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AN ABSTRACT DIFFERENTIAL INEQUALITY AND EIGENVALUES OF VARIATIONAL INEQUALITIES

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Abstract: A detailed analysis of properties of certain initial value problem for an abstract ordinary differential inequality is performed. The result obtained is used to give a new proof of a Miersemann's theorem on eigenvalues of variational inequalities in Hilbert spaces (see [1]). While the proof presented by E. Miersemann in [1] issues from certain Krasnoselskii s ideas (see [5]), our access draws from a method proposed by I.V. Skrypnik (see [6] or [7]).

Key words: Abstract differential inequality, eigenvalues of variational inequalities.

Classification: 35B32, 35P30, 49A29

Introduction. In this article a new proof of certain eigenvalue theorem by Miersemann (see [1]) is presented. In his proof Miersemann made use of the ideas of the original proof of Krasnoselskii potential bifurcation theorem (see [5]), while our method is inspired with the procedure proposed by I.V. Skrypnik to prove another important potential bifurcation result for the variational equations (see [6] - p. 161, Theorem 3.4 and [7] - p. 178, Theorem 12, respectively). (On the basis of these Skrypnik's ideas the author proved a small generalization of Krasnoselskii potential bifurcation theorem - see [9].)

Throughout this paper, H, A and K denotes a real Hilbert space, a continuous linear operator in H and a cone in H i.e. a nonempty, convex and closed subset of H such that for every $x \in K$: $\{t \cdot x;\ t \geq 0\} \subset K$, respectively.

In Section 1 solutions $k:\langle 0,\infty) \to K$ of the following abstract ordinary differential inequality are investigated -

(I.1) $(k'(t)-Ak(t)+(Ak(t),k(t))\cdot k(t)/\|k(t)\|^2,\ y-k(t)) \geq 0$

for all $y \in K$.

Section 2 contains a new proof of the following Miersemann´s eigenvalue result:

<u>Theorem 1</u>: Suppose that the operator A is selfadjoint and completely continuous. Let $\mu \in (0, \sup\{(Ax,x); \|x\|=1\})$ be an eigenvalue of A such that an eigenvector corresponding to μ lies in the interior of the cone K. Denote ν the least eigenvalue of A greater than μ.

Then there exist $\lambda \in (\mu, \nu\rangle$ and $x \in K \setminus \{0\}$ such that

(I.2) $(\lambda x - Ax, y - x) \geq 0$ for all $y \in K$.

(Thus, λ and x is an eigenvalue and an eigenvector, respectively, of the operator A with respect to the cone K.)

The eigenvector x is constructed as an accumulation value ($t \to \infty$) of an abstract function $k: t \in \langle 0, \infty) \to K$ satisfying an initial value problem for the differential inequality mentioned above with a conveniently chosen initial condition.

It is possible to use the method presented also to prove other Miersemann´s results on eigenvalues and bifurcation points of variational inequalities (see [2, 3, 4]).

1. <u>Auxiliary differential problems</u>. Let T and ρ be positive real numbers. Denote P the projection of the Hilbert space H onto the cone K;

(1.1) $\|x - Px\| = \inf\{\|x-y\|; y \in K\}$ for all $x \in K$.

In what follows, the standard function spaces $C(\langle 0,T\rangle, H)$, $C^1(\langle 0,T\rangle, H)$, $L^2(0,T,H)$ and $W^{2,1}(0,T,H)$ will be used.

<u>Remarks 1</u>: It is well known that:

1. For all $x \in H$ and $y \in K$

(1.2) $(x-Px, y-Px) \leq 0$,

(1.3) $(x-Px, y) \leq 0$ and $(x-Px, x) = 0$.

2. The mapping P is nonexpansive -

(1.4) for all $x, y \in H$: $\|Px - Py\| \leq \|x-y\|$.

3. $W^{2,1}(0,T,H)$ is continuously imbedded into $C(\langle 0,T\rangle, H)$.

It is easy to prove the following assertions:

4. If $k_n \to k$ in $W^{2,1}(0,T,H)$ then for all $t \in \langle 0,T\rangle$:

$:k_n(t) \longrightarrow k(t)$ in H.

5. Let $k \in L^2(0,T,H)$, $l \in C(\langle 0,T\rangle,H)$, $l(t) \in K$ for all $t \in \langle 0,T\rangle$. Then the statements introduced below are equivalent:

(1.5 - 6) For all $y \in K$ (continuous functions $\eta:\langle 0,T\rangle \to K$) there exists a set $M_y (M_\eta) \subset \langle 0,T\rangle$ of measure zero such that $(k(t),y-l(t)) \geq 0$ $((k(t),\eta(t)-l(t)) \geq 0)$ for all $t \in \langle 0,T\rangle \setminus M_y$ $(\langle 0,T\rangle \setminus M_\eta)$.

(1.7 - 8) There exists a set $M \subset \langle 0,T\rangle$ of measure zero such that $(k(t),y-l(t)) \geq 0$ $((k(t),\eta(t)-l(t)) \geq 0)$ for all $y \in K$ (continuous functions $\eta:\langle 0,T\rangle \to K$) and $t \in \langle 0,T\rangle \setminus M$.

Thus, we may write briefly -
$(k(t),y-l(t)) \geq 0$ for all $y \in K$ and almost all $t \in \langle 0,T\rangle$ or
$(k(t),\eta(t)-l(t)) \geq 0$ for all continuous functions $\eta:\langle 0,T\rangle \to K$ and almost all $t \in \langle 0,T\rangle$ - instead of each of the statements (1.5) - (1.8).

