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REMARKS ON PERIODIC SOLUTIONS, WITH PRESCRIBED ENERGY, FOR SINGULAR HAMILTONIAN SYSTEMS

Carlo GRECO

Abstract. In this paper we are searching for periodic solutions, with prescribed energy, of Hamiltonian systems $\dot{x}=H_y$, $\dot{y}=-H_x$ $(x,y\in\mathbb{R}^n)$, where $H(x,y)$ has the classical form: $H(x,y)=\frac{1}{2}|y|^2+V(x)$. We suppose that $V(x)\longrightarrow -\infty$ as $x\longrightarrow S$ $(S\subset\mathbb{R}^n)$, namely that the potential V is singular at $x\in S$.

Key words: Classical Hamiltonian systems, periodic solutions, singularities.

Classification: 34C25, 58F22

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§ 1. Introduction. Let S be a closed not empty subset of $\mathbb{R}^n$ $(n\geq 2)$, and let $V\in C^\infty(\mathbb{R}^n-S,\mathbb{R})$ be such that:

(1.1) $$V(x)\longrightarrow -\infty \text{ as } x\longrightarrow S;$$

(1.2) there exists a neighbourhood $\mathcal{N}$ of S, and a function $U\in C^1(\mathbb{R}^n-S,\mathbb{R})$, such that:

(i) $U(x)\longrightarrow -\infty$ as $x\longrightarrow S$

(ii) $-V(x)\geq |U'(x)|^2$ for every $x\in\mathcal{N}-S$

($|\cdot|$ is the norm in $\mathbb{R}^n$). The equation:

(1.3) $$\ddot{x}=-V'(x)$$

(where $\dot{x}=dx/dt$ and $V'(x)=$ the gradient of V at x) describes the notion of a dynamical system in a conservative force field (t is the time-variable, and V is the potential of this field). Because of (1.1), we say that the potential V is "singular" at $x\in S$; moreover, we observe that (1.2) is verified if, for instance, $V(x)=-1/|x|^\alpha$ with $\alpha\geq 2$, while it does not hold if $1\leq\alpha<2$. The main problems concerning (1.3), are to find periodic solutions

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of (1.3) with a prescribed period, or with a prescribed energy (if x(t) is a solution of (1.3), its energy is, of course: $h=\frac{1}{2}|\dot{x}(t)|^2+V(x(t))$).

The existence of periodic solutions of (1.3), with a prescribed period, was first investigated by Gordon [6] under the hypothesis (1.2). More recently, the same problem has been studied in [1],[2],[3],[8]. In [6], there are also some results of the existence of (non-periodic) solutions of (1.3), with prescribed energy which join two given points of $\mathbf{R}^n-S$ (see also [7]). In this paper we are searching for periodic solutions, with a given energy, of (1.3). To this end, we shall follow the method developed by Seifert in [12] and, more recently, used in [5],[9] in the case of a nonsingular potential. Then, we search for closed geodesics of the so-called "Jacobi metric" associated with the potential V.

Fix $h\in\mathbf{R}$, and set:

$N=\{x\in\mathbf{R}^n-S\,|\,V(x)\leq h\}$, $B=\{x\in\mathbf{R}^n-S\,|\,V(x)=h\}$;

let us consider $M=N\cup S$ and $Y=\{\omega\in C([0,1],M)\,|\,\omega(0),\omega(1)\in B\}$. In [9] it is proved that $H_0(Y,B,Z)\neq 0$ or $\pi_k(Y,B)\neq 0$ for some $k\geq 1$; in other words, there is an arcwise connected component α of Y different from B ($\alpha\in H_0(Y,B,Z)-\{0\}$), or there is a not trivial class β ($\beta\in\pi_k(Y,B)-\{0\}$) of continuous maps $f:D^k\rightarrow Y$ with $f(S^{k-1})\subset B$, where D^k is the disc in $\mathbf{R}^k$, and $S^{k-1}=\partial D^k$. Set $Y^*=\{\omega\in Y\,|\,\omega$ "does not cross" $S\}$; the first result of this paper concerns the case in which one of the following conditions is satisfied:

(1.4) $$H_0(Y^*,B,Z)\neq 0;$$

(1.5) $$\pi_k(Y^*,B)\neq 0 \text{ for some } k\in\mathbf{N}.$$

More precisely, the following theorem holds:

Theorem 1.1. Suppose that (1.1) and (1.2) hold, that M is compact, and that $V'(x)\neq 0$ for every $x\in B$. Then, if (1.4) or (1.5) is verified, there exists a periodic solution of (1.3), with energy h.

Remark 1.1. Let us observe that the hypotheses (1.4), (1.5) are verified, for example, if M is a ring-shaped domain $r_1\leq|x|\leq r_2$ or a torus, and S is a finite set. On the other hand, (1.4) and (1.5) do not hold if, for instance, $M=B_r(0)$ (the ball in $\mathbf{R}^n$) and $S=\{0\}$. Theorem 1.2 below just deals with such a situation.

Remark 1.2. Theorem 1.1 also holds for dynamical systems with kinetic energy $\frac{1}{2}\,a_{ij}(x)\dot{x}^i\dot{x}^j$, where $\{a_{ij}(x)\}$ is a positive definite matrix.

