5. Theorem. For any first countable compact X there is a zero-dimensional compact Y and a continuous onto mapping $f:Y\to X$ with $\chi(y,Y)<\zeta$ for all $y\in Y$.

Proof. Apply Theorem 1 and Result 4.

6. Corollary. For any first countable compact X with a PPC-base there is a zero-dimensional compact Y with $\chi(Y)=\omega$ which can be mapped continuously onto X.

7. Corollary. (A.V. Ivanov [1].) If CH is assumed, then any first countable compact space is a continuous image of a zero-dimensional first countable compact space.

We are going to prove in ZFC that first countable compact spaces have a PPC-base in case they belong to some wide classes extending thus the theorem of A.V. Ivanov within ZFC.

8. Theorem. If a first countable compact X belongs to one of the classes below:

(i) Corson (Eberlein) compact spaces;

(ii) linearly ordered spaces,

then X has a PPC-base.

Proof. For (i) it is sufficient to prove that the Σ-product of real lines has a PPC-base. Let $\Sigma = \{x \in R^{\tau} : \mathrm{supp}(x) = \{\alpha \in \tau : x_\alpha \neq 0\}$ is countable$\}$ and $B = \{M(\alpha_1,\dots,\alpha_n : O_1,\dots,O_n) : \alpha_1,\dots,\alpha_n \in \tau$, $O_1,\dots,O_n \in T^*(R)$ are rational intervals, $Fr(O_i) \not\ni 0$, $i=1,\dots,n\}$. Here $M(\alpha_1,\dots,\alpha_n : O_1,\dots,O_n) = \{x \in \Sigma : x(\alpha_i) \in O_i,\ i=1,\dots,n\}$ - the standard open set in Σ . If $\mathrm{ord}(P_B, x) > \omega$ for some $x \in \Sigma$, then by Δ-argument there is an uncountable $A \subset \tau$ such that $\mathrm{supp}(x) \supset A$ contradicting $x \in \Sigma$. Thus (i) is proved.

As to (ii) we shall establish even more, namely that every first countable compact LOTS X has a peripherally disjoint (in an obvious sense) base.

Note first that for any $x \in X$ either X is locally countable at x, or $|(a,b)| = \mathfrak{c}$ for each interval (a,b) containing x. Fix a numeration $\{x_\alpha : \alpha < \mathfrak{c}\}$ of the set X. Suppose intervals (a^n_α, b^n_α) are chosen for $\alpha < \beta < \mathfrak{c}$ and $n \in \omega$ so that

(8) $\{(a^n_\alpha, b^n_\alpha) : n \in \omega\}$ is a base of X at the point x_α ;

(9) if X is locally countable at x_α , then (a^n_α, b^n_α) are clopen for all $n \in \omega$;

(10) the family of boundaries of chosen intervals is disjoint.

If X is locally countable at x_β then pick any clopen interval base B_β at x_β :$B_\beta = \{(a^n_\beta, b^n_\beta) : n \in \omega\}$. If not, then let $A_\beta = \{a^n_\beta, b^n_\beta : \alpha < \beta, n \in \omega\}$. We will consider only the case when X is locally countable from the left at x_β ($\equiv$ there is an $x < x_\beta$ with $|(x, x_\beta)| \leq \omega$). All other possible cases are similar or simpler.

As $|A_\beta| < \mathfrak{c}$ reasoning as in proof of Theorem 1, we obtain a sequence $\{b^n_\beta : n \in \omega\} \subset X \setminus A_\beta$ with $b^n_\beta > x_\beta$ for all $n \in \omega$ and $b^n_\beta \to x_\beta$. Pick $a^n_\beta < x_\beta$ such that $a^n_\beta \notin Fr((a^n_\beta, x_\beta))$ and $\{(a^n_\beta, b^n_\beta) : n \in \omega\}$ is a base at x_β . The inductive step being done, we have got a base $B = \{(a^n_\alpha, b^n_\alpha) : \alpha < \mathfrak{c}, n \in \omega\}$ which is as promised, so our proof is complete.

9. Corollary. If a first countable space X belongs to one of the following classes:

(i) Corson compact spaces ;

(ii) Eberlein compact spaces;

(iii) continuous images of first countable compact LOTS, then there exists a zero-dimensional first countable compact space which can be mapped onto X continuously.

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