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ON CHROMATIC NUMBER OF PRODUCT OF GRAPHS

Lajos SOUKUP*

Abstract: It is shown that if ZFC is consistent, then so is ZFC + GCH + "There are two graphs, B and W, with cardinalities and chromatic numbers ω_2 such that the product of B and W has chromatic number ω ".

Key words: Chromatic number, product of graphs, consistency result.

Classification: 03E05, 03E35

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1. Introduction. The aim of this paper is to prove a theorem about the chromatic number of product of infinite graphs. Our set theoretical terminology is the standard one as can be found, e.g. in [5]. For example, we identify a cardinal number with the smallest ordinal having that cardinality, and use ω_0, ω_1, etc. instead of $\aleph_0, \aleph_1$.

Let us recall that given graphs $B=\langle U,E\rangle$, and $W=\langle V,F\rangle$ (for black and white, respectively) their product is defined as

$$B\times W=\langle U\times V, \{\{\langle g_0,h_0\rangle,\langle g_1,h_1\rangle\}: \{g_0,g_1\}\in E, \{h_0,h_1\}\in F\}\rangle.$$

That is, the set of vertices of $B\times W$ is the product of the set of the vertices of B and W and the set of edges is the product of the set of the edges.

S.T. Hedetniemi raised the following problem [4] : Given a natural number k, must the product of two k-chromatic graphs be also k-chromatic, or may this number be less than k?

The case k=3 is trivial, the product cannot be 2-chromatic.

M. El-Zahar and N. Sauer solved the problem for k=4 in [2]. In this case the chromatic number of the product must be 4. The problem for $k\geq 5$ is open.

A. Hajnal asked what happens for infinite cardinals. He succeeded in proving the following results, see [3]:

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Theorem. (1) If Chr $(B)=\omega_0$, Chr $(W)= k<\omega_0$, then Chr $(B\times W)=k$.

(2) If $\varkappa$ is a strongly compact cardinal, $\lambda<\varkappa$, and Chr $(B)=\varkappa$, Chr $(W)=\lambda$, then Chr $(B\times W)=\lambda$.

(3) There are two graphs, B and W on ω_1, such that Chr $(B)=$ $=$ Chr $(W)=\omega_1$, but Chr $(B\times W)=\omega_0$.

The problem how small the chromatic number of the product can be still remains open. Here we are going to give a partial answer by proving the following result.

Theorem. Con (ZF) implies Con (ZFC+GCH+ there are two graphs B and W on ω_2 such that Chr $(B)=$Chr $(W)=\omega_2$, but Chr $(B\times W)=\omega_0$).

2. A simple case. In order to make a bit easier to follow our construction, we present a proof for a weakened version of the main result, namely, we drop the assumption CH.

Theorem. Con (ZF) implies Con (ZFC + there are two graphs B and W on ω_2, such that Chr $(B)=$Chr $(W)=\omega_2=2^{\omega_0}$, but Chr $(B\times W)=\omega_0$).

Proof. Define the notion of forcing $Q=\langle Q,\leq\rangle$ as follows. Its underlying set Q consists of quadruples $\langle a,B,W,f\rangle$ where

(i) $a\in[\omega_2]^{<\omega_0}$, $B,W\subseteq[a]^2$, and f is a function, $f:a\times a\to\omega_0$,

(ii) $B\cap W=\emptyset$,

(iii) for each $\alpha,\beta\in a$ we have

$$f(\alpha,\beta)=\begin{cases}0 & \text{if } \alpha=\beta\\ >0 \text{ and even} & \text{if } \alpha<\beta\\ \text{odd} & \text{if } \alpha>\beta\end{cases}$$

(iv) if $\{\alpha,\beta\}\in B\cup W$ and $\gamma\in a$, $\alpha<\gamma$ then $f(\alpha,\gamma)\neq f(\beta,\gamma)$ and $f(\gamma,\alpha)\neq f(\gamma,\beta)$,

(v) for each $\{\alpha,\beta\}\in B$ and $\{\gamma,\delta\}\in W$, $f(\alpha,\gamma)\neq f(\beta,\delta)$.

The ordering on Q is as expected: if $p=\langle a^p,B^p,W^p,f^p\rangle\in Q$ and $g=\langle a^q,B^q,W^q,f^q\rangle\in Q$ then $p\leq q$ iff

$$a^q\subseteq a^p$$
$$B^q=B^p\cap[a^q]^2$$
$$W^q=W^p\cap[a^q]^2$$
$$f^q\subseteq f^p.$$

The elements of Q are the approximations of the edges of B and W, and the colouring of the product. It is easy to see that Q satisfies c.c.c. Now let $\mathcal{G}$

be V-generic over Q and put

$$\mathcal{B}=\bigcup\{B^p : p\in\mathcal{G}\}$$
$$\mathcal{W}=\bigcup\{W^p : p\in\mathcal{G}\}$$
$$F=\bigcup\{f^p : p\in\mathcal{G}\}$$

$\mathcal{B}$ and $\mathcal{W}$ are the sets of edges of graphs on ω_2 and their product has chromatic number at most ω_0 since F is a "good colouring" of $\mathcal{B}\times\mathcal{W}$ by ω_0 colours. On the other hand, for each $n\in\omega_0$ the complete graph on n vertices can be embedded into $\mathcal{B}\times\mathcal{W}$, thus Chr $(\mathcal{B}\times\mathcal{W})\geq\omega_0$.

Finally, Chr $(\mathcal{B})$=Chr $(\mathcal{W})=\omega_2$ follows from the following fact. For each $A\in[\omega_2]^{\omega_2}$ and $\{p_\alpha:\alpha\in A\}\subset Q$ in V there are two different elements $\alpha,\beta\in A$ and $q\leq p_\alpha, p_\beta$ with $\{\alpha,\beta\}\in B^q$.