6. Suppose that $k_n \to k$ in $L^2(0,T,H)$, $l \in L^2(0,T,H)$ and for all $n \in \mathbb{N}: (k_n(t),l(t)) \geq 0$ almost everywhere in $\langle 0,T\rangle$. Then $(k(t),l(t)) \geq 0$ almost everywhere in $\langle 0,T\rangle$.

Further needed properties of the function spaces mentioned above and the Bochner integral can be found in [8].

Lemma 1: Let $\xi \in C(\langle 0,T\rangle,H)$, $x \in H$. Then there exists the unique abstract function $k:\langle 0,T\rangle \to H$ such that

(1.9) $k \in C^1(\langle 0,T\rangle,H)$,

(1.10) $k'(t)=\xi(t)+\rho\cdot(Pk(t)-k(t))$ for all $t \in \langle 0,T\rangle$,

(1.11) $k(0)=x$.

Proof: Define the operator $U:C(\langle 0,T\rangle,H) \to C(\langle 0,T\rangle,H)$ by the formula:

(1.12) $Uk(t)=x+\int_0^t (\xi(\tau)+\rho\cdot(Pk(\tau)-k(\tau))d\tau$ for all $t \in \langle 0,T\rangle$.

Then

(1.13) for all $k,l \in C(\langle 0,T\rangle,H)$ and $t \in \langle 0,T\rangle: \|Uk(t)-Ul(t)\| =$
$= \|\int_0^t \rho\cdot((Pk(\tau)-Pl(\tau))-(k(\tau)-l(\tau)))\cdot \exp(-4\rho\tau)\cdot \exp(4\rho\tau)d\tau\| \leq \rho\cdot[\sup\{\|Pk(\tau)-Pl(\tau)\|\cdot\exp(-4\rho\tau);$
$\tau \in \langle 0,T\rangle\} + \sup\{\|k(\tau)-l(\tau)\|\cdot\exp(-4\rho\tau);\ \tau \in \langle 0,T\rangle\}]\cdot$

$\cdot\int_0^t \exp(4\rho\eta)d\eta \leq \sup\{\|k(\tau)-l(\tau)\|\cdot\exp(-4\rho\tau);\ \tau\in\langle 0,T\rangle\}\cdot$

$\cdot\ (\exp(4\rho t)-1)/2.$

Hence

(1.14) $\sup\{\|Uk(t)-Ul(t)\|\cdot\exp(-4\rho t);\ t\in\langle 0,T\rangle\} \leq \sup\{\|k(t)-$

$-l(t)\|\cdot\exp(-4\rho t);\ t\in\langle 0,T\rangle\}/2.$

According to the Banach fixed point theorem there is the unique solution of the equation $k=Uk$. Obviously, this equation and the problem (1.9), (1.10) and (1.11) are equivalent.

Lemma 2: Let $\xi\in C(\langle 0,T\rangle,H)$, $x\in K$. Denote by k the solution of the differential problem (1.9), (1.10) and (1.11). Put:

(1.15) $\Lambda = \max\{\|\xi(t)\|;\ t\in\langle 0,T\rangle\}.$

Then:

(1.16) $\max\{\|k(t)-Pk(t)\|;\ t\in\langle 0,T\rangle\} \leq \Lambda/\rho,$

(1.17) $\max\{\|k'(t)\|;\ t\in\langle 0,T\rangle\} \leq 2\cdot\Lambda,$

(1.18) $\max\{\|k(t)\|;\ t\in\langle 0,T\rangle\} \leq 2\cdot\Lambda\cdot T + \|x\|.$

Proof: Consider the continuous function

(1.19) $F:\langle 0,T\rangle \longrightarrow \mathbb{R}^1,\ F(t)=\|k(t)-Pk(t)\|$ for all $t\in\langle 0,T\rangle$.

Obviously, $F(0)=0<\Lambda/\rho$. Assume that

(1.20) there exists $t_0\in(0,T\rangle$ such that $F(t_0)>\Lambda/\rho$.

Denote:

(1.21) $t_1=\sup\{t\in\langle 0,t_0\rangle;\ F(t)\leq\Lambda/\rho\}.$

It is apparent that $0<t_1<t_0$, $F(t_1)=\Lambda/\rho$ and

(1.22) $F(t)>\Lambda/\rho$ for all $t\in(t_1,t_0\rangle$.

Thus

(1.23) $F(t_1)<F(t_0)$.

We shall show that the function F is decreasing on (t_1,t_0). This fact and the continuity of F on $\langle 0,T\rangle$ imply that $F(t_1) > > F(t_0)$, which contradicts (1.23).

Prove the monotonicity of F on (t_1,t_0). For all $t\in\langle 0,T\rangle$ define the function

(1.24) $G_t:\langle 0,T\rangle \longrightarrow \mathbb{R}^1,\ G_t(s)=\|k(s)-Pk(t)\|^2$ for all $s\in\langle 0,T\rangle$.

Obviously

(1.25) $G_t'(t)/2=(k'(t),k(t)-Pk(t))=(\xi(t),k(t)-Pk(t))-\rho\cdot F^2(t)\leq$
$\leq\|\xi(t)\|\cdot F(t)-\rho\cdot F^2(t)\leq\rho\cdot(\Lambda/\rho-F(t))\cdot F(t).$

Thus, if $F(t)>\Lambda/\rho$, then $G_t'(t)<0$ i.e. the function G_t is decreasing at the point t.

