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PARTITE CONSTRUCTION AND RAMSEYAN THEOREMS FOR SETS, NUMBERS AND SPACES

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Abstract: We present several general results in Ramsey theory for set systems, parameter sets, vector and affine spaces. We outline our amalgamation technique, known as Partite Construction, on a particular case of sparse Rado-families of numbers.

Key words: Ramsey theorems for systems, sparse families.

Classification: 05-02, 05C55, 05A17

Introduction. The following result is one of the most useful and fundamental combinatorial statements:

Finite Ramsey Theorem [24]. For every choice of positive integers t, a, b there exists a positive integer c such that $c \longrightarrow (b)^a_t$.

Here the undefined symbol $c \longrightarrow (b)^a_t$ is a shortened notation (due to Erdös and Rado) for the following statement:

For every partition of all a-element subsets of a set X of size c there exists a b-element subset B of C such that all a-element subsets of B belong to the same class of the partition.

This theorem has been generalized many times and some of these generalizations are both profound and difficult to prove. Motivated by general results due to Rado [25], Graham, Leeb, Rothschild [3] and others (cf. [5]), one of the main streams of the research was formed by efforts to prove the most general result which would imply all known (usually difficult) instances. This development for set systems culminated with the proof of the Ramsey theorem for systems [9],[1] which we state in the next section (after introducing the necessary notions).

Original proofs of these results were difficult and complex. However, some special cases were handled more efficiently by a systematic use of amalgamation of partite systems [15],[16].

Recently, we adopted this technique known as Partite Construction to yield a new proof of Ramsey Theorem for Systems together with its several strengthenings which cover virtually all known Ramsey theorems for special classes of set systems. These results are stated in Section 1.

Somehow surprisingly it appeared that the Partite Construction provides a proper setting for Ramsey theorems dealing with parameter sets, vector and affine spaces. This approach was found in [2] and we developed it to the full analogy of set systems. One should remark that although formally these results for spaces are similar to those for set systems, the details are much more delicate and the validity of these results was an open problem for several years (see e.g. [12]). We state some of these results in Section 2.

While the proofs and further details will appear in [20] and [21], we illustrate our technique on a particular case of Rado-families of numbers.

Section 1. Set Systems

1.1. A type $\Delta=(n_\delta;\ \delta\in\Delta)$ is a sequence of positive integers. A type will be fixed throughout this paper.

A system A of type Δ is a pair $(X,\mathcal{M})$ where X is a finite linearly ordered set, $\mathcal{M}=(\mathcal{M}_\delta;\ \delta\in\Delta)$, and $\mathcal{M}_\delta\subseteq\binom{X}{n_\delta}$. (Here $\binom{X}{k}$ denotes the set of all k-element subsets of X.) We will suppose (without loss of generality) that $\mathcal{M}_\delta\cap\mathcal{M}_{\delta'}=\emptyset$ for $\delta\neq\delta'$. Elements of sets $\mathcal{M}_\delta$ are called edges of A.

A is a subsystem of $B=(Y,\mathcal{N})$ if X is an ordered subset of Y and $M\in\mathcal{M}_\delta$ iff $M\in\mathcal{N}_\delta$ for every $M\subseteq X$ and $\delta\in\Delta$.

Isomorphisms are just monotone isomorphisms. A subsystem of B is isomorphic to A is called a copy of A in B. Denote by $\binom{B}{A}$ the set of all copies of A in B.

Now we can state:

1.2. **Ramsey Theorem for Systems** [9],[10],[14]: Let t be a positive integer A, B systems. Then there exists a system C such that

$$C\longrightarrow(B)^A_t$$

Moreover if A, B do not contain an irreducible system F then C may be chosen with the same property.

Here a system F is called irreducible if every pair of points of F is contained in an edge of F. The arrow $C\longrightarrow(B)^A_t$ has an analogous meaning as in the classical Erdös-Rado case: For every partition $\binom{C}{A}=\mathcal{A}_1\cup\ldots\cup\mathcal{A}_t$ there exists $B'\in\binom{C}{B}$ and an i such that $\binom{B'}{A}\subseteq\mathcal{A}_i$.