Obviously, this construction can be carried out for every regular cardinal in place of ω_2.

3. The proof of the main result. We use a generalization of a method of J.E. Baumgartner [1]. First of all we sketch the idea. The elements of the poset $\mathcal{P}$ we are going to force with are quadruples $\langle A,B,W,F\rangle$, where A is a countable subset of ω_2, B and W are edge-disjoint graphs on A approximating $\mathcal{B}$ and $\mathcal{W}$, and F is a set of functions, $F=\{F_x : x\in\omega^{<\omega}\}$. The union of $F_\emptyset$'s in the generic set will be a good colouring of $\mathcal{B}\times\mathcal{W}$.

The poset $\mathcal{P}$ will be ω-complete, therefore we need to show $\aleph_2$-c.c. As usual, we have to "amalgamate" p and $q\in\mathcal{P}$ whenever they satisfy certain assumptions. Assume π is a full isomorphism between $p=\langle A^p,B^p,W^p,F^p\rangle$ and $q=\langle A^q,B^q,W^q,F^q\rangle$. If $\alpha\in A^p\setminus A^q$, $\beta\in A^q\setminus A^p$, then we must define the "colour" of $\{\alpha,\beta\}$ in the amalgamated condition. Our idea is that $F^q_{\langle o\rangle}(\pi(\alpha),\beta)$ consist of the potential colours of $\{\alpha,\beta\}$. However, we need to define $F_{\langle o\rangle}(\alpha,\beta)$, too. Its candidates are the members of $F^q_{\langle o,o\rangle}(\pi(\alpha),\beta)$. In general, the elements of $F^q_{\langle k\rangle^\frown x}(\pi(\alpha),\beta))$ are the candidates to be elements of $F_x(\alpha,\beta)$.

Now we start the detailed construction with some notions. Let $\mathcal{U}=\{\alpha<\omega_2:\mathrm{cf}(\alpha)=\omega\}$, $\{f_n:n\in\omega\}$ be a set of functions from $\mathcal{U}$ into ω_2, such that for each $\alpha\in\mathcal{U}$, $\langle f_n(\alpha):n\in\omega\rangle$ is increasing and unbounded in α.

Let $\{S,G_n,H_n:n\in\omega\}$ be the following enumeration of ω: $S=0$, $G_n=2n+2$, $H_n==2n+1$.

If $\alpha,\gamma\in\mathcal{U}$, $\alpha<\gamma$, let $t(\alpha,\gamma)=\min\{n:\alpha<f_n(\gamma)\}$. If $n,k\in\omega$, let $T_{n,k}=\{S,G_m,H_\ell : m\geq n, \ell\geq k\}$ and $V_{n,k}={T_{n,k}}^{<\omega}$.

Definition 3.1. Let $\mathcal{P}_0 = \langle P_0, \leq \rangle$ be the partial ordered set whose underlying set P_0 consists of quadruples $\langle A,B,W,F \rangle$, where

(1) $A \in [\omega_2]^{\leq \omega_0}$, $B,W \subseteq [A]^2$, $F = \{F_x : x \in \omega^{<\omega}\}$;

(2) $B \cap W = \emptyset$;

(3) $F_\emptyset$ is a function from $[A]^2$ into $[\omega]^\omega$;

(4) If $x \in \omega^{<\omega}$, $x \neq \emptyset$ then F_x is a function from $A \times A$ into $[\omega]^\omega$.

The ordering on P_0 is as expected: if $p = \langle A^p, B^p, W^p, P^p \rangle \in P_0$, $q = \langle A^q, B^q, W^q, P^q \rangle \in P_0$, then $p \leq q$ iff

$A^q \subseteq A^p$

$B^q = B^p \cap [A^q]^2$

$W^q = W^p \cap [A^q]^2$

$F^q_x \subseteq F^p_x$ for each $x \in \omega^{<\omega}$.

Definition 3.2. Let $\mathcal{P}_1$ be the subset of $\mathcal{P}_0$ consisting of quadruples $p = \langle A,B,W,F \rangle \in \mathcal{P}_0$ satisfying conditions 1 - 5 below.

Condition 1. If $\{\alpha,\gamma\} \in B$, $\{\beta,\delta\} \in W$, $\alpha < \beta,\gamma,\delta$, $n = t(\alpha,\gamma)$, $k = t(\alpha,\delta)$, $x \in V_{n,k}$ then $F_\emptyset(\alpha,\beta) \cap F_x(\gamma,\delta) = \emptyset$.

Condition 2. If $\{\alpha,\gamma\} \in B$, $\beta \in A$, $\alpha < \beta,\gamma$, $x,y \in V_{0,0}$, $n = t(\alpha,\gamma)$, $k = t(\alpha,\beta)$ and for arbitrary $Z \in V_{n,k}$ and $i < n$ $\langle G_i \rangle^\frown x \neq Z^\frown y$ and $x \neq \langle G_i \rangle^\frown Z^\frown y$, then $F_x(\alpha,\beta) \cap F_y(\gamma,\beta) = \emptyset$.

Condition 3. If $\{\beta,\delta\} \in W$, $\gamma \in A$, $\beta < \gamma,\delta$, $x,y \in V_{0,0}$, $n = t(\beta,\gamma)$, $k = t(\beta,\delta)$ and for arbitrary $Z \in V_{n,k}$ and $i < k$ $\langle H_i \rangle^\frown x \neq Z^\frown y$ and $x \neq \langle H_i \rangle^\frown Z^\frown y$, then $F_x(\gamma,\beta) \cap F_y(\gamma,\delta) = \emptyset$.

