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ON THE MULTIPLE BIRKHOFF RECURRENCE THEOREM IN DYNAMICS

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Abstract: We prove the following extension of the Furstenberg-Weiss Multiple Birkhoff Recurrence Theorem: If the weight of a compact space X is less than $\underline{p}$ and if $\mathcal{S}$ is a countable commuting set of maps from X to X, then $[X,\mathcal{S}]$ has a multiple recurrent point. We also show that even for compact connected first countable spaces, the previous result is false if the weight is lifted.

Key words: Dynamical system, recurrent point, weight of space.

Classification: 54H20, 54A25, 03E65

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§ 0. **Introduction.** In this paper space means compact Hausdorff topological space. When X is a space, C(X,X) denotes the semigroup under composition of all continuous functions from X into X. A family $\mathcal{S}\subseteq C(X,X)$ is said to be commuting whenever $\forall f,g\in\mathcal{S}$, $f\circ g=g\circ f$.

For us, a (dynamical) system will be a pair $[X,\mathcal{S}]$, where $X\neq\emptyset$ is a space and $\emptyset\neq\mathcal{S}\subseteq C(X,X)$. In the case $\mathcal{S}=\{f\}$, the system $[X,\mathcal{S}]$ denoted by $[X,f]$ is traditionally called a discrete (dynamical) system.

A point $x\in X$ is said to be multiple recurrent in the system $[X,\mathcal{S}]$ provided that for each neighbourhood U of x, and for each finite set $G\subseteq\mathcal{S}$, there is an $n\in N$ (the positive integers) such that $\forall g\in G$, $g^n(x)\in U$. In the discrete system case, a multiple recurrent point is exactly that which is usually called a recurrent point. It is G. Birkhoff's theorem that each discrete system has a recurrent point (see [Bi] or [Fu],p. 20). H. Furstenberg and B. Weiss have proved the Multiple Birkhoff Recurrence Theorem (MBR). If X is a compact metric space and if $\mathcal{S}$ is finite and commuting, then $[X,\mathcal{S}]$ has a multiple recurrent point (see FW).

The main result shows the possibilities how to extend the MBR.

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In 2.3 we prove the following:
For a compact space X with $w(X)<\underline{p}$ and a countable commuting subset $\mathcal{S}\subseteq C(X,X)$ there is a multiple recurrent point in the system $[X,\mathcal{S}]$.

We also present an example (3.1) showing that some restrictions to the space X are necessary even for finite $\mathcal{S}$. Nevertheless, we obtain (2.5) a slightly weaker result true for all systems $[X,\mathcal{S}]$ with $\mathcal{S}$ commuting.

Notation: $\omega = N\cup\{0\}$. When X is a set, $|X|$ denotes the cardinality of X. When X is a space, the weight of $X, w(X)$ is the minimum cardinality of an open base for the topology of X. A family D of non-empty open sets of a space X is called a π-base provided that each non-empty open set contains a member of D. The well known cardinal characteristic $\underline{p}$ concerns families of subsets of ω.

$$\underline{p}=\min\{|\mathcal{A}|: \mathcal{A}\subseteq[\omega]^{\omega}, \bigcap\mathcal{A}'\in[\omega]^{\omega} \text{ for each finite } \mathcal{A}'\subseteq\mathcal{A} \text{ and } \forall B\in[\omega]^{\omega}(\exists A\in\mathcal{A})\ |B-A|=\omega\}.$$

Equivalently, $\underline{p}$ is the minimum cardinality possessed by a neighbourhood base of a non-empty nowhere dense subset of $\beta\omega-\omega$. It is F. Hausdorff's classical result that $\underline{p}\geq\omega_1$. Very important for us is Bell's Theorem [Be]:
Each compact separable space X cannot be covered by less than $\underline{p}$ nowhere dense subsets.

§ 1. **Preliminaries on minimal sets and systems.** Suppose $[X,\mathcal{S}]$ is a system. A set $A\subseteq X$ is said to be invariant in $[X,\mathcal{S}]$ provided that A is non-empty and for each $f\in\mathcal{S}$ is $f[A]\subseteq A$.

A set $M\subseteq X$ is said to be minimal in the system $[X,\mathcal{S}]$ provided it is a minimal element in the partially ordered, by inclusion, set of all closed invariant sets. When X is minimal in $[X,\mathcal{S}]$, then $[X,\mathcal{S}]$ is said to be a minimal system. Suppose $\mathcal{S}\subseteq C(X,X)$ is arbitrary, then $\langle\mathcal{S}\rangle$ denotes the set of all $g\in C(X,X)$ such that g is the composition of finitely many members of $\mathcal{S}$, so $\langle\mathcal{S}\rangle$ is a semigroup under composition.