Remark 1.3. For every $b\in B$, let us denote by $x_b(t)$ the solution of (1.3) such that $x_b(0)=b$ and $\dot{x}_b(0)=0$; $x_b(t)$ is, of course, constrained within the "potential well" N. If such a solution reaches B again at some $t=T_0$, then the function $x(t)$ such that $x(t)=x_b(t)$ if $t\in[0,T_0]$, and $x(t)=$ $=x_b(2T_0-t)$ if $t\in[T_0,2T_0]$, is a $2T_0$-periodic solution of (1.3), with energy h; it is called "brake orbit". As in [12],[5] and [9], the solutions obtained in Theorem 1.1 are, more precisely, brake orbits.

Remark 1.4. For general dynamical systems with singularities, we cannot expect the existence of brake orbits; if, for example, $S=\{0\}$ and V is spherically symmetric (that is $V(x)=V(|x|)$), M is a sphere, and the curve $x_b(t)$ coincides with the radius from b to 0, so it cannot give rise to a brake orbit. A periodic solution of (1.3), with energy h, which lies completely in the interior of N, is called "interior orbit". The existence of such orbits is examined in the next theorem.

Theorem 1.2. Suppose that $\mathbf{R}^n=\mathbf{R}^2$, and $S=\{0\}$. Suppose moreover that:

(1.6) $$\lim_{x\to 0} V(x)|x|^2=-\infty$$

(1.7) $$\lim_{|x|\to\infty} V(x)=\infty$$

(1.8) $$\liminf_{|x|\to\infty} |V'(x)|>0$$

(1.9) $$\limsup_{|x|\to\infty} |V'(x)|<\infty$$

(1.10) $$\lim_{|x|\to\infty} |V''(x)|=0.$$

Then, there exists $h_0\in\mathbf{R}$ such that, for every $h\geq h_0$, there exists an interior orbit (see Remark 1.4) of (1.3), with energy h.

§ 2. The geometrical framework. Fix $h\in\mathbf{R}$, suppose $V'(x)\neq 0$ for every $x\in B$, and consider the metric $ds^2=a(x)\,\delta_{ij}dx_idx_j$ on N, where $a(x)=h-V(x)$ (notice that ds is degenerate on B). We now define a coordinate system in a neighbourhood of B. Let $z^1,z^2,\dots,z^{n-1}$ be the local coordinates on B (we recall that B is an (n-1)-dimensional manifold); then, if $b\in B$, we can represent $x_b(t)$ (same notations as in Remark 1.3) by the n-1 coordinates of $b: z^1,z^2,\dots,z^{n-1}$ and z^n = the arc length of $x_b|_{[0,t]}$ with respect to ds:

$$z^n=\int_0^t a(x_b(t))^{1/2}|\dot{x}_b(t)|dt=\sqrt{2}\int_0^t a(x_b(t))dt.$$

So, if $\delta_1>0$ is sufficiently small, we get a neighbourhood of B in N, parametrized by $B\times[0,\sigma_1]$, such that $B_\sigma\equiv\{z^n=\sigma\}$ are parallel surfaces orthogonal to curves $z^1=\mathrm{const.},\dots,\ z^{n-1}=\mathrm{const.},z^n(s)=s$ (see [5],[9] for more details).

If $0<\sigma\le\sigma_1$, we set $N_\sigma=\mathring{N}-\{0<z^n<\sigma\}$ and $M_\sigma=N_\sigma\cup S$ (clearly $B_\sigma=\partial M_\sigma$).

The next step is to modify the metric ds; let us denote by $d(x)$ the (euclidean) distance in R^n of x from the set S of singularities. For $\rho>0$ (small), let $\chi_\rho\in C^\infty(R^n,[0,1])$ be such that $\chi_\rho(x)=1$ if $d(x)\le\rho/2,\chi(x)=0$ if $d(x)\ge\rho$, and consider the function $V_\rho(x)=(1-\chi_\rho(x))V(x)+\chi_\rho(x)m_\rho$, where $m_\rho=\min\{V(x)|x\in M,\ d(x)\ge\rho/2\}$. Then, we can define the new metric $ds^2_\rho=a_\rho(x)\,\delta_{ij}dx_idx_j$ on $M\equiv N\cup S$, where $a_\rho(x)=h-V_\rho(x)$; as shown by [12], § 6 (see also [11]), if $0<\sigma_3<\sigma_2<\sigma_1$, there exists a modified metric $d\tilde{s}_\rho$ on M_{σ_3} such that M_{σ_3} is geodesically convex with respect to $d\tilde{s}_\rho$, and $d\tilde{s}_\rho=ds_\rho$ on M_{σ_2}. Set $\Lambda_{\sigma_3}=\{\gamma\in C([0,1],M_{\sigma_3})|\gamma$ is piecewise smooth, and $\gamma(0),\gamma(1)\in B_{\sigma_3}\}$, and introduce the energy functionals $E_\rho,\ \tilde{E}_\rho=\Lambda_{\sigma_3}\to R$ with respect to ds_ρ and $d\tilde{s}_\rho$, namely:

$$E_\rho(\gamma)=\int_0^1 a_\rho(\gamma(t))|\dot\gamma(t)|^2dt,\quad \tilde{E}_\rho(\gamma)=\int_0^1|\dot\gamma(t)|^2_\sim dt$$

