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RANDOM FUNCTIONAL-DIFFERENTIAL INCLUSIONS WITH NONCONVEX RIGHT HAND SIDE IN A BANACH SPACE

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Abstract: In this paper we prove the existence of random solutions for stochastic functional-differential inclusions defined in a separable Banach space and with an orientor field which is nonconvex valued, lower semicontinuous and satisfies a compactness type hypothesis involving the Hausdorff measure of noncompactness. The proof is based on the "measurable selection method" which makes use of an earlier deterministic result that we proved. Our theorem extends the earlier results by Deimling, Ito, Ladde-Lakshmikantham and Nowak.

Key words and phrases: Functional-differential inclusion, measurable multifunction, lower semicontinuous multifunction, selection theorem, Hausdorff measure of noncompactness, Kamke function, separable Banach space.

Classification: 34G20

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1. Introduction. In this note we prove the existence of random solutions for a class of random functional-differential inclusions with a nonconvex valued orientor field defined in a separable Banach space.

The study of random generalized equations started with the work of Castaing [2] and since then there have been developed two basic approaches to the subject. The first is the so called "measurable selection approach", in which for each fixed value of the random parameter ω we solve the corresponding deterministic problem and then from all those solutions through a suitable measurable selection theorem we choose one that depends measurably in ω. This approach was used by Deimling [4] (for single valued differential equations in R^n) and by the author [13] (for functional-differential equations in Banach spaces). The second approach is the "random fixed point approach" and proceeds to the solution of the problem through the use of an appropriate random fixed point theorem. This method was adopted by Itoh [7] (for single valued differential equations), by Nowak [10] and Papageorgiou [11],

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[12] (for differential inclusions in R^n and a separable Banach space respectively) and by Phan Van Cuong [15] (for integral inclusions).

In this note we follow the first approach. Based on an earlier result proved by the author [14], we prove the existence of random solutions for the stochastic problem. This way we extend the earlier results on random differential equations (single valued and generalized) obtained by Deimling 4 , Ladde-Lakshmikantham [8] and Nowak [10].

2. Preliminaries. Throughout this note (Ω, Σ, μ) is a probability space and X is a separable Banach space. Also by $P_f(X)$ we will denote the family of nonempty, closed subsets of X and by $P_k(X)$ the family of nonempty, compact subsets of X. A multifunction $F:\Omega \to P_f(X)$ is said to be measurable if and only if for all $y \in X$, $\omega \to d(y,F(\omega))=\inf\{\|y-x\|: x \in F(\omega)\}$ is measurable. By S_F^1 we will denote the set of all integrable selectors of $F(\cdot)$ i.e. $S_F^1 = \{f \in L^1(X): f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$.

If Y, Z are Hausdorff topological spaces and $G:Y \to 2^Z \setminus \{\emptyset\}$ we say that $G(\cdot)$ is lower semicontinuous (l.s.c.) if and only if for all $U \subseteq Z$ open, $G^-(U)=\{y \in Y: G(y) \cap U \neq \emptyset$.

Let $P_b(X)$ be the family of bounded subsets of X. The Hausdorff (ball) measure of noncompactness $\beta: P_b(X) \to \mathbf{R}_+$ is defined by:

$$\beta(B)=\inf\{r>0: \text{B can be covered by finitely many balls of radius } r\}.$$

Finally, a function $w: [0,b] X\ \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a Kamke function if: (1) $t \to w(t,r)$ is measurable, (2) $r \to w(t,r)$ is continuous. (3) $|w(t,r)| \leq \varphi(t)$ a.e. and (4) $w \equiv 0$ is the only solution of the integral inequality $u(t) \leq \int_0^t w(s,u(s))ds$, $u(0)=0$.

3. The existence theorem. Let $T=[0,b]$, $T_0=[-r,0]$, $\hat{T}=[-r,b]$ $(r,b>0)$ and as already stated X is a separable Banach space.

If $x: [-r,b] \to X$ is a function, then for $t \in [0,b]$, $x_t: [-r,0] \to X$ denotes the past history of the function from time instant -r until the present time i.e. $x_t(s)=x(t+s)$, $s \in [-r,0]$.

Consider the following functional-differential inclusion defined on X:

$$(*) \qquad \left| \begin{array}{ll} x(t) \in F(t,x_t) & \text{a.e. on } T \\ x(u)=x_0(u) & u \in T_0 \end{array} \right|$$

where $x_0 \in C(T_0,X)$. By a solution of $(*)$ we understand an absolutely continuous function $x:\hat{T} \to X$ that satisfies $(*)$. In [14] the author, among other things, proved the following existence theorem concerning $(*)$.