Condition 4. If $\langle \beta,\delta \rangle \in W$, $\gamma \in A$, $\gamma < \beta,\delta$, $x,y \in V_{0,0}$ and for arbitrary $i \in \omega$ $\langle H_i \rangle^\frown x \neq y$ and $x \neq \langle H_i \rangle^\frown y$, then $F_x(\gamma,\beta) \cap F_y(\gamma,\delta) = \emptyset$.

Condition 5. If $\alpha,\beta \in A$, $x,y \in V_{0,0}$, $x \neq y$ then $F_x(\alpha,\beta) \cap F_y(\alpha,\beta) = \emptyset$.

If α,β,γ, x,y,n,k are such as in 2 above, we denote this fact by $b(\alpha,\beta,\gamma,x,y,n,k)$ and if they are such as in 3, we abbreviate this by writing $w(\beta,\gamma,\delta,x,y,n,k)$.

The notions <u>strongly closed</u>, <u>closed</u>, the lemma 1 and the method of Lemma 7 are due to J. Baumgartner [1].

If $\omega_1 \leq \alpha < \omega_2$, let $h_\alpha : \alpha \xrightarrow[\text{onto}]{1-1} \omega_1$. We say that $A \in [\omega_2]^{\leq\omega}$ is <u>strongly closed</u> iff $A \cap \omega_1 \in \omega_1$ and for each $\alpha \in A$ A is closed under h_α and h_α^{-1} and for each $\alpha \in A$ and $p \in \omega$ $f_p(\alpha) \in A$. For arbitrary $A \in [\omega_2]^{\leq\omega}$, scl(A) is the smallest strongly closed set containing A. If $p \in \mathcal{P}_1$, p is <u>closed</u> iff $A^p = \mathrm{scl}\, A^p \cap \mathcal{U}$. For $\mathcal{P}_1$ is σ-complete, the closed conditions form a dense subset of $\mathcal{P}_1$.

Lemma 1. If a, b are strongly closed and $a \cap \omega_1 = b \cap \omega_1$, then $a \cap b$ is an initial segment of both a and b.

Proof. Let $\xi = a \cap \omega_1 = b \cap \omega_1$, $v \in a \cap b$, $\eta \in a$, $\eta < v$. Then $h_\eta(v) \in a \cap \cap \omega_1 = b \cap \omega_1$. Thus $v = h_\eta^{-1}(h_\eta(v)) \in b$.

Definition 3. Let $p, q \in P_1$, p, q closed, $p = \langle A^p, B^p, W^p, F^p \rangle$, $q = \langle A^q, B^q, W^q, F^q \rangle$. We say that p <u>and</u> q <u>are isomorphic and</u> π <u>shows it</u>, in signs

$$p \cong_\pi q$$

iff the following conditions hold:

(a) $\pi : \mathrm{scl}\ A^p \xrightarrow[\text{onto}]{1-1} \mathrm{scl}\ A^q$, π is order preserving,

(b) $(\mathrm{scl}\ A^p) \cap \omega_1 = (\mathrm{scl}\ A^q) \cap \omega_1$,

(c) $\{\alpha, \beta\} \in B^p$ iff $\{\pi(\alpha), \pi(\beta)\} \in B^q$,

(d) $\{\alpha, \beta\} \in W^p$ iff $\{\pi(\alpha), \pi(\beta)\} \in W^q$,

(e) $F^p_x(\alpha, \beta) = F^q_x(\pi(\alpha), \pi(\beta))$,

(f) $t(\alpha, \beta) = t(\pi(\alpha), \pi(\beta))$,

(g) $\pi(f_k(\alpha)) = f_k(\pi(\alpha))$.

By Lemma 1, $D = A^p \cap A^q$ is an initial segment of both A^p and A^q.

At present we are ready to define the poset $\mathcal{P} = \langle P, \leq \rangle$, which adds the desired graphs to the ground model.

Definition 4. $\mathcal{P}$ consists of quintuples $p = \langle A, B, W, F^0, F^1 \rangle$, where both $p^0 = \langle A, B, W, F^0 \rangle$ and $p^1 = \langle A, W, B, F^1 \rangle$ are elements of $\mathcal{P}_1$. If $p, q \in P$, then let

$$p \leq q \text{ iff both } p^0 \leq q^0 \text{ and } p^1 \leq q^1.$$

If $p \in P$, let $p = \langle A^p, B^p, W^p, F^{0p}, F^{1p} \rangle$.

The notions of isomorphism, closedness are extended into elements of in a straightforward way.

So far we have defined a notion of forcing $\mathcal{P}$. To show that it works as expected, we need 3 technical lemmas, rather simple as stated but cumbersome to prove them. Using them we construct a generic model.

The lemmas below use some new notions. To begin with, if $\alpha, \beta \in A^p \cup A^q$, $D = A^p \cap A^q$, then let us denote by $E(\alpha, \beta) = E^{p,q}(\alpha, \beta)$ the set $\{S, G_n, H_\ell : D \subseteq f_n(\alpha),\ D \subseteq f_\ell(\beta)\}$. $E(\alpha, \beta)$ may be $\{S\}$. If $\alpha \in A^p \cup A^q$, then put

$$\tilde{\alpha} = \begin{cases} \pi(\alpha) & \text{if } \alpha \in A^p \\ \alpha & \text{otherwise.} \end{cases}$$

Definition 5. Assume $p \cong_{\mathcal{T}} q$. Let $t \in P_0$, $t = \langle A,B,W.F\rangle$. We say that t is (p,g)-good, iff the conditions (A) - (E) below are satisfied.

(A) $A = A^p \cup A^q$.