Let $t\in(t_1,t_0)$. According to (1.22) $F(t)>\Lambda/\rho$ and therefore there exists $\varepsilon>0$ such that

(1.26) $\|k(s)-Pk(t)\|^2=G_t(s)<G_t(t)=\|k(t)-Pk(t)\|^2=F^2(t)$ for all $s\in(t,t+\varepsilon)$.

Moreover

(1.27) $F^2(s)=\|k(s)-Pk(s)\|^2\leq\|k(s)-Pk(t)\|^2$

and thus $F(t)>F(s)$. The proof of (1.16) is complete. (1.17) and (1.18) can be obtained easily with the help of (1.16).

<u>Lemma 3</u>: Let $\xi\in C(\langle 0,T\rangle,H)$, $x\in K$. Then there exists precisely one abstract function

(1.28) $k\in W^{2,1}(0,T,H)$

such that

(1.29) $(k'(t)-\xi(t),y-k(t))\geq 0$ for all $y\in K$ and almost all $t\in\langle 0,T\rangle$,

(1.30) $k(0)=x$,

(1.31) $k(t)\in K$ for all $t\in\langle 0,T\rangle$.

For every $\rho>0$ denote by k_ρ the solution of (1.9), (1.10) and (1.11). Then $\lim_{\rho\to\infty}k_\rho=k$ in $W^{2,1}(0,T,H)$ weakly.

Proof: According to the fifth part of Remarks 1 (1.29) may be replaced by

(1.32) $(k'(t)-\xi(t),\eta(t)-k(t))\geq 0$ for all continuous functions $\eta:\langle 0,T\rangle\to K$ and almost all $t\in\langle 0,T\rangle$.

1. Uniqueness. Let the abstract functions k and l solve the problem (1.28), (1.32), (1.30) and (1.31). Then:

(1.33) $(k'(t)-\xi(t),l(t)-k(t))\geq 0$ and $(l'(t)-\xi(t),k(t)-l(t))\geq 0$ almost everywhere in $\langle 0,T\rangle$.

Adding these inequalities we have:

(1.34) $(k'(t)-l'(t),l(t)-k(t))\geq 0$ for almost all $t\in\langle 0,T\rangle$.

Thus for all $t \in \langle 0,T\rangle$

(1.35) $\|k(t)-l(t)\|^2 = \|k(0)-l(0)\|^2 + 2\cdot\int_0^t (k'(\tau)-l'(\tau),k(\tau)-$
$-l(\tau))d\tau \leq 0.$

Hence we get that $k(t)=l(t)$ for all $t \in \langle 0,T\rangle$.

2. According to Lemma 2 the set $\{k_\rho;\rho \in \langle 0,\infty)\}$ is bounded in $W^{2,1}(0,T,H)$. Hence there exist sequences $\{\rho_n\}_{n=1}^{+\infty} \subset \mathbb{R}^1$ and $\{k_n\}_{n=1}^{+\infty} \subset W^{2,1}(0,T,H)$ with the following properties:

(1.36) the sequence $\{\rho_n\}_{n=1}^{+\infty}$ is positive, increasing and boundless;

(1.37) for all $n \in \mathbb{N}: k_n$ is the solution of (1.9), (1.10) and (1.11) with $\rho = \rho_n$;

(1.38) $\{k_n\}_{n=1}^{+\infty}$ tends to an abstract function k in $W^{2,1}(0,T,H)$ weakly.

Thus

(1.39) $k_n(t) \rightharpoonup k(t)$ in H for all $t \in \langle 0,T\rangle$ -

see the fourth part of Remarks 1. From (1.39) it follows immediately that $k(0)=x$.

3. For every $\rho > 0$ denote $M(\rho) = \{y \in H; \|y-Py\| \leq \Lambda/\rho\}$.

(1.40) The set $M(\rho)$ is weakly closed

since it is convex and closed. Fix $p \in \mathbb{N}$. According to Lemma 2 for all positive integers $m > p$ and all $t \in \langle 0,T\rangle$

(1.41) $\|k_m(t)-Pk_m(t)\| \leq \Lambda/\rho_m \leq \Lambda/\rho_p$

and therefore $k_m(t) \in M(\rho_p)$. From (1.39), (1.40) and the last statement it follows that for all $t \in \langle 0,T\rangle : k(t) \in M(\rho_p)$ i.e. $\|k(t)-Pk(t)\| \leq \Lambda/\rho_p$. Passing to the limit ($p \to \infty$) in this inequality, we obtain that $k(t) \in K$.

4. Take an $\eta \in C(\langle 0,T\rangle,H)$ such that for all $t \in \langle 0,T\rangle : \eta(t) \in K$. In virtue of (1.36), (1.37) and (1.3) we have:

(1.42) $(k_n'(t)-\xi(t),\eta(t)) = -\rho_n\cdot(k_n(t)-Pk_n(t),\eta(t)) \geq 0$ for all $t \in \langle 0,T\rangle$ and $n \in \mathbb{N}$.

The facts introduced above imply that

(1.43) $(k'(t)-\xi(t),\eta(t)) \geq 0$ almost everywhere in $\langle 0,T\rangle$ -

see the sixth part of Remarks 1.

5. Making use of (1.36), (1.37) and (1.3) we have:

(1.44) $$\|k_n(t)\|^2 - \|x\|^2 - 2\cdot\int_0^t (\xi(\tau),k_n(\tau))d\tau = 2\cdot\int_0^t (k_n'(\tau) - \xi(\tau),k_n(\tau))d\tau = -2\cdot\rho_n \int_0^t (k_n(\tau)-Pk_n(\tau),k_n(\tau))d\tau = -2\cdot\rho_n \int_0^t \|k_n(\tau)-Pk_n(\tau)\|^2 d\tau \leq 0$$ for all $n \in \mathbb{N}$ and $t \in \langle 0,T\rangle$.