Theorem 1 [14]: <u>If</u> $F:T\times C(T_0,X)\to P_k(X)$ is a multifunction s.t.

(1) $(t,y)\to F(t,y)$ is measurable,

(2) for all $t\in T$, $y\to F(t,y)$ is l.s.c.,

(3) $|F(t,y)|\le a(t)+b(t)\|y\|_\infty$ a.e. with $a(\cdot),b(\cdot)\in L^1$,

(4) for every $B\in C(T_0,X)$ nonempty and bounded we have:

$\beta(F(t,B))\le w(t,\beta(B))$ a.e. where $w(\cdot,\cdot)$ is a Kamke function,

<u>then</u> $(*)$ admits a solution.

Here we will consider the following random version of $(*)$, with the random parameter belonging in a probability space. So we have the following multivalued Cauchy problem:

$$(**)\qquad \left|\begin{array}{l} \dot{x}(\omega,t)\in F(\omega,t,x(\omega,t)) \quad \mu\times\lambda\text{-a.e.} \\ x(\omega,u)=x_0(\omega,u) \text{ for all } (\omega,u)\in\Omega\times T_0 \end{array}\right|$$

where λ is the Lebesgue measure on T. By a random solution of $(**)$ we understand a stochastic process $x:\Omega\times\hat{T}\to X$ with absolutely continuous realizations s.t. for μ-almost all $\omega\in\Omega$, $x(\omega,\cdot)$ solves the corresponding deterministic functional-differential inclusion.

We have the following existence theorem concerning $(**)$.

Theorem 2: <u>If</u> $F:\Omega\times T\times C(T_0,X)\to P_k(X)$ is a multifunction s.t.

(1) $(\omega,t,x)\to F(\omega,t,x)$ is measurable,

(2) for every $(\omega,t)\in\Omega\times T$, $x\to F(\omega,t,x)$ is l.s.c.,

(3) $|F(\omega,t,x)|\le a(\omega,t)+b(\omega,t)\|x\|_\infty$ $\mu\times\lambda$-a.e. with $a(\cdot,\cdot)$, $b(\cdot,\cdot)\in$ $\in L^1(\Omega\times T)$,

(4) for every $B\subseteq C(T_0,X)$ nonempty and bounded we have: $\beta(F(\omega,t,B)\le w(\omega,t,\gamma(B))$ $\mu\times\lambda$-a.e. where for every $\omega\in\Omega$, $w(\omega,\cdot,\cdot)$ is a Kamke function.

(5) $x_0:\Omega\times T_0\to X$ is measurable in ω, continuous in $r\in T_0$

<u>then</u> $(*)$ admits a random solution.

Proof: Consider the following functions:

$$p:\Omega\times\hat{T}\times C(\hat{T},X)\times L^1(X)\to X$$

defined by

$$p(\omega,t,x,f) = \begin{cases} x(t)-x_0(\omega,0)-\int_0^t f(s)ds \text{ for } t\in T \\ x(t)-x_0(\omega,t) \text{ for } t\in T_0 \end{cases}$$

and $q:\Omega\times C(\hat{T},X)\times L^1(X)\to \mathbf{R}$ defined by:

$$q(\omega,x,f)=d(f,S^1_{F(\omega,\cdot,x\cdot)}).$$

From the above definitions, it is easy to see that:

$\omega \longrightarrow p(\omega,t,x,f)$ is measurable and $(t,x,f) \longrightarrow p(\omega,t,x,f)$ is continuous

Therefore Lemma III-14 of Castaing-Valadier [3] tells us that $(\omega,t,x,f) \longrightarrow p(\omega,t,x,f)$ is measurable. Let $\{t_m\}_{m\geq 1}$ be dense in $\hat{T}$ and define

$$\hat{p}(\omega,x,f)=\sup_{m\geq 1} p(\omega,t_m,x,f).$$

Clearly then $(\omega,x,f) \longrightarrow \hat{p}(\omega,x,f)$ is jointly measurable.