(B) $B = B^p \cup B^q$.

(C) $W = W^p \cup W^q$.

(D) For each $x \in V_{0,0}$ $F_x = F_x^p \cup F_x^q \cup F_x'$, where $\mathrm{dom}\,(F_x') = \mathrm{dom}\,(F_x) \setminus \setminus (\mathrm{dom}\,F_x^p \cup \mathrm{dom}\,F_x^q)$ and for each $\langle \alpha,\beta\rangle \in \mathrm{dom}\,(F_x')$ $F_x'(\alpha,\beta) \subseteq \bigcup\{F^q_{\langle t\rangle^\frown x}(\tilde{\alpha},\tilde{\beta}) : t \in E^{p,q}(\alpha,\beta)\}$.

(E) For each $x,y \in V_{0,0}$ if $\langle\alpha,\beta\rangle \in \mathrm{dom}\,(F_x')$ and $\langle\gamma,\delta\rangle \in \mathrm{dom}\,(F_y')$ then $F_x'(\alpha,\beta) \cap F_y'(\gamma,\delta) \neq \emptyset$ implies $x=y$ and $\langle\alpha,\beta\rangle = \langle\gamma,\delta\rangle$.

Obviously, if $p \cong_{\mathcal{T}} q$, then there are (p,q)-good elements of $\mathcal{P}_0$. The first lemma we have promised, is the following one.

Lemma 2. If t is (p,q)-good, then $t \in \mathcal{P}_1$.

Proof. The general form of a condition is the following

$(\forall x,y \in V_{0,0})(\forall \langle\alpha,\beta\rangle \in \mathrm{dom}\,F_x$ and $\langle\gamma,\delta\rangle \in \mathrm{dom}\,F_y)$ if ... then $F_x(\alpha,\beta) \cap F_y(\gamma,\delta) = \emptyset)$.

We say that $F_x(\alpha,\beta)$ is new, if $\langle\alpha,\beta\rangle \in \mathrm{dom}\,F_x'$. It is clear from the isomorphism of p and q and the condition (E) of the (p,q)-goodness that if one of the conditions 1 - 5 fails in t, then we can assume that either $F_x(\alpha,\beta)$ or $F_y(\gamma,\delta)$ is new, but not both.

Let us vetify conditions 1 - 5 one by one. Let $D = A^p \cap A^q$.

Condition 1. Let $\{\alpha,\gamma\} \in B$, $\{\beta,\delta\} \in W$, $\alpha < \beta,\gamma,\delta$ $n = t(\alpha,\gamma)$, $k = t(\alpha,\delta)$, $x \in V_{n,k}$. As we remarked, exactly one of $F_x(\gamma,\delta)$ and $F_\emptyset(\alpha,\beta)$ must be new. If $F_\emptyset(\alpha,\beta)$ is new, then $\alpha,\beta \notin D$. Without loss of generality we may assume $\alpha \in A^p \setminus D$, $\beta \in A^q \setminus D$. Because $\alpha < \gamma,\delta$, hence $\gamma,\delta \notin D$. But $\{\alpha,\gamma\} \in B = B^p \cup B^q$ and $\{\beta,\delta\} \in W = W^p \cup W^q$, thus $\gamma \in A^p \setminus D$ and $\delta \in A^q \setminus D$. Hence $F_x(\gamma,\delta)$ is also new, a contradiction. Thus $F_x(\gamma,\delta)$ is new and, for example, $\gamma \in A^p \setminus D$, $\delta \in A^q \setminus D$. Since $\{\beta,\delta\} \in W = W^p \cup W^q$, $\beta \in A^q$. As $F_\emptyset(\alpha,\beta)$ is old, $\alpha \in A^q$. As $\gamma \in A^p \setminus D, \{\alpha,\gamma\} \in B$, α must be in A^p. Thus $\alpha \in D$. Thus $E^{p,q}(\gamma,\delta) \subseteq T_{n,k}$.

By the definition of (p,q)-goodness we have

$$F_x'(\gamma,\delta) \subseteq \bigcup\{F^q_{\langle t\rangle^\frown x}(\tilde{\gamma},\tilde{\delta}) : t \in E^{p,q}(\gamma,\delta)\} \subseteq \bigcup\{F^q_{\langle t\rangle^\frown x}(\gamma,\delta) : t \in T_{n,k}\} \subseteq$$
$$\subseteq \bigcup\{F^q_y(\tilde{\gamma},\tilde{\delta}) : y \in V_{n,k}\}.$$

Since $t(\alpha,\gamma)=t(\alpha,\tilde{\gamma})$ and $t(\alpha,\sigma)=t(\alpha,\tilde{\sigma})$ hence applying condition 1 for q we get that every member on the right side is disjoint from $F^q_\emptyset(\alpha,\tilde{\beta})$. For $F_\emptyset(\alpha,\beta)=F^q_\emptyset(\alpha,\tilde{\beta})$ and $F_x(\gamma,\sigma)=F'_x(\gamma,\sigma)$ hence $F_x(\gamma,\sigma)\cap F_\emptyset(\alpha,\beta)=\emptyset$.