From (1.39), (1.44) and (1.38) it follows:

(1.45) $$\|k(t)\|^2 \doteq \liminf_{n\to\infty} \|k_n(t)\|^2 = \lim_{n\to\infty}[\|x\|^2 + 2\cdot\int_0^t (\xi(\tau), k_n(\tau))d\tau] = \|x\|^2 + 2\cdot\int_0^t (\xi(\tau),k(\tau))d\tau$$ on the interval $\langle 0,T\rangle$.

Hence $\int_0^t (k'(\tau)-\xi(\tau),k(\tau))d\tau \leq 0$. Owing to this fact and the validity of (1.43) with $\eta=k$ we have that $(k'(t)-\xi(t), k(t))=0$ for almost all $t\in\langle 0,T\rangle$. Finally, the subtraction of the last equation from the inequality (1.43) leads to the relation

(1.46) $(k'(t)-\xi(t),\eta(t)-k(t)) \geq 0$ almost everywhere in $\langle 0,T\rangle$.

Thus

(1.47) k solves (1.28), (1.29), (1.30) and (1.31).

6. Let $k_\rho \not\rightharpoonup k$ in $W^{2,1}(0,T,H)$. It is obvious that then there exist sequences $\{\hat\rho_n\}_{n=1}^{+\infty} \subset \mathbb{R}^1$ and $\{\hat k_n\}_{n=1}^{+\infty} \subset W^{2,1}(0,T,H)$ with the following properties:

(1.48) the sequence $\{\hat\rho_n\}_{n=1}^{+\infty}$ is positive, increasing and boundless;

(1.49) for all $n \in \mathbb{N}$: $\hat k_n$ solves the problem (1.9), (1.10) and (1.11);

(1.50) $\{\hat k_n\}_{n=1}^{+\infty}$ tends to an abstract function $\hat k \neq k$ in $W^{2,1}(0,T,H)$ weakly.

Repeating the procedure described above we obtain that

(1.51) $\hat k$ solves the problem (1.28), (1.29), (1.30) and (1.31).

The conjunction of the statements $k \neq \hat k$, (1.47) and (1.51) contradicts the uniqueness result.

<u>Lemma 4</u>: For i=1 and 2 let $\xi_i \in C(\langle 0,T\rangle,H)$ and $x_i \in K$. Denote by $k_i = \Phi_T(\xi_i,x_i)$ the solution of the problem (1.28), (1.29), (1.30) and (1.31) with $\xi=\xi_i$ and $x=x_i$ for i=1 and 2. Define the

function $f_T:\langle 0,\infty)\to \mathbb{R}^1$ as follows:

(1.52) $f_T(c)=c^{-1}\cdot(1-\exp(-2\cdot c\cdot T))$ for all $c\in(0,\infty)$,

(1.53) $f_T(0)=\lim_{c\to 0+} f_T(c)=2\cdot T$.

Then for every nonnegative number c

(1.54) $\sup\{\|k_1(t)-k_2(t)\|\cdot\exp(-c\cdot t);\ t\in\langle 0,T\rangle\}\leq$
$\leq f_T(c)\cdot\sup\{\|\xi_1(t)-\xi_2(t)\|\cdot\exp(-c\cdot t);\ t\in\langle 0,T\rangle\}+$
$+\|x_1-x_2\|.$

Thus, the mapping $\Phi_T:C(\langle 0,T\rangle,H)\times K\to C(\langle 0,T\rangle,H)$ is Lipschitz continuous.

Proof: For the sake of brevity let us write k,ξ and x instead of k_1-k_2, $\xi_1-\xi_2$ and x_1-x_2, respectively. For almost every $t\in\langle 0,T\rangle$

(1.55) $(k_1'(t)-\xi_1(t),-k(t))\geq 0$ and $(k_2'(t)-\xi_2(t),k(t))\geq 0$.

Adding these inequations we obtain

(1.56) $(k'(t)-\xi(t),k(t))\leq 0$ and thus $(k'(t),k(t))\leq(\xi(t),$
$k(t))$ almost everywhere in $\langle 0,T\rangle$.

Hence we have:

(1.57) $\|k(t)\|^2=\|k(0)\|^2+2\cdot\int_0^t(k'(\tau),k(\tau))d\tau\leq\|x\|^2+2\cdot\int_0^t(\xi(\tau),$
$k(\tau))\cdot\exp(-2\cdot c\cdot\tau)\ \exp(2\cdot c\cdot\tau)d\tau\leq\|x\|^2+2\cdot\sup\{\|\xi(\tau)\|\cdot$
$\cdot\exp(-c\cdot\tau);\ \tau\in\langle 0,T\rangle\}\cdot\sup\{\|k(\tau)\|\cdot\exp(-c\cdot\tau);\ \tau\in\langle 0,T\rangle\}\cdot$
$\cdot\int_0^t\exp(2\cdot c\cdot z)dz=\|x\|^2+\sup\{\|\xi(\tau)\|\cdot\exp(-c\cdot\tau);\ \tau\in\langle 0,T\rangle\}\cdot$
$\cdot\sup\{\|k(\tau)\|\cdot\exp(-c\cdot\tau);\ \tau\in\langle 0,T\rangle\}\cdot\exp(2\cdot c\ t)\cdot f_t(c)$
for all $t\in\langle 0,T\rangle$ and $c\geq 0$.