The amalgamation technique (Partite Construction) outlined below in Section 3 yields some further strengthenings among which we list the following:

1.3. **Hom-connected graphs.** The above proof yields immediately the following stronger results which we state now. First we give an auxiliary definition. Let $B=(X,\mathcal{M})$ be a set system. A set $Y\subset X$ is called a cut of B if there is a partition of X-Y into two disjoint sets Y_1, Y_2 such that no pair $\{y_1,y_2\}$, $y_1\in Y_1$, $y_2\in Y_2$, is covered by an edge of B. We shall also consider the cut Y as a subsystem of B determined by Y.

A system B is called hom A-connected iff no cut Y of B has a homomorphism into A. Here a homomorphism is an edge preserving mapping. (Hom K_k-connected graph coincides with the notion of chromatically k-connected graph, see [16].) It is also convenient to recall the following notion [9]: Given a (possibly infinite) set $\mathcal{F}$ of systems denote by Forb($\mathcal{F}$) the set of all those systems A which do not contain any system $F\in\mathcal{F}$ as a subsystem. Now we have

Theorem (Ramsey Theorem with Forbidden Subsystems). Let $\mathcal{F}$ be a set of Hom A-connected systems. Then for every positive t and every $B\in$ Forb($\mathcal{F}$) there exists $C\in$ Forb($\mathcal{F}$) such that $C\longrightarrow(B)^A_t$.

The Partite Construction is very convenient for constructing sparse Ramsey graphs. This is not surprising as one of the byproducts of the partite construction is a new easy construction of highly chromatic graphs without short cycles [11], cf. [7],[6]. There are various ways how to define the sparseness and we list two of them.

1.4. **Sparse Ramsey Theorems - Ramsey Families.** We say that $\mathcal{B}\subseteq\binom{C}{B}$ is a Ramsey family if for every partition

$$\binom{C}{A}=\mathcal{A}_1\cup\ldots\cup\mathcal{A}_t,$$

there exists $B'\in\mathcal{B}$ such that $\binom{B'}{A}\subseteq\mathcal{A}_i$ for some i. We denote this by $C\xrightarrow{\mathcal{B}}(B)^A_t$. We associate with $\mathcal{B}$ a uniform hypergraph $H^A_{\mathcal{B}}=(X,E)$ where $X=\binom{C}{A}$ and

$$E=\left\{\binom{B}{A}\,;\ B\in\mathcal{B}\right\}.$$

We have

Theorem (Sparse Ramsey families). For every A, B and positive integers t, ℓ there exists C and a system $\mathcal{B}\subseteq\binom{C}{B}$ such that

1) $C\xrightarrow{\mathcal{B}}(B)^A_t$

2) The hypergraph $H^A_{\mathcal{B}}$ has no cycles of length $\leq\ell$.

1.5. **Sparse Ramsey theorems - Cycles in copies.** We have the following

Theorem (sparse copies). Let B be a Hom A-connected system. Let t, ℓ be positive integers. Then there exists a set system C with the following properties:

1) $C \longrightarrow (B)^A_t$

2) The hypergraph $H^A_{\binom{C}{B}}$ has no cycles of length $\leq \ell$.

1.6. **Linearity.** We say that a system B is A-linear if every two copies of A in B intersect in at most one vertex. Typical examples of linear systems are Steiner systems. In [18] we proved Ramsey theorem for Steiner systems. More generally we have:

Theorem. Let A be a system, t a positive integer. Then for every A-linear B there exists A-linear C such that

$$C \longrightarrow (B)^A_t$$

Section 2: Spaces

2.1. Let $A=\{a_1,\dots,a_q\}$ be a fixed finite set (alphabet), let $B\subseteq A$ be non empty. For non-negative integers $k\leq n$ we will define special subsets P_k, called k-parameter sets, of the cartesian product A_n in the following way (cf. [4],[3]):

For disjoint, non empty subsets $\omega_1,\dots,\omega_k$ of $[n]=\{1,2,\dots,n\}$, define P_k as the set of all those $(x_1,\dots,x_n)\in A^n$ such that

(i) If $u,v\in\omega_j$ for some j then $x_u=x_v$;

(ii) If $u\in[n]-\bigcup_j \omega_j$ then $x_u=a_u$ - a fixed element of B.

In a certain sense, P_k is the combinatorial analogue of a k-dimensional affine space over a q-element field (at least, when q is a prime power). Observe that $|P_k|=q^k$ for $k\geq 0$. A set $X\subseteq A^n$ is said to be an i-parameter subset of P_k if X is an i-parameter set in A^n and $X\subseteq P_k$. A discussion of various properties of k-parameter sets can be found in [4].