($|\cdot|_\sim$ is the $d\tilde{s}_\rho$-norm). Since $d\tilde{s}_\rho$ is obtained by multiplying ds_ρ by a real function ≥1, we have $E_\rho(\gamma)\le\tilde{E}_\rho(\gamma)$. The main reason for considering the geodesic convex metric $d\tilde{s}_\rho$, is to define a curve shortening procedure on M_{σ_3}: let us denote by $\tilde{d}$ the distance on M_{σ_3} with respect to $d\tilde{s}_\rho$. Then, there exists $\eta>0$ such that: 1^o) if $\tilde{d}(x,y)\le\eta$, there exists a unique shortest geodesic arc which joins x to y; 2^o) if $\tilde{d}(x,B_{\sigma_3})\le\eta$, there exist a unique point $r(x)\in B_{\sigma_3}$, and a unique shortest geodesic arc which joins x to $r(x)$. Fix $K>0$, and let $\tilde{\Lambda}^K=\{\gamma\in\Lambda_{\sigma_3}|\tilde{E}_\rho(\gamma)\le K\}$; choose $m\in N$ in such a way that, if $\gamma\in\tilde{\Lambda}^K$, and $|t'-t''|\le1/m$, then $\tilde{d}(\gamma(t'),\gamma(t''))\le\eta$. For any $\gamma\in\tilde{\Lambda}^K$, we denote by $\mathcal{D}\gamma$ the curve obtained from γ in the following way: 1^o step. We join the points $r(\gamma(1/m))$, $\gamma(1/m)$, $\gamma(2/m),\dots,\gamma((m-1)/m)$, $r(\gamma((m-1)/m))$ by the shortest geodesic arcs. 2^o step. We consider the centres $C_1,\dots,C_m$ of these arcs, and join $r(C_1)$, $C_1,C_2,\dots,C_m$, $r(C_m)$, as before, by the shortest geodesic arcs. Then, the map $\mathcal{D}:\tilde{\Lambda}^K\to\tilde{\Lambda}^K$ is continuous and $\tilde{E}$-decreasing; moreover $\tilde{E}_\rho(\mathcal{D}\gamma)=\tilde{E}_\rho(\gamma)>0$ if and only if γ is a geodesic of $d\tilde{s}_\rho$ which starts from and reaches B_{σ_3} orthogonally (see [5],[9]). As we shall see later, by the curve shortening procedure we can obtain the geodesic of $d\tilde{s}_\rho$;

then, we can get a geodesic of ds by the limiting procedure by [12] (see also [9] and [5], p. 88). We close this section with the sketch of it.

Suppose that, for every σ_3, there exists a geodesic $\gamma \in \Lambda_{\sigma_3}$ of ds_ρ, such that the euclidean distance dist(Im(γ),S) of Im(γ) from the set S of singularities, is $\geq \rho$. Since $ds_\rho = ds_\rho$ on M_{σ_2}, the part of γ which lies in M_{σ_2}, gives rise, after a reparametrization, to a solution $x:[0,T] \to M_{\sigma_2}$ of (1.3), with $x(0)$, $x(T) \in M_{\sigma_2}$. As $\sigma_2, \sigma_3 \to 0$, we get a sequence $x_n(t)$, $t \in [0,T_n]$, of solutions of (1.3), such that $V(x_n(0)) \to h, V(x_n(T_n)) \to h$; by [13], we know that $0 < c_1 \leq T_n \leq c_2$, where c_1 and c_2 do not depend on n. Let us consider a subsequence, still denoted by $(x_n)_n$, such that $T_n \to T_0 \in [c_1,c_2]$, and $x_n(0) \to b \in B$. Then, for the solution $x_b(t)$, we have $V(x_b(T_0)) = \lim_{n\to\infty} V(x_n(T_n)) = h$, so x_b reaches B at the time T_0, and it gives rise, according to Remark 1.3, to a $2T_0$-periodic solution of (1.3).

§ 3. Proof of Theorem 1.1. We start with a lemma.

Lemma 3.1. Let V be such that (1.1), (1.2) hold, and fix K, $\varepsilon > 0$, $h \in \mathbb{R}$. Then, there exists $r > 0$ such that, if $0 < \rho < r$, and $\Gamma \subset \{\gamma \in C([0,1], M_{\sigma_3}) \mid \gamma$ is piecewise smooth$\}$ verifies the conditions:

(3.1) $\quad \mathrm{Im}(\gamma) \cap \{x \mid d(x) \geq \varepsilon\} \neq \emptyset$ and

(3.2) $\quad \int_0^1 a_\rho(\gamma)|\dot\gamma|^2 dt \leq K$

for every $\gamma \in \Gamma$, then we have dist(Im(γ),S) $\geq \rho$ for every $\gamma \in \Gamma$.