Accordingly:

(1.58) $[\sup\{\|k(t)\|\cdot\exp(-c\cdot t);\ t\in\langle 0,T\rangle\}]^2\leq\|x\|^2+\sup\{\|\xi(t)\|\cdot$
$\cdot\exp(-c\cdot t);\ t\in\langle 0,T\rangle\}\cdot\sup\{\|k(t)\|\cdot\exp(-c\cdot t);\ t\in\langle 0,T\rangle\}\cdot$
$\cdot f_T(c)\leq\sup\{\|k(t)\|\cdot\exp(-c\cdot t);\ t\in\langle 0,T\rangle\}\cdot[\|x\|+f_T(c)\cdot$
$\cdot\sup\{\|\xi(t)\|\cdot\exp(-c\cdot t);\ t\in\langle 0,T\rangle\}$.

In what follows, D is the operator defined on H as:

(1.59) $D(x)=Ax-(Ax,x)\cdot x/\|x\|^2$ for all $x\in H\setminus\{0\}$ and $D(0)=0$.

Obviously, Dx is the orthogonal projection Ax on $[\mathcal{L}\{x\}]^{\perp}$

and thus

(1.60) $(Dx,x)=0$ for all $x \in H$.

Further, D_T will denote the operator given on $C(\langle 0,T\rangle,H)$ as:

(1.61) $(D_T k)(t)=D(k(t))$ for all $t \in \langle 0,T\rangle$.

Both the operators D and D_T are continuous.

Lemma 5: Let $x \in K$. Then there exists the unique abstract function

(1.62) $k \in W^{2,1}(0,T,H)$

satisfying the conditions:

(1.63) $(k'(t)-Dk(t),y-k(t)) \geq 0$ for all $y \in K$ and almost all $t \in \langle 0,T\rangle$,

(1.64) $k(0)=x$,

(1.65) $k(t) \in K$ for all $t \in \langle 0,T\rangle$.

Proof: 1. The auxiliary result -

(1.66) $\|Dx-Dy\| \leq 6\|A\| \cdot \|x-y\|$ for all $x,y \in H$ -

will be proved only under the additional conditions $x \neq 0$, $y \neq 0$. (The proof for the remaining cases is very simple.) Without loss of generality we may suppose that $\|y\| \leq \|x\|$. Obviously

(1.67) $$Dx-Dy=A(x-y)-(A(x-y),x)\cdot x/\|x\|^2 -$$
$$-(Ay,x-y)\cdot x/\|x\|^2-(Ay,y)\cdot(x-y)/\|x\|^2 -$$
$$-(Ay,y)\cdot y\cdot(\|y\| - \|x\|)(\|y\| + \|x\|)/(\|x\|^2\cdot\|y\|^2).$$

Hence:

(1.68) $$\|Dx-Dy\| \leq \|A\|\cdot\|x-y\| + \|A\|\cdot\|x-y\|\cdot\|x\|^2/\|x\|^2 +$$
$$+ \|A\|\cdot\|y\|\cdot\|x-y\|\cdot\|x\|/\|x\|^2 + \|A\|\cdot\|y\|^2\cdot\|x-y\|/\|x\|^2 +$$
$$+ \|A\|\cdot\|y\|^3\cdot\|y-x\|\cdot(\|y\| + \|x\|)/(\|x\|^2\|y\|^2) \leq 6\cdot\|A\|\cdot\|x-y\|.$$

2. Define the operator $E_T : C(\langle 0,T\rangle,H) \longrightarrow C(\langle 0,T\rangle,H)$ by the formula:

(1.69) $E_T = \Phi_T(\cdot,x) \circ D_T$.

In virtue of Lemma 4 and the estimate (1.66) we have:

(1.70) $$\sup\{\|(E_T k_1)(t)-(E_T k_2)(t)\|\cdot\exp(-12\cdot\|A\|\cdot t);\ t \in \langle 0,T\rangle\} \leq$$
$$\leq f_T(12\cdot\|A\|)\cdot\sup\{\|(D_T k_1)(t)-(D_T k_2)(t)\|\cdot\exp(-12\cdot\|A\|\cdot t);$$

$t \in \langle 0,T\rangle\} \leq 6 \cdot \|A\| \cdot f_T(12 \cdot \|A\|) \cdot \sup \{\|k_1(t)-k_2(t)\| \cdot$
$\cdot \exp(-12 \cdot \|A\| \cdot t);\ t \in \langle 0,T\rangle\} \leq \sup \{\|k_1(t)-k_2(t)\| \cdot$
$\cdot \exp(-12 \cdot \|A\| \cdot t);\ t \in \langle 0,T\rangle\}/2.$

According to the Banach fixed point theorem there exists the unique $k \in C(\langle 0,T\rangle,H)$ such that $k=E_T k$. It is easily seen that the last equation and the problem (1.62), (1.63), (1.64) and (1.65) are equivalent.

From Lemma 5 it follows immediately:

<u>Lemma 6</u>: For every $x \in K$ there exists the unique abstract function

(1.71) $k:\langle 0,\infty) \longrightarrow K$

such that

(1.72) $k/\langle 0,t\rangle \in W^{2,1}(0,t,H)$ for all $t \in (0,\infty)$,

(1.73) $(k'(t)-Dk(t),y-k(t)) \geq 0$ for all $y \in K$ and almost all $t \in \langle 0,\infty)$,

(1.74) $k(0)=x$.

With help of Lemma 4, the estimate (1.66) and elementary ε, δ-considerations, the following result can be readily derived:

<u>Lemma 7</u>: The mapping $k:(t,x) \in \langle 0,\infty) \times K \longmapsto k(t,x) \in K$, where for every $x \in K$, $k(\cdot,x)$ denotes the solution of the problem (1.71), (1.72) and (1.73) acquiring the value x at the point t=0, is continuous.