When q is a prime power and $A=GF(q)$, a more common substructure of A^n is that of a k-dimensional affine (or vector) space over GF(q). Since, similarly as in [2], we will be treating both k-parameter sets and k-dimensional spaces in A^n simultaneously, we will call them both k-spaces in A^n (although when we use this term, we will always have one particular interpretation in mind). We shall denote the set of k-spaces in A^n by $\binom{A^n}{k}$ and their number by $\left[{n \atop k}\right]$. X will be called a subspace of A^n if $X\in\binom{A^n}{k}$ for some k, in which case k is called the dimension of X, denoted by dim X.

The following is the space analogy of Ramsey theorem (first conjectured by Rota):

2.2. **Theorem** (Ramsey theorem for spaces [3],[4]): For all integers t, a, b with $0 \leq a \leq b$ there exists an integer N such that if $\binom{A^n}{a}_N = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$ ia an arbitrary partition of all a-dimensional subspaces of A^N into t classes, then there is always a b-dimensional subspace X of A^N with $\binom{X}{a} \subseteq \mathcal{A}_i$ for some i.

(Note that a=0, b=1, A=B is the well-known Hales-Jewett theorem [8].)

2.3, Let a type $(n_\delta;\ \delta \in \Delta)$, be fixed.

A space system **A** is a pair $(V, \mathcal{S})$ where V is a space, $\mathcal{S} = (\mathcal{S}_\delta;\ \delta \in \Delta)$ and $\mathcal{S}_\delta \subseteq \binom{V}{n_\delta}$ - the set of all n_δ-dimensional subspaces of V.

Elements of V are called points, elements of $\bigcup_\delta \mathcal{S}_\delta$ edges of **A**.

A system will be always considered with a canonical linear ordering of its points (such as lexicographic, see [22] for details).

We say that the system **A** is a subsystem of the system $\mathbf{B} = (U, \mathcal{T})$ if V is an (ordered) subspace of U and $\mathcal{T}_\delta \cap \binom{V}{n_\delta} = \mathcal{S}_\delta$ for every $\delta \in \Delta$. Denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all subspaces of **B** which are isomorphic to **A**.

Using these concepts, we may formulate our main result:

2.4. **Theorem** (Ramsey theorem for space systems): Let t be a positive integer, **A**, **B** systems. Then there exists a system **C** with the following properties:

(i) $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_t$

(ii) **C** contains an irreducible system **F** if **B** contains **F** as a subsystem.

Here the undefined notions have the following meaning:

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_t$ is the classical Erdös-Rado partition arrow which is a shortened notation for the following statement: For every partition $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$ there exists $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}'}{\mathbf{A}} \subseteq \mathcal{A}_i$ for an i.

A system **F** is irreducible if every pair x,y of points belongs to an edge of **F**.

The amalgamation technique (Partite Construction) outlined below in Section 3 yields some further strenghtenings among which we list the following:

2.5. **Hom-connected systems.** First we give two auxiliary definitions:

Let $\mathbf{B} = (V, \mathcal{T})$ be a system. A space R is called a cut of **B** if there is a partition of V-R into two disjoint sets V_1 and V_2 such that no pair $\{v_1, v_2\}$, $v_1 \in V_1$, $v_2 \in V_2$ is contained in an edge of **B**.

We shall consider R together with all edges of **B** contained in R; this will be denoted by **R** - thus **R** is a system.

$\mathbf{B}$ is said to be lin $\mathbf{A}$ -connected if there is no cut $\mathbf{R}$ of $\mathbf{B}$ for which there is a linear map $\mathbf{R} \to \mathbf{A}$.

It is also convenient to recall the following notion [9]: Given a (possibly infinite) set $\mathcal{F}$ of systems, denote by Forb($\mathcal{F}$) the set of all those systems $\mathbf{A}$ which do not contain any system $\mathbf{F} \in \mathcal{F}$ as a subsystem. Now we have:

Theorem (Ramsey theorem for forbidden subspaces). Let $\mathcal{F}$ be a set lin-$\mathbf{A}$ connected systems. Then for every positive t and every $\mathbf{B} \in$ Forb($\mathcal{F}$) there exists $\mathbf{C} \in$ Forb($\mathcal{F}$) with $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_t$.