Proof. Since (3.1) is still verified if ε is decreased, we can assume $\{x \mid d(x) \leq \varepsilon\} \subset \mathcal{N}$. Set $A = \max\{|U(x)| \mid d(x) = \varepsilon\}$, and choose $r \in]0,\varepsilon[$, in such a way that $V(x) \leq 2h$ and $|U(x)| > \sqrt{2K} + A$ for $d(x) \leq r$ (see (1.1), $(1.2)_i$). Let ρ and Γ be as in the statement of the lemma, let $\gamma \in \Gamma$, and suppose, by contradiction, that dist(Im(γ),S) $< \rho$. Then, there exists an interval $[t',t''] \subset [0,1]$ such that $\gamma([t',t'']) \subset \{x \mid \rho \leq d(x) \leq \varepsilon\}$, $d(\gamma(t')) = \rho$ and $d(\gamma(t'')) = \varepsilon$. Because of (3.2) and $(1.2)_{ii}$, we have:

$$K \geq \int_0^1 a_\rho(\gamma)|\dot\gamma|^2 dt \geq \int_{t'}^{t''} a_\rho(\gamma)|\dot\gamma|^2 dt = \int_{t'}^{t''} (h-V(\gamma))|\dot\gamma|^2 dt \geq$$
$$\geq -\frac{1}{2}\int_{t'}^{t''} V(\gamma)|\dot\gamma|^2 dt \geq \frac{1}{2}\int_{t'}^{t''} |U'(\gamma)|^2|\dot\gamma|^2 dt \geq \frac{1}{2}\left|\int_{t'}^{t''} U'(\gamma)\dot\gamma\, dt\right|^2 =$$
$$= \frac{1}{2}|U(\gamma(t'')) - U(\gamma(t'))|^2,$$

so $|U(\gamma(t'))| \le |U(\gamma(t'))-U(\gamma(t''))|+|U(\gamma(t''))| \le \sqrt{2K}+A$; but this is impossible, since $d(\gamma(t'))=\rho< r$.

Proof of Theorem 1.1. Set $\Lambda^*_{\sigma_3}=\{\gamma\in\Lambda_{\sigma_3} | \gamma$ does not cross $S\}$; we have two cases. 1^0 case: suppose that (1.4) holds. Since $H_0(Y^*,B,Z) \approx \approx H_0(\Lambda^*_{\sigma_3},B_{\sigma_3},Z)$, there exists $\alpha_0 \in H_0(\Lambda^*_{\sigma_3},B_{\sigma_3},Z)-\{0\}$.

Let us consider $\omega_0\in\alpha_0$, and set $K=\int_0^1 a(\omega_0)|\dot\omega_0|^2dt$, $\varepsilon=\frac{1}{2}\operatorname{dist}(S,B_{\sigma_3})$. Finally, we take r as in Lemma 3.1, and fix $\rho\in]0,r[$. Then the set $\Gamma=\{\gamma\in\alpha_0|\tilde E_\rho(\gamma)\le K\}$, verifies (3.1) and (3.2) (we recall that $E_\rho(\gamma)\le\tilde E_\rho(\gamma)$), therefore $\operatorname{dist}(\operatorname{Im}(\gamma),S)\ge\rho$ for every $\gamma\in\Gamma$. Set $c=\inf\{\tilde E_\rho(\gamma)|\gamma\in\Gamma\}$, and observe that $c>0$; otherwise there would exist a sequence $(\gamma_n)_n\subset\alpha_0\subset\Lambda_{\sigma_3}$ such that the arc length of γ_n with respect to ds goes to zero. Then, for large n, γ_n clearly cannot belong to α_0, so we have a contradiction. Let us consider now a minimizing sequence $(\gamma_n)_n\subset\Gamma$ ($\lim_{n\to\infty}\tilde E_\rho(\gamma_n)=c$); since $\mathcal{D}\gamma_n\in\Gamma$ and $\tilde E_\rho(\mathcal{D}\gamma_n)\le\tilde E_\rho(\gamma_n)$ ($\mathcal{D}$ is the curve shortening procedure on $(M_{\sigma_3},d\tilde s_\rho)$), we have: $\lim_{n\to\infty}\tilde E_\rho(\mathcal{D}\gamma_n)= =\lim_{n\to\infty}\tilde E_\rho(\gamma_n)=c>0$.

Therefore (see [10], Appendix), a subsequence of $(\gamma_n)_n$ converges to a geodesic $\gamma\in\Lambda_{\sigma_3}$ of $d\tilde s_\rho$, with $\operatorname{dist}(\operatorname{Im}(\gamma),S)>\rho$. Notice that $\operatorname{Im}(\gamma)$ is not completely contained in $M-M_{\sigma_2}$; for if not, we would have $\operatorname{Im}(\gamma_n)\subset M-M_{\sigma_1}$ for large n, so we can project γ_n on B_{σ_3} along the curves z^1=const.,..., z^{n-1}= =const., $z^n(s)=s$. But this is impossible, since $\gamma_n\in\alpha_0$. Finally, by the limiting procedure sketched in Section 2, we get the result.