<u>Lemma 8:</u> Let $x \in K$ and let k be the solution of the problem (1.71), (1.72), (1.73) and (1.74). Then:

(1.75) $\|k(t)\| = \|x\|$ for all $t \in \langle 0,\infty)$,

(1.76) $\|k'(t)\|^2=(k'(t),Dk(t))=(k'(t),Ak(t))$ for almost all $t \in \langle 0,\infty)$.

Moreover,

(1.77) if A is a selfadjoint operator then $(Ak(t),k(t)) \geq$
$\geq (A\varkappa,x)$ for all $t \in \langle 0,\infty)$ and $\int_0^{+\infty} \|k'(t)\|^2 dt < +\infty$.

Proof: The condition (1.73) may be also expressed as follows:

(1.78) there exists a set $M \subset \langle 0,\infty)$ of measure zero such that $(k'(t)-Dk(t),\eta(t)-k(t)) \geq 0$ for all continuous functions $\eta:\langle 0,\infty) \longrightarrow K$ and all $t \in \langle 0,\infty) \setminus M$.

1. Inserting $\eta=2\cdot k$ and $\eta=k/2$ into the inequality (1.78) we get:

(1.79) $(k'(t)-Dk(t),k(t))=0$ for almost all $t\in\langle 0,\infty)$. From (1.60) and the last equation it follows:

(1.80) $(k'(t),k(t))=0$ almost everywhere in $\langle 0,\infty)$.

Hence

(1.81) $\|k(t)\|^2-\|x\|^2=2\cdot\int_0^t(k'(\tau),k(\tau))d\tau=0$ on $\langle 0,\infty)$.

2. Let us extend the abstract function k on the whole real axis as follows:

(1.82) $k(t)=k(0)(=x)$ for all $t\in(-\infty,0)$.

Put:

(1.83) $\hat{M}=\{t\in\langle 0,\infty);\ \text{non}\ [\lim_{h\to 0}(h^{-1}\cdot(k(t+h)-k(t)))=k'(t)]\}$
$\cup\ M$.

Because $k'\in L^2(0,T,H)$ and meas(M)=0, we have that

(1.84) $\text{meas}(\hat{M})=0$.

Thus, for all $t\in\langle 0,\infty)\setminus\hat{M}$

(1.85) $(k'(t)-Dk(t),k'(t))=\lim_{h\to 0+}(k'(t)-Dk(t),h^{-1}\cdot(k(t+h)-$
$-k(t)))\geq 0$

and at the same time

(1.86) $(k'(t)-Dk(t),k'(t))=\lim_{h\to 0-}(k'(t)-Dk(t),h^{-1}\cdot(k(t+h)-$
$-k(t)))\leq 0$.

The inequalities (1.85) and (1.86) imply that

(1.87) $\|k'(t)\|^2=(Dk(t),k'(t))$ almost everywhere in $\langle 0,\infty)$.

The validity of the equality $(Dk(t),k'(t))=(Ak(t),k'(t))$ for almost all $t\in\langle 0,\infty)$ can be verified by a simple account which makes use of (1.80).

3. Owing to the symmetry of A and (1.76)

(1.88) for every $t\in\langle 0,\infty)$: $(Ak(t),k(t))-(Ax,x)=2\cdot\int_0^t\|k'(\tau)\|^2d\tau$.

Furthermore, the expression $(Ak(t),k(t))-(Ax,x)$ is bounded by $2\cdot\|A\|\cdot\|x\|^2$ independently of t.

2. Proof of Theorem 1. We start from a simple auxiliary assertion which will be useful in our proof of Theorem 1.

Lemma 9: Let $\wp$ be a positive number. Suppose that sequences of elements from H- $\{x_n\}_{n=1}^{+\infty}$ and $\{y_n\}_{n=1}^{+\infty}$ - and elements y and z of H satisfy the following requirements:

(2.1) $\{x_n\}_{n=1}^{+\infty}$ tends weakly to the zero element of H,

(2.2) $\{y_n\}_{n=1}^{+\infty} \subset K \cap S(0,\wp)$ +),

(2.3) $\{y_n\}_{n=1}^{+\infty}$ tends weakly to y,

(2.4) $\{Ay_n\}_{n=1}^{+\infty}$ tends strongly to z,

(2.5) $(y,z) > 0$,

(2.6) $(x_n - Dy_n, v - y_n) \geq 0$ for every $n \in \mathbb{N}$ and every $v \in K$.

Then $y \in K$, $\|y\| = \wp$, $\{y_n\}_{n=1}^{+\infty}$ tends strongly to y, z=Ay and

(2.7) $(\lambda \cdot y - Ay, v - y) \geq 0$ for all $v \in K$,

where

(2.8) $\lambda = \wp^{-2} \cdot (z,y)$.

Proof: Since K is a weakly closed set, the weak limit of the sequence $\{y_n\}_{n=1}^{+\infty} \subset K$ - i.e. the element y - belongs to K. Putting $v = y + y_n$ into the inequality (2.6) we obtain:

(2.9) $0 \leq (x_n - Dy_n, y) = (x_n, y) - (Ay_n, y) + (Ay_n, y_n) \cdot (y_n, y)/\|y_n\|^2$.

Passing to the limit in the last relation we have:

(2.10) $0 \leq -(z,y) + (z,y) \cdot \|y\|^2/\wp^2$.