Although we do not in general assume (as Furstenberg) that $\mathcal{S}$ is either commuting or finite, the proofs of each of the following lemmas are either similar to the discrete system case and/or straightforward - so they are partially left to the reader (note that compactness is only necessary in 1.1 (i) and 1.2 (iii)).

1.1. **Lemma.** Suppose $[X,\mathcal{S}]$ is a system. Then the following two statements are true:

(i) If $A \subseteq X$ is closed and invariant in $[X, \mathcal{G}]$ then there is an $M \subseteq A$ minimal in $[X, \mathcal{G}]$.

(ii) M is minimal in $[X, \mathcal{G}]$ iff M is minimal in $[X, \langle \mathcal{G} \rangle]$.

Suppose that $(\Sigma, \cdot)$ is any semigroup. Let us call $S \subseteq \Sigma$ syndetic in Σ orovided there is a finite set $F \subseteq \Sigma$ such that for each $g \in \Sigma$ there exists $f \in F$ so that $f.g \in S$.

1.2. **Lemma.** Suppose $[X, \mathcal{G}]$ is a system and M is a closed invariant set in $[X, \mathcal{G}]$. Then the following three statements are equivalent:

(i) M is a minimal set in $[X, \langle \mathcal{G} \rangle]$.

(ii) $\forall x \in M$, $cl(\{f(x) : f \in \langle \mathcal{G} \rangle\}) = M$.

(iii) $\forall x \in M$ and for each non-empty open $U \subseteq M$ $\{f \in \langle \mathcal{G} \rangle : f(x) \in U$ is a syndetic set in $\langle \mathcal{G} \rangle$.

Proof. We show (ii) $\longleftrightarrow$ (iii).

($\longrightarrow$) We claim that $\bigcup \{f^{-1}[U] : f \in \langle \mathcal{G} \rangle\} = M$ for each non-empty open set U. When $M - \bigcup \{f^{-1}[U] : f \in \langle \mathcal{G} \rangle\} \neq \emptyset$, then this set must be invariant. Therefore from the compactness of M we get $f_1, \ldots, f_k$ so that $\bigcup_{i=1}^{k} f_i^{-1}[U] = M$. For $g \in \langle \mathcal{G} \rangle$ and $g(x) = y$ we obtain f_j so that $y \in f_j^{-1}[U]$ and finally $f_j \circ g(x) \in U$, so that $S = \{f \in \langle \mathcal{G} \rangle : f(x) \in U\}$ is syndetic.

($\longleftarrow$) Let $cl(\{f(x) : f \in \langle \mathcal{G} \rangle\}) \subsetneq M$. Then $U = M - cl(\{f(x) : f \in \langle \mathcal{G} \rangle\})$ is open, but $\{f \in \langle \mathcal{G} \rangle : f(x) \in U\}$ is syndetic, that means non-empty.

§ 2. **The main result.** We shall use the following combinatorial fact originally published in [Ra] (also see [GRS], pp. 38-39).

2.1. **Lemma.** (Gallai´s theorem) Suppose that $k \in N$ and suppose $\mathcal{P}$ is a finite partition of ω^k. If $E \subseteq \omega^k$ is finite, then $\exists P \in \mathcal{P}$, $\exists r \in \omega^k$, $\exists n \in N$, such that $\forall s \in E$, $r + n \cdot s \in P$.

2.2. **Theorem.** Suppose that $[X, \mathcal{G}]$ is a minimal system, X is separable, $w(x) < \underline{p}$, and suppose $\mathcal{G}$ is commuting and $|\mathcal{G}| < \underline{p}$. Then the set of all multiple recurrent points is a dense subset of X.

Proof: Let $\mathcal{B}$ denote the family of all non-empty open sets of X. Fix $G = \{g_1, \ldots, g_k\} \subseteq \mathcal{G}$. For every $B \in \mathcal{B}$ define

$$D(B,G) = \{V \cap \bigcap_{i=1}^{k} g_i^{-n}[V] : n \in N, V \in \mathcal{B} \text{ and either } V \subseteq B \text{ or } V \cap B = \emptyset\}.$$

To prove the following claim, we need only the assumption that $[X,\mathcal{G}]$ is minimal and $\mathcal{G}$ is commuting.

Claim: $D(B,G)$ is a π-base of X for each $B\in\mathcal{B}$. To see this, fix $B\in\mathcal{B}$ and suppose $U\in\mathcal{B}$ is arbitrary.