Condition 2. Let α,γ,β,n,k,x,y be such as expected. As above, it can be seen that $F_y(\gamma,\beta)$ must be new and α must lie in D. Hence $E^{p,q}(\gamma,\beta)\subseteq$ $\subseteq T_{n,k}$.
Now

$F'_y(\gamma,\beta)\subseteq\bigcup\{F^q_{\langle t\rangle^\frown y}(\tilde{\gamma},\tilde{\beta}):t\in E^{p,q}(\gamma,\beta)\}$. We must check that each $F^q_{\langle t\rangle^\frown y}(\tilde{\gamma},\tilde{\beta})$ appearing in the right side is disjoint from $F_x(\alpha,\beta)=F^q_x(\tilde{\alpha},\tilde{\beta})=$ $=F^q_x(\alpha,\tilde{\beta})$. We want to apply condition 2 for q. But $b(\alpha,\beta,\gamma,x,y,n,k)$ holds and $t\in E^{p,q}(\gamma,\beta)\subseteq T_{n.k}$. Hence $b(\alpha,\tilde{\beta},\tilde{\gamma},x,$ t y,n,k) holds, too, therefore by condition 2,

$$F^q_x(\alpha,\tilde{\beta})\cap F^q_{\langle t\rangle^\frown y}(\gamma,\tilde{\beta})=\emptyset. \text{ thus}$$

$$F_x(\alpha,\beta)\cap F_y(\gamma,\beta)=\emptyset.$$

Condition 3. Let β,γ,σ,x,y,n,k be such as expected i.e. $w(\beta,\gamma,\sigma,x,y,n,k)$. Now $F_y(\gamma,\sigma)$ must be new and $\beta\in D$. Hence

$$F'_y(\gamma,\sigma)\subseteq\bigcup\{F^q_{\langle t\rangle^\frown y}(\tilde{\gamma},\tilde{\sigma}):t\in E^{p,q}(\gamma,\sigma)\}.$$

Let s be an arbitrary member of $E^{p,q}(\gamma,\sigma)$. Since $\beta\in D$, we have $s\in T_{n,k}$. From $w(\beta,\gamma,\sigma,x,y,n,h)$ we get $w(\beta,\gamma,\sigma,x,\langle s\rangle^\frown y,n,k)$. Applying condition 3 for q,

$$F^q_x(\tilde{\gamma},\beta)\cap F^q_{\langle s\rangle^\frown y}(\tilde{\gamma},\tilde{\sigma})=\emptyset.$$

Therefore

$$F_x(\gamma,\beta)\cap F_y(\gamma,\sigma)=\emptyset.$$

Condition 4. In this case it is impossible that exactly one of $F_x(\gamma,\beta)$ and $F_y(\gamma,\sigma)$ is new.

Condition 5. Obviously, $F_x(\alpha,\beta)$ and $F_y(\alpha,\beta)$ are new at the same time. The lemma 2 is proved.

Lemma 3. Let $p,q\in\mathcal{P}_1$, $p\cong_\pi q$, $\nu<\mu<\omega_2$, $\ell\in\omega$, $D=A^p\cap A^q$, $\nu\in A^p\setminus D$, $\mu\in A^q\setminus D$, $\pi(\nu)=\mu$, $D\subset f_\ell(\nu)$, $D\subset f_\ell(\mu)<\nu$. Let $t=\langle A,B,W,F\rangle$ be a (p,q)-good element of $\mathcal{P}_0$ such that

$\langle\alpha,\beta\rangle\in\operatorname{dom}F'_x$ implies
if $\alpha\notin\{\nu,\mu\}$ then $F_x(\alpha,\beta)\subset F^q_{\langle s\rangle^\frown x}(\tilde{\alpha},\tilde{\beta})$

if $\alpha \in \{\nu, \mu\}$ then $F_x(\alpha, \beta) \subset F^q_{\langle G_\ell \rangle ^\frown x}(\tilde{\alpha}, \tilde{\beta})$.

Then $r = \langle A, B \cup \{\nu, \mu\}, W, F \rangle \in \mathcal{P}_1$.

Proof. Assume on the contrary that $r \notin \mathcal{P}_1$. We know $t \in \mathcal{P}_1$, and the difference between r and t is only one edge, $\{\nu, \mu\}$. Therefore we must check only cases when edge $\{\nu, \mu\}$ acts in conditions 1 - 5.

Condition 1. Let $\alpha, \beta, \gamma, \delta$, x,n,k as expected. In this case α must be ν, and γ must be μ. Since $f_\ell(\mu) < \nu < f_n(\mu)$, $\ell < n$.

(i) $F_\emptyset(\alpha, \beta)$ is new. Since $F'_\emptyset(\alpha, \beta) \subset F^q_{\langle G_\ell \rangle}(\tilde{\alpha}, \tilde{\beta}) = F^q_{\langle G_\ell \rangle}(\tilde{\gamma}, \tilde{\beta})$, therefore it is enough to prove

$$F^q_{\langle G_\ell \rangle}(\tilde{\gamma}, \tilde{\beta}) \cap F^q_x(\tilde{\gamma}, \tilde{\delta}) = \emptyset .$$

If $\tilde{\gamma} < \tilde{\beta}, \tilde{\delta}$ then because $\ell < n$ and $x \in V_{n,k}$ we can apply condition 4 for q to obtain it. If $\tilde{\beta} < \tilde{\gamma}, \tilde{\delta}$, then because $\ell < n$, $x \in V_{n,k}$ $t(\alpha, \gamma) = n \leq t(\tilde{\beta}, \tilde{\gamma})$, we can use the condition 3 for q and obtain the desired result.

If $\tilde{\delta} < \tilde{\gamma}, \tilde{\beta}$, then because $\ell < n \leq t(\tilde{\delta}, \tilde{\gamma})$ we can apply Condition 3.