From (2.10) and (2.5) we get immediately: $\|y\| \geq \wp$. However (2.3) implies that $\|y\| \leq \liminf_{n \to \infty} \|y_n\| = \wp$ and hence $\|y\| = \wp$. From the facts $y_n \rightharpoonup y$ and $\|y_n\| \to \|y\|$ it follows that $y_n \to y$. Hence owing to the continuity of A we have: $Ay_n \to Ay = z$. Thus $\lambda = (Ay,y)/\|y\|^2$.

Finally, for all $v \in K$

(2.11) $(\lambda \cdot y - Ay, v - y) = (Ay,y) \cdot (y, v-y)/\|y\|^2 - (Ay, v-y) =$

$= \lim_{n \to \infty} [(x_n, v-y_n) + (Ay_n, y_n)(y_n, v-y_n)/\|y_n\|^2 - (Ay_n, v-y_n)] =$

$= \lim_{n \to \infty} (x_n - Dy_n, v - y_n) \geq 0$.

In what follows, we use the following notations:

1. A is a linear, selfadjoint and completely continuous operator.

+) $a \in H$, $b > 0$ $S(a,b) = \{x \in H; \|x - a\| = b\}$

2. $\{\lambda_n\}_{n=1}^{p}$ ($p \in \mathbb{N} \cup \{+\infty\}$) is the nonincreasing sequence containing all positive eigenvalues of A.
3. $\{u_n\}_{n=1}^{p}$ is an orthonormal system in H; for all $n \in \mathbb{N}$, $n \leq p$, u_n is an eigenvector of A corresponding to the eigenvalue λ_n.

Definition 1: Let R be a metric space.
1. Let $M_1, M_2 \subset R$. Suppose that a continuous mapping $f: M_1 \times \langle 0,1\rangle \to R$ such that $f(x,0)=x$ for all $x \in M_1$ and $f(M_1,1)=M_2$ exists. Then we say that the set M_2 is a continuous deformation of the set M_1 within R.
2. Let $M \subset R$. We say that the set M is contractible within R if there exists an $a \in R$ such that the set $\{a\}$ is a continuous deformation of the set M within R.

The basic properties of the notions defined above are summarized for example in [9].

Proof of Theorem 1: Let m be a positive integer such that $\lambda_{m-1} = \nu$ and $\lambda_m = \mu$ - thus

(2.12) $\lambda_i > \lambda_m$ for all $i=1,2,\dots,m-1$.

1. Further the following notations will be used:

(2.13) $H_1 = \mathcal{L}(\{u_1, u_2, \dots, u_{m-1}\})$,

(2.14) P_1 is the orthogonal projection H onto H_1,

(2.15) $R = \{z \in H;\ P_1 z \neq 0\}$.

Suppose that $u_m \in \operatorname{int}(K)$. Then there exists a $\delta > 0$ such that $S(u_m, \delta) \subset K$. Put:

(2.16) $F = \{(1+\delta^2)^{-1/2} \cdot (u_m + \sum_{i=1}^{m-1} \alpha_i \cdot u_i);\ \alpha_i \in \mathbb{R}^1$ for $i=1,2,\dots, m-1,\ \sum_{i=1}^{m-1} \alpha_i^2 = \delta^2\}$.

Obviously:

(2.17) $F \subset K \cap S(0,1) \cap R$.

A simple account using among others (2.12) yields:

(2.18) $(Ax,x) > \lambda_m$ for all $x \in F$.

2. It will be shown that

(2.19) the set F is not contractible within R.

According to Lemma 9 from [9]

(2.20) the set $P=S(0,\delta\cdot(1+\delta^2)^{-1/2})\cap H_1$ is not contractible within R.

Furthermore,

(2.21) the set F is a continuous deformation of the set P within R.

The deformation mapping can be given on $\langle 0,1\rangle\times P$ as: $f(t,x)=$ $=x+t\cdot(1+\delta^2)^{-1/2}\cdot u_m$. From (2.20) and (2.21) it follows (2.19) in virtue of Lemma 8 from [9].

3. Further we shall prove that

(2.22) for all $t\in(0,\infty)$ the set $k(t,F)$ is not contractible within R.

(For the definition of the symbol $k(\cdot,\cdot)$ see Lemma 7.) Fix $x\in F$ and $t\in(0,\infty)$. Denote $k=k(\cdot,x)$. According to Lemma 8

(2.23) $(Ak(t),k(t))\geq(Ax,x)$.

From (2.23), (2.18), (2.17) and the first part of Lemma 8 it follows:

(2.24) $(Ak(t),k(t))>\lambda_m\cdot\|k(t)\|^2=\lambda_m\cdot\|P_1k(t)\|^2+\lambda_m\cdot\|(I-P_1)k(t)\|^2$.

Furthermore,

(2.25) $(Ak(t),k(t))=(AP_1k(t),P_1k(t))+(A(I-P_1)k(t),(I-P_1)k(t))\leq$
$\leq\lambda_1\cdot\|P_1k(t)\|^2+\lambda_m\cdot\|(I-P_1)k(t)\|^2$.

Finally, comparing the estimates (2.24) and (2.25) we get that

(2.26) $\|P_1k(t)\|^2>0$ i.e. $P_1k(t)\neq 0$.

Now it is readily seen that

(2.27) $k(t,F)$ is a continuous deformation of F within R for all $t\in(0,\infty)$ -

the deformation is realized by the mapping $k(\cdot,\cdot)/\langle 0,t\rangle\times F$.
From (2.19) and (2.27) it follows (2.22).