Define $V=B\cap U$ if $B\cap U\neq\emptyset$, otherwise define $V=U$ if $B\cap U=\emptyset$. Pick up $x\in X$. Since $[X,\mathcal{G}]$ is minimal, we can apply 1.1(ii) and 1.2(iii). Hence the set $S=\{g\in\langle\mathcal{G}\rangle:g(x)\in V\}$ is syndetic in the abelian semigroup $\langle\mathcal{G}\rangle$. Let $F\subseteq\langle\mathcal{G}\rangle$ be the associated finite set. For each $f\in F$ define

$$P_f=\{v=\langle v_1,\dots,v_k\rangle\in\omega^k:f\circ g_1^{v_1}\circ\dots\circ g_k^{v_k}\in S\}.$$

Then $\{P_f:f\in F\}$ is a finite covering of ω^k, because S is syndetic in $\langle\mathcal{G}\rangle$. Now we apply Gallai's theorem for the finite set $E=\{0,1\}^k$ and therefore for some $f\in F$ there is a $v\in\omega^k$ and $n\in N$ so that for each $e=\langle e_1,\dots,e_k\rangle\in E$,

$$f\circ g_1^{v_1+ne_1}\circ\dots\circ g_k^{v_k+ne_k}\in S.$$

Denote $h=f\circ g_1^{v_1}\circ\dots\circ g_k^{v_k}$, then for each i, $g_i^n\circ h\in S$. It means that $h(x)\in V$ and also $g_i^n(h(x))\in V$ for each $i=1,\dots,k$. So $h(x)\in V\cap\bigcap_{i=1}^k g_i^{-n}[V]$ and then $V\cap\bigcap_{i=1}^k g_i^{-n}[V]$ is a non-empty open subset of U and the claim is proved.

To prove 2.2 let $\mathcal{A}\subset\mathcal{B}$ be a base for the topology of X such that $|\mathcal{A}|<\underline{p}$. For each $A\in\mathcal{A}-\{\emptyset\}$ and finite $G\subseteq\mathcal{G}$ put $N(A,G)=X-\bigcup D(A,G)$. According to the claim, $N(A,G)$ is nowhere dense. As X is separable and $|\mathcal{G}|<\underline{p}$, we may apply Bell's Theorem to find an $x\in X-\bigcup\{N(A,G):A\in\mathcal{A},G\subseteq\mathcal{G}$ is finite$\}$. So for each $A\in\mathcal{A}$, A a neighbourhood of x and finite $G\subseteq\mathcal{G}$ there exists $V\in D(A,G)$ such that $x\in V$ and $V\subseteq A$. This implies that there exists $n\in N$ such that $g^n(x)\in A$ for each $g\in G$.

2.3. **Corollary.** Suppose that X is a compact Hausdorff space with $w(x)<\underline{p}$. If $\mathcal{G}\subseteq C(X,X)$ is countable and commuting, then the system $[X,\mathcal{G}]$ has a multiple recurrent point.

Proof: From 1.1(i) take a minimal set M of $[X,\mathcal{G}]$. Then $w(M)<\underline{p}$ and by 1.2(ii), M is separable. So 2.2 applies.

2.4. **Corollary.** (The Multiple Birkhoff Recurrence Theorem) Suppose X is a compact metrizable space and $\mathcal{G}\subseteq C(X,X)$ is countable and commuting. Then $[X,\mathcal{G}]$ has a multiple recurrent point.

Proof: Since $\underline{p}$ is an uncountable cardinal and $w(X)\leq\omega$, then 2.3 applies.

Let us agree to call $x \in X$ multiple non-wandering in the system $[X, \mathcal{S}]$ provided that for each neighbourhood U of x and for each finite $G \subseteq \mathcal{S}$, $\exists\, u \in U$, $\exists\, n \in N$, $\forall f \in G$, $f^n(u) \in U$. This is a slight generalization of Birkhoff's notion of non-wandering point [Bi]. From the claim of the proof 2.2 we obtain immediately .

2.5. **Theorem.** Suppose that $[X, \mathcal{S}]$ is a minimal system, and $\mathcal{S}$ is commuting. Then each point of X is a multiple non-wandering in $[X, \mathcal{S}]$.

There is a much longer purely topological proof of 2.2 and 2.3 which makes no use of Gallai's Theorem. In fact, Gallai's Theorem can be obtained (see [Fu]) as a corollary of the Multiple Birkhoff Recurrence Theorem.