(ii) $F_x(\gamma, \delta)$ is new. Since $F'_x(\gamma, \delta) \subset F^q_{\langle G_\ell \rangle ^\frown x}(\tilde{\gamma}, \tilde{\delta})$ hence it is enough to prove

$$F^q_\emptyset(\tilde{\gamma}, \tilde{\beta}) \cap F^q_{\langle G_\ell \rangle ^\frown x}(\tilde{\gamma}, \tilde{\delta}) = \emptyset$$

because $F_\emptyset(\alpha, \beta) = F^q_\emptyset(\tilde{\alpha}, \tilde{\beta}) = F^q_\emptyset(\tilde{\gamma}, \tilde{\beta})$. But $\alpha < \beta, \delta$, hence $\tilde{\gamma} = \tilde{\alpha} < \tilde{\beta}, \tilde{\delta}$. For $G_\ell \neq H_i$, we can apply Condition 3 in q to obtain the desired result.

Condition 2. Let $\{\alpha, \gamma\} \in B$, $\beta \in A$, x,y,n,k as expected. Then $\alpha = \nu$ and $\gamma = \mu$. Since $f_\ell(\mu) < \nu < f_n(\mu)$, $\ell < n$.

(i) $F_x(\alpha, \beta)$ is new. Since $F'_x(\alpha, \beta) \subset F^q_{\langle G_\ell \rangle ^\frown x}(\tilde{\alpha}, \tilde{\beta}) = F^q_{\langle G_\ell \rangle ^\frown x}(\tilde{\gamma}, \tilde{\beta})$, we need

$$F^q_{\langle G_\ell \rangle ^\frown x}(\tilde{\gamma}, \tilde{\beta}) \cap F^q_y(\tilde{\gamma}, \tilde{\beta}) = \emptyset .$$

For $b(\alpha, \beta, \gamma, x, y, n, k)$ and $\ell < n$, $\langle G_\ell \rangle ^\frown x \neq y$, thus what we have hoped, is really true.

(ii) $F_y(\gamma, \beta)$ is new. Since $F'_y(\gamma, \beta) \subset F^q_{\langle G_\ell \rangle ^\frown y}(\tilde{\gamma}, \tilde{\beta}) = F^q_{\langle G_\ell \rangle ^\frown y}(\tilde{\alpha}, \tilde{\beta})$, we need $\langle G_\ell \rangle ^\frown y \neq x$. For $b(\alpha, \beta, \gamma, x, y, n, k)$ and $\ell < n$, it is clear.

In the remaining cases, the edge $\{\nu, \mu\}$ cannot act, thus the lemma 3 is proved.

Lemma 4. Let $p, q \in \mathcal{P}_1$, $p \cong_\pi q$, $\nu < \mu < \omega_2$, $\ell \in \omega$, $D = A^p \cap A^q$, $\nu \in A^p \setminus D$, $\mu \in A^q \setminus D$, $\pi(\nu) = \mu$. $D \subset f_\ell(\nu)$, $D \subset f_\ell(\mu) < \nu$. Let $t = \langle A, B, W, F \rangle$ be a (p,q)-good element of $\mathcal{P}_0$ such that $\langle\alpha,\beta\rangle\in \operatorname{dom} F'_x$ implies

if $\beta\notin\{\nu,\mu\}$ then $F_x(\alpha,\beta)\subset F^q_{\langle S\rangle^\frown x}(\tilde\alpha,\tilde\beta)$

if $\beta\notin\{\nu,\mu\}$ then $F_x(\alpha,\beta)\subset F^q_{\langle H_\ell\rangle^\frown x}(\tilde\alpha,\tilde\beta)$.

Then $r=\langle A,B,W\cup\{\nu,\mu\},F\rangle\in\mathcal{P}_1$.

Proof. Assume on the contrary that $r\notin\mathcal{P}_1$. Keeping in mind that the difference between r and t is only one edge, $\{\nu,\mu\}$, we must check only cases when the edge $\{\nu,\mu\}$ acts in conditions 1 - 5. In the condition 2 and 5 the edge $\{\nu,\mu\}$ cannot act.

Condition 1. Let $\alpha,\beta,\gamma,\delta$,x,n,k as expected. Now $\{\beta,\delta\}$ must be $\{\nu,\mu\}$

(i) $F_\emptyset(\alpha,\beta)$ is new. Since $F_\emptyset(\alpha,\beta)\subseteq F^q_{\langle H_\ell\rangle}(\tilde\alpha,\tilde\beta)=F^q_{\langle H_\ell\rangle}(\tilde\alpha,\tilde\beta)$ we must prove $F^q_{\langle H_\ell\rangle}(\tilde\alpha,\tilde\beta)\cap F^q_x(\tilde\gamma,\tilde\delta)=\emptyset$. For $n=t(\tilde\alpha,\tilde\gamma)$, $k=t(\tilde\gamma,\tilde\delta)$, $x\in V_{n,k}$ we get $b(\tilde\alpha,\tilde\delta,\tilde\gamma;\langle H_\ell\rangle,x,n,k)$. Thus we can apply condition 2 in q to obtain what we had to prove.

(ii) $F_x(\gamma,\delta)$ is new. Since $F_x(\gamma,\delta)\subset F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\delta)=F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\beta)$, we must prove $F^q_\emptyset(\tilde\alpha,\tilde\beta)\cap F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\beta)=\emptyset$. For $\alpha<\tilde\gamma,\tilde\beta$ and $t(\tilde\alpha,\tilde\gamma)=t(\alpha,\gamma)=$ $=n$ we can see $b(\tilde\alpha,\tilde\beta,\tilde\gamma,\emptyset,\langle H_\ell\rangle^\frown x,n,t(\tilde\alpha,\tilde\beta))$. Indeed, for arbitrary $j\in\omega$ and $z\in V_{0,0}$ $\langle G_j\rangle\neq z^\frown\langle H_\ell\rangle^\frown x$ and $\emptyset\neq\langle G_j\rangle^\frown z^\frown\langle H_\ell\rangle^\frown x$. Thus $F^q_\emptyset(\tilde\alpha,\tilde\beta)\cap F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\beta)=$ $=\emptyset$ by condition 2.