2^0 case: suppose that (1.5) holds. Let $\beta_0\in\pi_k(\Lambda^*_{\sigma_3},B_{\sigma_3})-\{0\}$ (notice that $\pi_k(\Lambda^*_{\sigma_3},B_{\sigma_3})\approx\pi_k(Y^*,B)$), choose $f_0\in\beta_0$, set

$K=\max\{\int_0^1 a(\omega)|\dot\omega|^2dt|\omega\in\operatorname{Im}(f_0)\}$, $\varepsilon=\frac{1}{2}\operatorname{dist}(S,B_{\sigma_3})$, and take r,ρ as in Lemma 3.1. Then, for every $\gamma\in\Gamma\equiv\{\gamma\in\operatorname{Im}(f)|f\in\beta_0, \tilde E_\rho(\gamma)\le K\}$, we have $\operatorname{dist}(\operatorname{Im}(\gamma),S)\ge\rho$. Set $\Phi=\{f\in\beta_0|\tilde E_\rho(\gamma)\le K$ for every $\gamma\in\operatorname{Im}(f)\}$, and $c=\inf_{f\in\Phi}\max\{\tilde E_\rho(\gamma)|\gamma\in\operatorname{Im}(f)\}$. As before it is not difficult to check that $c>0$. Let us consider $(f_n)_n\subset\Phi$ such that $\max\{\tilde E_\rho(\gamma)|\gamma\in\operatorname{Im}(f_n)\}$ goes to c as $n\to\infty$. Then, since $\mathcal{D}\circ f_n\in\Phi$, we have:

$c\le\max\{\tilde E_\rho(\mathcal{D}\gamma)|\gamma\in\operatorname{Im}(f_n)\}\le\max\{\tilde E_\rho(\gamma)|\gamma\in\operatorname{Im}(f_n)\}$, therefore there exists $(\gamma_n)_n$ such that $\gamma_n\in\operatorname{Im}(f_n)$, and $\tilde E_\rho(\mathcal{D}\gamma_n)\to c$.

Since $\tilde{E}_\rho(\mathcal{D}\gamma_n) \le \tilde{E}_\rho(\gamma_n) \le \max\{\tilde{E}_\rho(\gamma) | \gamma \in \mathrm{Im}(f_n)\}$, we also have $\tilde{E}_\rho(\gamma_n) \longrightarrow$ $\longrightarrow c$; so, a subsequence of $(\gamma_n)_n$ converges to a geodesic γ of $d\tilde{s}_\rho$, with $\mathrm{dist}(\mathrm{Im}(\gamma), S) \ge \rho$, which start from and reach B_{σ_3} orthogonally. From [9], we have that the curves $z^1=$const.,...,$z^{n-1}=$const., and $\sigma_3 \le z^n \le \sigma_1$, are geodesic of $d\tilde{s}_\rho$. Therefore, the part of γ contained in $M-M_{\sigma_1}$ coincides with one of such curves, and $\mathrm{Im}(\gamma)$ is not completely contained in $M-M_{\sigma_2}$.
At this point, we can use the same argument as in the 1^0 case.

§ 4. Proof of Theorem 1.2. The aim of this section is to prove Theorem 1.2, so we assume, from now on, that $\mathbf{R}^n=\mathbf{R}^2$ and $S=\{0\}$. Let $P_{\sigma_3}=\{\gamma \in C([0,1], M_{\sigma_3}) | \gamma$ is piecewise smooth, and $\gamma(0)=\gamma(1)\}$, and $\Gamma_{\sigma_3}=\{\gamma \in P_{\sigma_3} | \gamma$ is homotopically not trivial in $\mathbf{R}^2-\{0\}\}$.

Let us consider the manifold M_{σ_3}, with the boundary B_{σ_3} and metric $d\tilde{s}_\rho$ (for some $\rho > 0$ fixed); since M_{σ_3} is geodesically convex, we can still use the curve shortening procedure on M_{σ_3} as in Section 2, but in this section, we apply it to the closed curves $\gamma \in P_{\sigma_3}$. In fact, if $K>0$, for every $\gamma \in$ $\in P_{\sigma_3}$ with $\tilde{E}_\rho(\gamma) \le K$, there exists a closed curve, which we still denote (as in Section 2) by $\mathcal{D}\gamma$, homotopic to γ with an $\tilde{E}_\rho$-decreasing homotopy. Moreover, if $\lim_{n\to\infty} \tilde{E}_\rho(\gamma_n) = \lim_{n\to\infty} \tilde{E}_\rho(\mathcal{D}\gamma_n) > 0$, then a subsequence of $(\gamma_n)_n$ converges to a closed geodesic of $d\tilde{s}_\rho$ (see [10], Appendix).