2.6. **Sparse Ramsey Theorems - Ramsey Families.** We say that $\mathcal{B} \subseteq \binom{\mathbf{C}}{\mathbf{B}}$ is a Ramsey family if for every partition

$$\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$$

there exists $\mathbf{B}' \in \mathcal{B}$ such that $\binom{\mathbf{B}'}{\mathbf{A}} \subseteq \mathcal{A}_i$ for some i. We denote this by $\mathbf{C} \xrightarrow{\mathcal{B}} (\mathbf{B})^{\mathbf{A}}_t$.

Recall the definition of the hypergraph $H^{\mathbf{A}}_{\mathcal{B}}$ introduced in Section 1. We have:

Theorem (Sparse Ramsey Families). For every systems $\mathbf{A}$, $\mathbf{B}$ and positive integers t, ℓ there exists a system $\mathbf{C}$ and family $\mathcal{B} \subseteq \binom{\mathbf{C}}{\mathbf{B}}$ such that

1) $\mathbf{C} \xrightarrow{\mathcal{B}} (\mathbf{B})^{\mathbf{A}}_t$.

2) The Hypergraph $H^{\mathbf{A}}_{\mathcal{B}}$ contains no cycles of length $\leq \ell$.

2.7. **Sparse Ramsey Theorems - Cycles in Copies.** We have the following:

Theorem (Ramsey theorem with forbidden cycles in copies): Let t, ℓ be positive integers , $p \leq a \leq b$. Put $\mathbf{A} = \left(A^a, \binom{A^a}{p}\right)$, $\mathbf{B} = \left(A^b, \binom{A^b}{p}\right)$ and exclude the possibility a=0, b=1, $|A|=2$. Then there exists $\mathcal{S} \subset \binom{A^X}{p}$ such that the system $\mathbf{C} = (A^X, \mathcal{S})$ satisfies

1) $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_t$

2) The hypergraph $\left(\binom{A^X}{a}, H^{\mathbf{A}}_{\binom{\mathbf{C}}{\mathbf{A}}}\right)$ contains no cycles of length $\leq \ell$.

Remark. The theorem fails to be true for a=0, b=1 and $|A|=2$, as, in this case we deal with a perfect graph.

A special case of this theorem for $A = \{0,1\}$, $B = \{0\}$ and p=a=1 has been conjectured in [19]. This case corresponds to the Finite Union Theorem which is known to be equivalent to Finite-Union-Theorem [25]. We prove this particular case directly in Section 3.

Another particular case is the existence of sparse Van der Waerden sets. This case corresponds to $A=\{0,1,2,\ldots,k\}$, $B=A$, $p=a=0$, $b=1$. It follows that there are infinitely many minimal Van der Waerden sets, thus generalizing [26].

Section 3. Sparse Rado Sets

3.1. The following theorem was proved by R. Rado [25].

Theorem . For all positive integers t, n there exists a set of integers A such that for every partition

$$A=\bigcup_{i=1}^{t} A_i$$

there exists $a_1,a_2,\ldots,a_n$ and an integer j such that $\sum_{i\in\omega} a_i \in A_j$ for every $0\neq\omega\subset[1,n]$.

If $A=\{a_1,a_2,\ldots,a_n\}$ is a set of integers we put

$$\Sigma A=\{\sum_{i\in\omega} a_i,\ 0\neq\omega\subseteq[1,n]\}.$$

For the sake of brevity we call the set ΣA an n-<u>gon.</u>

Definition. Let A be a set of positive integers, $t>1$. Denote by H_n^A the set system $(A,\mathcal{M}_n^A)$ where $M\in\mathcal{M}_n^A$ if and only if M is an n-gon.

3.2. We prove the following generalization of [19] which is a particular case of Theorem 2.7.

Theorem (Sparse Rado Sets). For every $t,n\geq 2$, $\ell\geq 2$ there exists a set of integers A such that the following holds:

1) For every partition of A into t classes one of the classes contains an n-gon;

2) the hypergraph H_n^A does not contain cycles of length $\leq\ell$;

3) every triple $x,y,x+y$ in A is a subset of some $M\in\mathcal{M}_n^A$.

3.3. Let $\mathcal{X}$ be a family of pairwise disjoint sets. Let $\Sigma\mathcal{X}$ denote the family of all sets of the form $\bigcup_{X\in\mathcal{X}'} X$; $\mathcal{X}'\subset\mathcal{X}$. If $|\mathcal{X}|=n$, we shall call the set $\Sigma\mathcal{X}$ n-<u>gon</u>. Using this notion we can formulate

Theorem (Finite Union Theorem) [5]. For every positive integers n, t there exists a positive integer N such that for every partition of all subsets of $[N]=\{1,\ldots,N\}$ into t classes one of the classes contains a monochromatic n-gon.