Condition 3. Let β,γ,δ,x,y,n,k as expected. Now $\beta=\nu$ and $\delta=\mu$

(i) $F_x(\gamma,\beta)$ is new. Thus $F'_x(\gamma,\beta)\subset F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\beta)=F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\delta)$. Since $f_\ell(\mu)<\nu<f_k(\mu)$, $\ell<k$. Thus $\langle H_\ell\rangle^\frown x\neq y$ by $w(\beta,\gamma,\delta,x,y,n,k)$, therefore

$F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\delta)\cap F^q_y(\tilde\gamma,\tilde\delta)=\emptyset$

(ii) $F_y(\gamma,\delta)$ is new. Since $F'_y(\gamma,\delta)\subset F^q_{\langle H_\ell\rangle^\frown y}(\tilde\gamma,\tilde\delta)=F^q_{\langle H_\ell\rangle^\frown y}(\tilde\gamma,\tilde\beta)$ and $x\neq\langle H_\ell\rangle^\frown y$ because $\ell<k$,

$F^q_{\langle H_\ell\rangle^\frown y}(\tilde\gamma,\tilde\beta)\cap F^q_x(\tilde\gamma,\tilde\beta)=\emptyset$.

Condition 4. Let β,γ,δ,x,y, as expected. Now $\{\beta,\delta\}=\{\nu,\mu\}$.

(i) $F_x(\gamma,\beta)$ is new. Since $F'_x(\gamma,\beta)\subset F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\beta)=F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\delta)$ and $\langle H_\ell\rangle^\frown x\neq y$, $F^q_{\langle H_\ell\rangle^\frown x}(\tilde\gamma,\tilde\delta)\cap F^q_y(\tilde\gamma,\tilde\delta)=\emptyset$

(ii) $F_x(\gamma,\delta)$ is new. Since $F'_y(\gamma,\delta)\subset F^q_{\langle H_\ell\rangle^\frown y}(\tilde\gamma,\tilde\delta)=F^q_{\langle H_\ell\rangle^\frown y}(\tilde\gamma,\tilde\beta)$ and $x\nsubseteq\langle H_\ell\rangle^\frown y$, $F^q_x(\tilde\gamma,\tilde\beta)\cap F^q_{\langle H_\ell\rangle^\frown y}(\tilde\gamma,\tilde\beta)=\emptyset$.

This completes the proof of Lemma 4.

We are going to use the following notions. If G is V-generic over $\mathcal{P}$, let

$$\mathcal{A}=\bigcup\{A^p:p\in G\}$$

$$\mathcal{B}=\bigcup\{B^p:p\in G\}$$

$\mathcal{W}=\bigcup\{W^p:p\in G\}$, and if $x\in\omega^{<\omega}$, i=0,1,

$$F^i_x=\bigcup\{F^{ip}_x:p\in G\}.$$

If i=0,1, let f^i be a choice function for $F^i_\emptyset$, that is,

$$f^i:[\mathcal{A}]^2\longrightarrow\omega$$
$$f^i(\alpha,\beta)\in F^i_\emptyset(\alpha,\beta).$$

Let us define the function f as follows:

$$\mathrm{Dom}\ (f)=\mathcal{A}\times\mathcal{A}$$

$$f(\alpha,\beta)=\begin{cases}0 & \text{if } \alpha=\beta\\ 2\cdot f^0(\alpha,\beta)+1 & \text{if } \alpha<\beta\\ 2\cdot f^1(\beta,\alpha)+2 & \text{if } \alpha>\beta\end{cases}$$

We claim that in $V^{\mathcal{P}}$, $\mathcal{B}$ and $\mathcal{W}$ are ω_2-chromatic graphs on $\mathcal{A}=\mathcal{U}$, and f is a good colouring of $\mathcal{B}\times\mathcal{W}$. To see it we need some observation.

Lemma 5. For arbitrary $\alpha\in\mathcal{U}$, $D_\alpha=\{p\in\mathcal{P}_1\ \alpha\in A^p\}$ is dense in $\mathcal{P}_1$.

Proof. Let $p=\langle A,B,W,F\rangle\in\mathcal{P}_1$. We may assume $\alpha\notin A^p$. Let $r=\langle A\cup\{\alpha\}$, $B,W,G\rangle\in\mathcal{P}_0$ such that $r\le p$. If $\{G_x(\alpha,\nu),\ G_x(\nu,\alpha):x\in\omega^{<\omega},\ \nu\in A\cup\{\alpha\}\}$ consists of pairwise disjoint subsets of ω, then it is easy to see that $r\in P_1$.

Lemma 6. If CH holds, $\mathcal{P}$ satisfies ω_2-c.c.

Proof. Let $\{p_\alpha:\alpha<\omega_2\}\subset\mathcal{P}$. Since the closed elements of $\mathcal{P}$ form a dense subset, we may assume every p_α is closed. Since $2^\omega=\omega_1$ there are only ω_1 isomorphic types of elements of $\mathcal{P}$. Thus there are $\mu<\beta<\omega_2$, $p_\alpha\cong p_\beta$. Then, by Lemma 2 p_α and p_β are compatible.

Lemma 7. If CH holds, then $V^{\mathcal{P}} \models \mathrm{Chr}(\mathcal{B})=\mathrm{Chr}(\mathcal{W})=\omega_2$.