§3. **Example.** Since the Birkhoff Recurrence Theorem, originally proved for compact metric spaces, is true for each discrete system, one might conjecture that the same is the case for the Multiple Birkhoff Theorem. However, we have the following counter-example.

3.1. **Example.** There is a compact connected first countable space X with a homeomorphism $h: X \to X$ such that $[X, \{h, h^{-1}\}]$ has no multiple recurrent points.

Proof: Let X be the annulus $\{re^{2\pi i\Theta} : 1 \leq r \leq 2,\ \Theta \text{ is a real}\}$ in the plane. If $1 < r < 2$, a basic neighbourhood of $re^{2\pi i\Theta}$ will have the form

$\{se^{2\pi i\Theta} : 0 < |s-r| < \varepsilon\}$ where $\varepsilon < \min\{r-1, 2-r\}$.

A basic neighbourhood of $e^{2\pi i\Theta}$ will have the form, for ε, $0 < \varepsilon < 1$,

$L_{\varepsilon,\Theta} \cup \{se^{2\pi i\Theta} : 1 \leq s < 1+\varepsilon\}$

where $L_{\varepsilon,\Theta} = \{re^{2\pi i\varphi} : 1 \leq r \leq 2,\ 0 < \Theta - \varphi < \varepsilon\}$.

A basic neighbourhood of $2e^{2\pi i\Theta}$ will have the form, for ε, $0 < \varepsilon < 1$,

$R_{\varepsilon,\Theta} \cup \{se^{2\pi i\Theta} : 2-\varepsilon < s \leq 2\}$ where $R_{\varepsilon,\Theta} = \{re^{2\pi i\varphi} : 1 \leq r \leq 2,\ 0 < \varphi - \Theta < \varepsilon\}$.

Obviously X is first countable. It is easy to show that X is compact and connected. Now arbitrarily choose an irrational α, $0 < \alpha < 1$ and define a rotation $h: X \to X$ by

$$h(re^{2\pi i\Theta}) = re^{2\pi i(\Theta+\alpha)}.$$

Clearly, h is a homeomorphism. On the other hand if $\varepsilon < \frac{1}{2}$, then $\forall \Theta, \varphi\ \forall n \in N$, $\forall r$, $1 \leq r \leq 2$ $h^n(re^{2\pi i\Theta}) \in L_{\varepsilon,\varphi}$, if $h^{-n}(re^{2\pi i\Theta}) \in R_{\varepsilon,\varphi}$.

Since for all real Θ and $\forall n \in N$, $\Theta + n\alpha \neq \Theta \neq \Theta - n\alpha$ (mod the integers) no point of X is multiple recurrent in $[X,\{h,h^{-1}\}]$.

Our example is the simplest of several exhibiting the failure, in general, of the Multiple Birkhoff Recurrence Theorem. The first two authors of this paper have a compact X, $w(X)=2^{\omega}$ with a homeomorphism h such that $[X,\{h,h^2\}]$ has no multiple recurrent point - this is especially interesting in the light of the result [ES]:
If x is recurrent in a discrete system $[X,f]$, then, $\forall n\in N$, x is recurrent in the system $[X,f^n]$. The third author jointly with J. Pelant have found a system $[X, \{f,g\}]$ with f and g commuting homeomorphism such that $[X,f]$ and $[X,g]$ have no recurrent point in common. All of these examples will appear elsewhere.

Question 1: Is it provable in ZFC that there is a system $[X,\mathcal{G}]$, $\mathcal{G}$ finite and commuting and $w(X)=\underline{p}$ such that there is no multiple recurrent point in $[X,\mathcal{G}]$?

Question 2: Suppose X is the Cantor set. Is there a commuting $\mathcal{G}\subseteq C(X,X$ such that $[X,\mathcal{G}]$ has no multiple recurrent point?

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References

[Be] M. BELL: On the combinatorial principle P(c), Fund. Math. 144(1981), 149-157.

[Bi] G.D. BIRKHOFF: Dynamical Systems, AMS Colloq. Publ. 9(1927).

[ES] P. ERDÖS and A.H. STONE: Some remarks on almost periodic transformations, Bull. AMS 53(1945), 126-130.

[Fu] H. FURSTENBERG: Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton Univ. Press (1981).

[FW] H. FURSTENBERG and B. WEISS: Topological dynamics and combinatorial number theory, J. d'Analyse Math. 34(1978), 61-85.

[GRS] R. GRAHAM, B. ROTHSCHILD and J. SPENCER: Ramsey Theory, John Wiley (1980).

[RA] R. RADO: Note on combinatorial analysis, Proc. London Math. Soc. 48 (1945), 122-160.

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