On the other hand for $q(\cdot,\cdot,\cdot)$ we have:

$$q(\omega,x,f)=d(f,S^1_{F(\omega,\cdot,x\cdot)})=\inf\{\|f-g\|_1 : g\in S^1_{F(\omega,\cdot,x\cdot)}\}=$$

$$=\inf\{\int_0^b \|f(s)-g(s)\|ds : g\in S^1_{F(\omega,\cdot,x\cdot)}\} = \int_0^b \inf\{\|f(s)-z\| : z\in F(\omega,s,x_s)\}\, ds =$$

$$= \int_0^b d(f(s),F(\omega,s,x_s))ds.$$

Consider the map $h:\hat{T}\times C(\hat{T},X)\longrightarrow \hat{T}\times C(T_0,X)$ defined by:

$$h(s,x)=(s,x_s).$$

Clearly $h(\cdot,\cdot)$ is continuous. Hence the map $h_1:(\omega,s,x)\longrightarrow(\omega,x,s_s)$ is measurable from $\Omega\times\hat{T}\times C(\hat{T},X)$ into $\Omega\times\hat{T}\times C(T_0,X)$. Furthermore since by hypothesis (1) $F(\cdot,\cdot,\cdot)$ is jointly measurable, for every $y\in X$ we have that h_2: $:(\omega,s,z)\longrightarrow d(y,F(\omega,s,z))$ is measurable. Then

$$h_2\circ h_1:(\omega,s,x)\longrightarrow d(y,F(\omega,s,x_s))$$

is measurable and since $y\longrightarrow d(y,F(\omega,s,x_s))$ is continuous, we conclude that $(\omega,s,x)\longrightarrow d(f(s),F(\omega,s,x_s))$ is measurable. Thus from Fubini's theorem we deduce that:

$$(\omega,x,f)\longrightarrow q(\omega,x,f)=d(f,S^1_{F(\omega,\cdot,x\cdot)})=\int_0^b d(f(s),F(\omega,s,x_s))ds$$

is measurable. Now consider the multifunction

$R:\Omega\longrightarrow 2^{C(\hat{T},X)\times L^1(X)}$ defined by:

$$R(\omega)=\{(x,f)\in C(\hat{T},X)\times L^1(X):\hat{p}(\omega,x,f)=0,\ q(\omega,x,f)=0\}.$$

From Theorem 1 we know that for all $\omega\in\Omega$, $R(\omega)\neq\emptyset$. Also using the measurability of $\hat{p}(\cdot,\cdot,\cdot)$ proved earlier, we get that:

$$GrR=\{(\omega,x,f)\in\Omega\times C(\hat{T},X)\times L^1(X):(x,f)\in R(\omega)\}\in$$

$$\in\Sigma\times B(C(\hat{T},X)\times L^1(X))=\Sigma\times B(C(\hat{T},X))\times B(L^1(X))$$

Apply Theorem 3 of Saint-Beuve [16] to get $\lambda_1:\Omega\longrightarrow C(\hat{T},X)$ and λ_2: $:\Omega\longrightarrow L^1(X)$ both measurable s.t. $(\lambda_1(\omega),\lambda_2(\omega))\in R(\omega)$ μ-a.e.

Set $\lambda_1(\omega)(t)=x(\omega,t)$. Clearly $x_1(\cdot,\cdot)$ is a Carathéodory function (i.e. measurable in ω , continuous in t). Also from Lemma 16 p. 196 of Dunford-Schwartz [5] we know that there exists $f\in L^1(\Omega\times T,X)$ s.t. $f(\omega,\cdot)=\lambda_2(\omega)$ μ-a.e. Then

$$x(\omega,t)=x_0(\omega,0)+\int_0^t f(\omega,s)ds \quad \mu\text{-a.e. for all } t\in T,$$

$$x(\omega,u)=x_0(\omega,u) \text{ for } u\in T_0 .$$

Invoking Lemma 2.2.1 of Ladde-Lakshmikantham [8] we conclude that $x(\cdot,\cdot)$ is the desired random solution of $(**)$.

Q.E.D.

Remark: Our result extends Theorem 5.1 of Nowak [10], which to our knowledge is the only other existence result for random differential inclusions wich nonconvex orientor field existing in the literature. In his result Nowak had $X=R^n$ and $F(\omega,t,x)$ was Hausdorff Lipschitz in the x-variable, while the system had no memory (i.e. r=0). So our theorem is a significant extension in several directions of the work of Nowak. Also Nowak's result was a random version of an earlier deterministic result by Himmelberg-Van Vleck [6]. Our theorem even in the absence of randomness (i.e. no ω dependence) is more general than the result of Himmelberg-Van Vleck. Also it generalizes the finite dimensional deterministic results of Bressan [1] and Lojasiewicz [9].

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