The idea of the proof of Theorem 1.2 is to start from a curve $\eta \in \Gamma_{\sigma_3}$, and to consider the sequence: $\gamma_0=\eta$, $\gamma_{n+1}=\mathcal{D}\gamma_n$. If we choose a very small ρ, we get a closed geodesic γ of $d\tilde{s}_\rho$ such that $\mathrm{dist}(\mathrm{Im}(\gamma),0) \ge \rho$. On the other hand, γ is not contained in $M_{\sigma_3}-M_{\sigma_2}$, provided M is sufficiently large (that is, provided the energy level h is sufficiently high, see (1.7)). Since $d\tilde{s}_\rho=ds$ on $\{x | \ |x| \ge \rho,\ x \in M_{\sigma_2}\}$, γ is a closed geodesic of ds. Then it gives rise, by a reparametrization of the time, to a solution of (1.3) with energy h. To carry out this programme, we need some lemmas. Set $\|\gamma\|_0=\max\{|\gamma(t)| \,|\, t \in [0,1]\}$; the lemma 4.1 is due to [6].

Lemma 4.1. We have $\int_0^1 |\dot\gamma|^2 dt \ge \|\gamma\|_0^2$ for every $\gamma \in \Gamma_{\sigma_3}$.

Proof. Let $\gamma \in \Gamma_{\sigma_3}$, and suppose $\int_0^1 |\dot\gamma|^2 dt < \|\gamma\|_0^2$; then, since

$|\gamma(t')-\gamma(t'')| \le \int_0^1 |\dot\gamma|^2 dt \le (\int_0^1 |\dot\gamma|^2 dt)^{1/2}$, there exists a disc $D \subset \mathbb{R}^2 - \{0\}$ such that $\mathrm{Im}(\gamma) \subset D$. Therefore γ is homotopically trivial, and we have a contradiction.

Lemma 4.2. Suppose that (1.6) holds, and let $\eta \in \Gamma_{\sigma_3}$, $h \in \mathbb{R}$, and $K > 0$. Then, there exists $r_1 > 0$ such that, if $0 < \rho < r_1$, and if $H: [0,s_0] \to \Gamma_{\sigma_3}$ is a continuous function which verifies the conditions:

(4.1) $\quad H(0) = \eta$

(4.2) $\quad \int_0^1 a_\rho(\gamma)|\dot\gamma|^2 dt \le K$ for every $\gamma \in \mathrm{Im}(H)$,

we have $\|\gamma\|_0 \ge r_1$ for every $\gamma \in \mathrm{Im}(H)$.

Proof. Fix $c > 2K$, and choose a $0 < r_1 < \|\eta\|_0$ so small in such a way that $V(x) \le -c/|x|^2 \le 2h$ for $0 < |x| \le r_1$; let ρ and H as in the statement of the lemma, and set, for simplicity, $\gamma_s \equiv H(s)$ $(s \in [0,s_0])$. We argue by contradiction and suppose that $\|\gamma_\tau\|_0 < r_1$ for some $\tau \in [0,s_0]$. Since $\rho < r_1 < \|\eta\|_0 = \|\gamma_0\|_0$, and since $s \to \|\gamma_s\|_0$ is continuous, there exists $s \in [0,s_0]$ such that $\rho < \|\gamma_s\|_0 < r_1$. We claim that $V_\rho(\gamma_s(t)) \le -c/\|\gamma_s\|_0^2$ for every $t \in [0,1]$; in fact, if we fix $t \in [0,1]$ and choose $x,y \in \mathbb{R}^2$ such that $x = \gamma_s(t)$, and $|y| = \|\gamma_s\|_0$, we have two cases. 1^0 case: $|x| \le \rho/2$. Then $V_\rho(x) = m_\rho \le V(y) \le -c/|y|^2$. 2^0 case: $|x| > \rho/2$. Then $V_\rho(x) = (1-\chi_\rho(|x|))V(x) + \chi_\rho(|x|)m_\rho \le (1-\chi_\rho(|x|))V(x) + \chi_\rho V(x) = V(x) \le -c/|x|^2 \le -c/|y|^2$, so the claim is proved. Finally we observe that, since $m_\rho \le 2h$ and $V(\gamma_s(t)) \le 2h$, we have $V_\rho(\gamma_s(t)) \le 2h$ $(t \in [0,1])$. Then the inequalities (see (4.2) and Lemma 4.1):

$$K \ge \int_0^1 a_\rho(\gamma_s)|\dot\gamma_s|^2 dt = \int_0^1 (h - V_\rho(\gamma_s))|\dot\gamma_s|^2 dt \ge$$

$$\ge -\frac{1}{2}\int_0^1 V_\rho(\gamma_s)|\dot\gamma_s|^2 dt \ge \frac{c}{2\|\gamma_s\|_0^2}\int_0^1 |\dot\gamma_s|^2 dt \ge \frac{c}{2},$$

give $c \le 2K$, so we have a contradiction.

Lemma 4.3. Suppose that (1.6) holds, let $\eta \in \Gamma_{\sigma_3}$, $h \in \mathbb{R}$, and set $K = \int_0^1 a(\eta)|\dot\eta|^2 dt$. Then, there exists $r_2 > 0$ such that, if $0 < \rho < r_2$, there exists a closed geodesic $\gamma \in \Gamma_{\sigma_3}$ of ds_ρ^2, such that $\tilde{\xi}_\rho(\gamma) \le K$ and $\mathrm{dist}(\mathrm{Im}(\gamma),0) \ge \rho$.