Proof. Assume on the contrary that $p \in \mathcal{P}$ and $p \Vdash$ "$\underline{h}: \mathcal{U} \to \omega_1$ is a good colouring of $\mathcal{B}$". Let $\{p_\alpha : \alpha \in \mathcal{U}\}$, $g: \mathcal{U} \to \omega_1$ be such that

p_α's are closed, $\alpha \in A^{p_\alpha}$, $p_\alpha \leq p$ and

$p_\alpha \Vdash$ "$\underline{h}(\alpha)=\check{g}(\alpha)$".

Since there are only $\omega_1=2^\omega$ isomorphic types of the elements of $\mathcal{P}$, there is a stationary subset S of U and there are $\xi, \eta, \tau \in \omega_1$ such that:

(i) $(\forall \alpha, \beta \in S)$ p_α and p_β are isomorphic and $\pi_{\alpha,\beta}$ shows it,

(ii) $(\forall \alpha \in S)$ $g(\alpha)=\tau$,

(iii) $(\forall \alpha \in S)$ $A^{p_\alpha} \cap \omega_1 = \eta$,

(iv) $(\forall \alpha \in S)$ α is the ξ^{th} element of A^{p_α}.

Since S is stationary and for each $\alpha \in S$ $\langle f_n(\alpha): n \in \omega\rangle$ is unbounded in α, there is an $n \in \omega$ such that f_n is not essentially bounded on S, that is, for each $\beta < \omega_2$ $\{\alpha \in S: f_n(\alpha) > \beta\}$ is stationary in ω_2.

Thus there are $\alpha < \gamma < \omega_2$: both $f_n^{-1}(\alpha) \cap S$ and $f_n^{-1}(\gamma) \cap S$ are stationary. Let $\nu, \mu \in S$, $\nu < \mu$ such that $f_n(\nu)=\gamma$, $f_n(\mu)=\alpha$. By (iv) $\pi(\nu)=\mu$. By the definition of isomorphism

$$\pi(\gamma)=\pi(f_n(\nu))=f_n(\pi(\nu))=f_n(\mu)=\alpha.$$

Since $D=A^{p_\nu} \cap A^{p_\mu}$ is an initial segment of both A^{p_ν} and A^{p_μ}, $D \subset \alpha$ and $D \subset \gamma$. For $f_n(\mu)=\alpha<\gamma=f_n(\nu)<\nu$, $f_n(\mu)<\nu$.

Thus we can apply Lemma 3 for p_ν^0, p_μ^0, ν, μ and n, and Lemma 4 for p_ν^1, p_μ^1, ν, μ and n. Hence we obtain $p=\langle A,B,W,F^0,F^1\rangle$ such that $p \in \mathcal{P}$ and $p \leq p_\nu, p_\mu$ and $\{\nu,\mu\} \in B$. But

$p \Vdash \underline{h}(\nu)=\underline{h}(\mu)=\tau \wedge \{\nu,\mu\} \in B \wedge \underline{h}$ is a good colouring of $\mathcal{B}$.

Contradiction. Thus $\mathrm{Chr}(\mathcal{B})=\omega_2$. Similarly, $\mathrm{Chr}(\mathcal{W})=\omega_2$,

Proof of main result. Assume the CH and let us regard $V^{\mathcal{P}}$. By Lemma 6 $\mathcal{P}$ satisfies $\aleph_2$-c.c. Since $\mathcal{P}$ is σ-closed, CH remains true and the cardinalities of V and $V^{\mathcal{P}}$ are the same. By Lemma 7

$$V^{\mathcal{P}} \models \text{"}\mathrm{Chr}(\mathcal{B})=\mathrm{Chr}(\mathcal{W})=\omega_2\text{"}.$$

By Lemma 5, $\mathcal{U}=\mathcal{A}$.

Let $\alpha, \beta, \gamma \in \mathcal{U}$, $\{\alpha,\gamma\} \in \mathcal{B}$, $\{\beta,\delta\} \in \mathcal{W}$. Assume on the contrary

$f(\alpha,\beta)=f(\gamma,\delta)$. Since $\mathcal{B}'\cap\mathcal{W}=\emptyset$, $\alpha\neq\beta$ or $\gamma\neq\delta$. Since $f^{-1}\{0\} = \{(\nu,\nu):\nu\in\mathcal{U}\}$, $\alpha\neq\beta$ and $\gamma\neq\delta$. Since $f(\alpha,\beta)$ is odd iff $\alpha>\beta$, we can see $\alpha<\beta$ iff $\gamma<\delta$. Let $p=\langle A,B,W,F^0,F^1\rangle\in\mathcal{G}$ such that $\alpha,\beta,\gamma,\delta\in A$.

(i) $\alpha<\beta$. Thus $\gamma<\delta$. We may assume $\alpha<\gamma$. Since $\alpha<\beta,\gamma,\delta$, by condition 1 for $p^0=\langle A,B,W,F^0\rangle$, $F^0_\emptyset(\alpha,\beta)\cap F^0_\emptyset(\gamma,\delta)=\emptyset$. But $f(\alpha,\beta)=2\cdot f^0(\alpha,\beta)+2$, $f(\gamma,\delta)=2\cdot f^0(\gamma,\delta)+2$, $f^0(\alpha,\beta)\in F^0_\emptyset(\alpha,\beta)$, $f^0(\gamma,\delta)\in F^0_\emptyset(\gamma,\delta)$, thus $f(\alpha,\beta)\neq f(\gamma,\delta)$.

(ii) $\alpha>\beta$. Similarly, using p^1 instead of p^0. Therefore f really shows Chr $(\mathcal{B}\times\mathcal{W})=\omega_0$. On the other hand, for each $n\in\omega$ the complete graph on n vertices can be embedded into $\mathcal{B}\times\mathcal{W}$, thus Chr $(\mathcal{B}\times\mathcal{W})\geq\omega$.

This completes the proof of the main result.

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