3.4. Finite Union Theorem is known to imply Rado Finite Sum Theorem by a simple coding:

Fix K a large number and to every subset $M \subset [N]$ assign the number $K(M) = \sum_{i \in M} K^i$. Set

$$K[N] = \{K(M),\ M \subseteq [N]\}.$$

It appears that the set K[N] has for every $K \geq 2$ the following properties:

Theorem (Rado Finite Sum Theorem) . For every positive integers n, t, s there exists a set X of positive integers with the following properties:

1) For every partition of X into t classes one of the classes contains an n-gon.

2) If a, b, c are positive integers smaller than s and x, y, z are distinct elements of X, then

$$(*)\qquad \begin{array}{l} ax+by=cz \text{ iff either} \\ a=b=c \text{ and } x+y=z \text{ or} \\ a+b=c \text{ and } x=y=z. \end{array}$$

Proof: Put K=2s and X=K[N]. Observe that the size of X depends on n, t only. Also for $K \neq K'$ the sets K[N] and K'[N] have the same additive structure.

3.5. A similar comment applies to the fact that it suffices to prove Rado Finite Sum Theorem for vectors with integral entries and also for vectors with rational coordinates. These conventions make our constructions easier to formulate.

The rest of the paper is devoted to the proof of Theorem 3.2.

3.6. We construct sparse Rado sets by a variant of our Partite construction. First fix n, t, ℓ positive integers. Choose X as in the above Theorem 3.4 (for s=2). Put $X = \{x_1, \dots, x_q\}$. We shall construct certain sets of vectors with rational coordinates. These sets will be denoted by $P^0, P^1, \dots, P^k, \dots, P^q$; we call these sets <u>pictures</u> (cf. [16]).

The construction proceeds by induction on k.

3.7. <u>Construction of picture</u> P^0. Let $\{X_1, \dots, X_r\}$ be the family of all n-gons in X (as $s > 1$ sets X_i are in 1-1 correspondence with (set-system) n-gons in [N], with X= K[N]).

Let P^0 consist of all vectors of the form

$$(x, \varepsilon^1_{i(1)}, \varepsilon^2_{i(2)}, \dots, \varepsilon^r_{i(r)})$$

where $x \in X$ and all ε^i_j entries are zero with exception of one entry, say $\varepsilon^j_{i(j)}$ which belongs to the set X_j and $\varepsilon^j_{i(j)} = x$.

For $\vec{x} = (x, \vec{\varepsilon}) \in P^0$ we write $\pi(\vec{x}) = x$.

3.8. In the induction step let P^k be given. Assume that P^k consists of positive integral vectors. Let again the elements of P^k be of the form $\vec{x}=(x,e)$, $x=\pi(\vec{x})$.

Consider the set $Y=\{\vec{x}\in P^k;\ \pi(\vec{x})=x_k\}$. Now we invoke Theorem 2.6 specialized to Hales-Jewett theorem (note that this special case may be proved by probabilistic means, see [27]):

Let H be a large number and $\mathcal{L}$ a family of lines $\mathcal{L}\subseteq\binom{Y^H}{1}$ such that

(i) for every partition Y^H into t classes $\mathcal{A}_1\cup\ldots\cup\mathcal{A}_t$ one of the classes contains a monochromatic line $L\in\mathcal{L}$;

(ii) $\mathcal{L}$ contains no cycles of length $\leq \ell$.

Let s_k be larger than any entry of a vector from P^k. Now assume that we (possibly) change the set X so that it satisfies the condition ($*$) of Theorem 3.4 for s_k. (This we may assume as we change neither the additive structure of X nor the size of X.)

Now define the picture P^{k+1} to consist of all vectors of the form

$$\vec{x}=(x,\vec{x}_1,\ldots,\vec{x}_H)$$

where $\vec{x}$ satisfies one of the two possibilities

1) $x=x_k$ and $(\vec{x}_1,\ldots,\vec{x}_H)\in Y^H$.

2) There exists a line $L\in\mathcal{L}$ determined by $\emptyset\neq\omega\subseteq[H]$, elements $\vec{x}_i^{\,o}\in Y$, $i\in[H]-\omega$ and $\vec{x}_j\in P^k$, $j\neq k$, such that the following holds:

$$\pi(\vec{x}_j)=x_j=x$$

$$\vec{x}_i=\frac{x_j}{x_k}\,\vec{x}_i^{\,o}\ \text{for } i\in[H]-\omega$$

$$\vec{x}_i=\vec{x}_j\ \text{for } i\in\omega\,.$$

This concludes the definition of P^{k+1}.