Proof. Let $r_1 > 0$ be as in Lemma 4.2, set $\varepsilon = r_1$, and choose $r > 0$ as in Lemma 3.1. Then, we fix $r_2 \in]0,r[$ with $r_2 < \mathrm{dist}(\mathrm{Im}(\eta),0)$, $\rho \in]0,r_2[$, and consider the sequence: $\gamma_0 = \eta$, $\gamma_{n+1} = \mathcal{D}\gamma_n$, where $\mathcal{D}$ is the curve shortening procedure on M_{σ_3}, with respect to $d\tilde{s}_\rho$. We have that:

(4.3) $\qquad \gamma_n \in \Gamma_{\sigma_3}$ and $\text{dist}(\text{Im}(\gamma_n),0) \geq \rho$ for every $n \in \mathbf{N}$.

In fact, if $n \in \mathbf{N}$ is fixed, there exists a homotopy $H \in C([0,1], P_{\sigma_3})$ such that $H(0) = \eta$, $H(1) = \gamma_n$, and $\tilde{E}_\rho(\gamma) \leq K$ for every $\gamma \in \text{Im}(H)$. We claim that

(4.4) $\qquad \text{dist}(\text{Im}(\gamma),0) \geq \rho$ for every $\gamma \in \text{Im}(H)$;

clearly (4.4) implies (4.3). In order to prove (4.4), we set, for simplicity, $\eta_s = H(s)$ ($s \in [0,1]$), and suppose, by contradiction, $\text{dist}(\text{Im}(\eta_\tau),0) < \rho$ for some $\tau \in [0,1]$. Since $\text{dist}(\text{Im}(\eta),0) > r_2 > \rho$, there exists $s_0 \in [0,\tau]$ such that $\eta_s \in \Gamma_{\sigma_3}$ (that is it is homotopically not trivial in $\mathbf{R}^2 - \{0\}$) for every $s \in [0,s_0]$, and $\text{dist}(\text{Im}(\eta_{s_0}),0) < \rho$. Then the continuous function $H:[0,s_0] \to \Gamma_{\sigma_3}$ verifies (4.1) and (4.2) (we recall that $E_\rho(\gamma) \leq \tilde{E}_\rho(\gamma)$), so we have $\|\eta_s\|_0 \geq r_1$ for every $s \in [0,s_0]$. On the other hand, since $\rho < r_2 < r$, and since the set $\Gamma = H([0,s_0])$ verifies (3.1) (we recall that $\varepsilon = r_1$) and (3.2), because of Lemma 3.1 we have $\text{dist}(\text{Im}(\eta_s),0) \geq \rho$ for every $s \in [0,s_0]$. In particular, $\text{dist}(\text{Im}(\eta_{s_0}),0) \geq \rho$, so we have a contradiction. At this point, by standard argument (see [10]), we know that a subsequence of $(\gamma_n)_n$ converges to a closed geodesic γ of $d\tilde{s}_\rho$, with $\gamma \in \Gamma_{\sigma_3}$ and $\text{dist}(\text{Im}(\gamma),0) \geq \rho$ because of (4.3). Therefore, the lemma is proved.

Proof of Theorem 1.2. Because of (1.8), (1.9) and (1.10), there exist $R,H_1,H_2 > 0$ such that, for every $x \in \mathbf{R}^2$, $|x| \geq R$:

(4.5) $\qquad H_1 \leq |V'(x)|^2 - 2|V''(x)|$, and $|V'(x)|^2 + 2|V''(x)| \leq H_2$.

For any $b \in \mathbf{R}^2 - \{0\}$ we denote, as in Section 2, by $x_b(t)$ the solution of the Cauchy problem:

$$\begin{cases} \ddot{x} = -V'(x), \\ x(0) = b, \\ \dot{x}(0) = 0; \end{cases}$$

notice that, because of the standard existence theorem (see [4], Th. 1.2), and the assumption (1.9), there exists $t_0 > 0$ such that, for every $|b| > R$, $x_b(t)$ exists on $[0,t_0]$. Observe that, from (1.6) and (1.9), we have:

(4.6) $\quad V(x) \leq c|x|$ for every $x \neq 0$,

where $c > 0$ is a suitable (large) constant. Then, we set:

$$t_1 = \min\{t_0, \sqrt{2/H_2}\}, \quad \lambda = H_1 t_1 / 3\sqrt[3]{12},$$

and choose a piecewise smooth closed curve η such that η is homotopically not trivial in $\mathbf{R}^2 - \{0\}$. Clearly, because of (1.7), there exists $h_0 \in \mathbf{R}$ such that, for every $h \geq h_0$ we have $\mathrm{Im}(\eta) \subset \{x | V(x) \leq h-1\}$, and:

(4.7) $\quad V(x) \geq h-1$ implies $|x| \geq R$;

(4.8) $$0 < h \int_0^1 |\dot{\eta}|^2 dt - \int_0^1 V(\eta) |\dot{\eta}|^2 dt \leq \frac{\lambda}{c^2}(h-1)^2.$$