We claim that P^q has the desired properties. This follows from the following four claims.

3.9. <u>Claim 1:</u> In the induction step (construction of P^{k+1}) the following holds:

$$(x,\vec{x}_1,\ldots,\vec{x}_H)+(y,\vec{y}_1,\ldots,\vec{y}_H)=(z,\vec{z}_1,\ldots,\vec{z}_H)$$

iff $x+y=z$ and there exists a line $L\in\mathcal{L}$ determined by ω and constant coordinates $\vec{x}_i^{\,o}$, $i\notin\omega$ such that $\vec{x}_i+\vec{y}_i=\vec{z}_i$ for $i\in\omega$.

Thus any sum $(x,\vec{x}_1,\ldots,\vec{x}_H)+(y,\vec{y}_1,\ldots,\vec{y}_H)=(z,\vec{z}_1,\ldots,\vec{z}_H)$ in P^{k+1} corresponds to the uniquely determined sums $\vec{x}_i+\vec{y}_i=\vec{z}_i$ in P^k.

Proof of Claim 1. We check the definition of P^{k+1} . Note that for $j\notin\omega$

we have $\vec{x}_j = \frac{x}{x_k}\vec{x}^o_j$, $\vec{y}_j = \frac{y}{x_k}\vec{x}^o_j$, $\vec{z}_j = \frac{z}{x_k}\vec{x}^o_j$.

It follows that the above condition is sufficient. The necessity follows from the property (2) of Theorem 3.4, i.e. from the assumption on X:

Assume that for some i none of the above possibilities occur. Then we have one of the following four cases:

1) $\frac{x}{x_k}\vec{x}^o_i + \frac{y}{x_k}\vec{y}^o_i = \vec{z}_i$

2) $\frac{x}{x_k}\vec{x}^o_i + \vec{y}_i = \frac{z}{x_k}\vec{z}^o_i$

3) $\vec{x}_i + \vec{y}_i = \frac{z}{x_k}\vec{z}^o_i$

4) $\vec{x}_i + \frac{y}{x_k}\vec{y}^o_i = \vec{z}_i$.

In each of these cases we get a contradiction with the choice of X.

3.10. For a line $L \in \mathcal{L}$ denote by P_L the set of all vectors $\vec{x} \in P^{k+1}$ of the form $(x, \vec{x}_1, \ldots, \vec{x}_H)$ where $\vec{x}_i = \frac{x}{x_k}\vec{x}^o_i$ for $i \notin \omega$ and $x_i = x_j$ for $i,j \in \omega$. It follows from Claim 1 that the additive structure of P_L is the same as that of P^k.

Claim 2. $|P_L \cap P_{L'}| \leq 1$ and $P_L \cap P_{L'} \subseteq Y$ for any pair of distinct lines $L, L' \in \mathcal{L}$.

Proof. Check the construction.

Claim 3. The family $\{P_L; L \in \mathcal{L}\}$ does not contain cycles of length $\leq \ell$.

Proof: Use the fact that $\mathcal{L}$ does not contain short cycles. It follows from Claim 3 by induction on k that the hypergraph $H_n^{P^q}$ does not contain cycles of length $\leq \ell$.

3.11. It remains to be proved that for every partition of P^q into t classes one of the classes contains an n-set together with all sums. This is proved (as usual for the Partite Construction cf.[16]) by the backward induction $k=q, q-1, \ldots, 1, 0$: Using the construction of P^q there exists a line L such that the set P_L (with the same additive structure as P^{q-1}) has all vectors with projection x_q in one class of the partition.

Continue this argument for P_L (i.e. P^{q-1}). This leaves us with a subset P of P^q with the same additive structure as P^o such that all vectors with the same projection are in one class of the partition. Now we use the choice of X to get an n-gon in one class of the partition.

This finishes the proof of Sparse Rado Sum Theorem 3.2.

3.11. **Remark.** The above proof is a typical example of the Partite Construction . The details of the amalgamation procedure are similar. However, if we do not decompose singletons then we need a more complicated argument, known as Partite Lemma, to establish the Ramsey object for the subobject induced by sets with the same projection. See papers [15],[16],[20],[21] for more details.

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