Fix $h \geq h_0$, set $a(x) = h - V(x)$ $(x \neq 0)$, $M = \{0\} \cup \{x | V(x) \leq h\}$, and $B = \partial M$. Then M is compact and $V'(x) \neq 0$ for every $x \in B$ (see (4.7), (4.5)). Moreover we have: $(d^2/dt^2)(a(x_b(t))) = (d/dt)(a'(x_b(t))\dot{x}_b(t)) = a'(x_b(t))\ddot{x}_b(t) + a''(x_b(t))[\dot{x}_b(t), \dot{x}_b(t)]$, namely:

(4.9) $$\frac{d^2}{dt^2} a(x_b(t)) = |V'(x_b(t))|^2 - V''(x_b(t))[\dot{x}_b(t), \dot{x}_b(t)].$$

We claim that

(4.10) $\quad a(x_b(t)) \leq 1$ for every $b \in B$ and $t \in [0, t_1]$:

otherwise, there would exist $b \in B$ and $\tau \in [0, t_1[$ such that $a(x_b(t)) \leq 1$ on $[0, \tau]$, and $a(x_b(\tau)) = 1$.

Then, since $\frac{1}{2}|\dot{x}_b(t)|^2 + V(x_b(t)) = h$, from (4.9) and (4.5) we have: $(d^2/dt^2)(a(x_b(t))) \leq |V'(x_b(t))|^2 + 2|V''(x_b(t))|(h - V(x_b(t))) \leq |V'(x_b(t))|^2 + 2|V''(x_b(t))| \leq H_2$ for every $t \in [0, \tau]$. Then $a(x_b(t)) \leq \frac{1}{2} H_2 t^2$ on $[0, \tau]$, so $1 = a(x_b(\tau)) \leq \frac{1}{2} H_2 \tau^2 < \frac{1}{2} H_2 t_1^2$; but this is not possible because of our choice of t_1. At this point, we go back to the construction of the neighbourhood $\{z^n \leq \sigma_1\}$ of B, as sketched in Section 2, and observe that we can take σ_1 = the minimum of arc length of $x_b([0, t_1])$ with respect to ds, that is:

$$\sigma_1 = \min\{\sqrt{2} \int_0^1 a(x_b(t)) dt | b \in B\}.$$

If $b \in B$, from (4.9), (4.10) and (4.5), we have:

$(d^2/dt^2)\ a(x_b(t)) \geq |V'(x_b(t))|^2 - 2|V''(x_b(t))|(h-V(x_b(t))) \geq |V'(x_b(t))|^2 -$
$-2|V''(x_b(t))| \geq H_1$. Therefore:

(4.11) $\quad a(x_b(t))$ is increasing and $a(x_b(t)) \geq \frac{1}{2} H_1 t^2$ for every $t \in [0,t_1]$.

In particular, since $\sigma_1 = \sqrt{2} \int_0^{t_1} a(x_b(t))dt$ for some $b \in B$, we have $\sigma_1 \geq \sqrt{2}\, H_1 t_1^3/6$. Set $\sigma_3 = \sigma_1/3$; we claim that $a(x) \geq \lambda$ on B_{σ_3}. In fact, if $x \in B_{\sigma_3}$, there exist $b \in B$ and $\tau \in [0,t_1]$, such that $x = x_b(\tau)$, and $\sigma_3 =$ $= \sqrt{2} \int_0^{\tau} a(x_b(t))dt$. From (4.11), we have $\sigma_3 \geq \sqrt{2}\, H_1 \tau^3/6$; on the other hand, by the mean value theorem, $\sigma_3 = \sqrt{2}\, a(x_b(\xi))\tau \leq \sqrt{2}\, a(x_b(\tau))\tau = \sqrt{2}\, a(x)\tau$; therefore $\sigma_3^3 \leq (\sqrt{2}\, a(x))^3 6 \quad \sigma_3/\sqrt{2}\, H_1$, so $a(x)^3 \geq H_1 \sigma_3/12 = H_1 \sigma_1^2/9\cdot 12 \geq$ $\geq H_1^3 t_1^3/3^3\cdot 12$, and the claim follows. Let us now set $K = \int_0^1 a(\eta)|\dot{\eta}|^2 dt$, and consider r_2, ρ and the closed geodesic $\gamma \in \Gamma_{\sigma_3}$ of $d\tilde{s}_\rho$ as in Lemma 4.3. We know that $\mathrm{dist}(\mathrm{Im}(\gamma),0) \geq \rho$. On the other hand, we have (see Lemma 4.1):

$$\|\gamma\|_0^2 \leq \int_0^1 |\dot{\gamma}|^2 dt \leq \frac{1}{\lambda} \int_0^1 a(\gamma)|\dot{\gamma}|^2 dt \leq \frac{1}{\lambda}\ \tilde{E}(\gamma) \leq K/\lambda\ ,$$

so, because of (4.6) and (4.8), $V(\gamma(t)) \leq c|\gamma(t)| \leq c\sqrt{K/\lambda} \leq h-1$. Therefore $\mathrm{Im}(\gamma) \subset \{x \in M_{\sigma_1} \mid |x| \geq \rho\}$, and γ is a closed geodesic of the Jacobian metric ds.

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