4. Let us prove that

(2.28) for all $t\in(0,\infty)$ an $x_t\in F$ such that $k(t,x_t)\in\mathcal{L}(\{u_{m-1}\})+$ $+H_1^{\perp}$ has to exist.

Suppose that for a $t\in(0,\infty)$ the set $k(t,F)\cap(\mathcal{L}(\{u_{m-1}\})+H_1^{\perp})$ is empty. Hence the set $P_1k(t,F)\cap\mathcal{L}(\{u_{m-1}\})$ is also empty. This fact implies that the set $k(t,F)$ is contractible within R (see [9], Lemma 9), which contradicts (2.22).

5. Choose an increasing and boundless sequence of positive

numbers $\{t_n\}_{n=1}^{+\infty}$. For all $n \in \mathbb{N}$ let $x_n \in F$ and $k(t_n,x_n) \in$ $\in \mathcal{L}(\{u_{m-1}\})+H\frac{1}{l}$. Because the set F is compact, without loss of generality it may be supposed that the sequence $\{x_n\}_{n=1}^{+\infty}$ converges to an $x \in F$. According to Lemma 7 for every positive number t the sequence $\{k(t,x_n)\}_{n=1}^{+\infty}$ tends to $k(t,x)$ in H. For the sake of brevity let us write $\hat{k}$ instead of $k(\cdot,x)$. The abstract function $\hat{k}$ fulfils the condition (1.73) and thus:

(2.29) $(\hat{k}'(t)-D\hat{k}(t),v-\hat{k}(t)) \geq 0$ for all $t \in \langle 0,\infty) \setminus M$ and all $v \in K$,

where

(2.30) $M \subset \langle 0,\infty)$, $\text{meas}(M)=0$.

According to Lemma 8 $\int_0^{+\infty} \| \hat{k}'(t)\|^2 dt < +\infty$. In virtue of (2.30) and the last statement we have that

(2.31) $\text{meas}(N_n = \{t \in \langle 0,\infty); \text{non}\, [\|\hat{k}'(t)\| \leq 1/n]\} \cup M) < +\infty$ for all $n \in \mathbb{N}$.

Now let us construct a numeral sequence $\{\hat{t}_n\}_{n=1}^{+\infty}$ in the following way:

1. Put $\hat{t}_0=1$
2. For $n \geq 1$ put

(2.32) $A_n = \langle \hat{t}_{n-1}+1,\infty) \setminus N_n$.

According to (2.31) $A_n \neq \emptyset$. Choose an arbitrary element of A_n and denote it by $\hat{t}_n$.

The sequence $\{\hat{t}_n\}_{n=1}^{+\infty}$ is increasing and boundless. Since for all $n \in \mathbb{N}: \hat{t}_n \notin N_n$ i.e. $\|\hat{k}'(\hat{t}_n)\| \leq 1/n$,

(2.33) the sequence $\{\hat{k}'(\hat{t}_n)\}_{n=1}^{+\infty}$ tends to the zero element of H.

According to the first part of Lemma 8

(2.34) for all $n \in \mathbb{N}: \|\hat{k}(\hat{t}_n)\| = \|x\| = 1$.

Owing to this fact and the complete continuity of A

(2.35) there exists a sequence $\{\tau_n\}_{n=1}^{+\infty}$ chosen from $\{\hat{t}_n\}_{n=1}^{+\infty}$ such that $\{\hat{k}(\tau_n)\}_{n=1}^{+\infty}$ converges weakly in H - to some y - and $\{A\hat{k}(\tau_n)\}_{n=1}^{+\infty}$ converges strongly in H to Ay.

Further by virtue of (2.35), (1.77) and (2.18) we have:

(2.36) $(Ay,y) = \lim_{m\to\infty} (A\hat{k}(\tau_n),\hat{k}(\tau_n)) \geq (Ax,x) > \lambda_m = \mu > 0$.

Finally, for all $n \in \mathbb{N}$: $\tau_n \notin M$ which guarantees that

(2.37) $(\hat{k}'(\tau_n)-D\hat{k}(\tau_n), v-\hat{k}(\tau_n)) \geq 0$ for every $n \in \mathbb{N}$ and $v \in K$ - see (2.29).

The validity of the assertions (2.33), (2.34), (2.35), (2.36) and (2.37) makes it possible to use Lemma 9 for the sequences $\{\hat{k}'(\tau_n)\}_{n=1}^{+\infty}$ and $\{\hat{k}(\tau_n)\}_{n=1}^{+\infty}$. The application of Lemma 9 mentioned above leads to the conclusion which reads:

(2.38) $y \in K \cap S(0,1)$, $\{\hat{k}(\tau_n)\}_{n=1}^{+\infty}$ tends strongly to y and for all $v \in K$: $(\lambda \cdot y - Ay, v-y) \geq 0$, where $\lambda = (Ay,y)$.

Thus, according to (2.36)

(2.39) $\lambda > \lambda_m = \mu$.

6. It remains to prove that

(2.40) $\lambda \leq \lambda_{m-1} = \nu$.

Consider the sequences $\{t_n\}_{n=1}^{+\infty}$, $\{\tau_n\}_{n=1}^{+\infty}$ and $\{x_n\}_{n=1}^{+\infty}$ defined in the foregoing part of the proof. Fix $p \in \mathbb{N}$ and $\varepsilon > 0$. Since $\{k(\tau_p, x_n)\}_{n=1}^{+\infty}$ tends to $\hat{k}(\tau_p)$ and A is a continuous operator, $\{(Ak(\tau_p,x_n), k(\tau_p,x_n))\}_{n=1}^{+\infty}$ tends to $(A\hat{k}(\tau_p), \hat{k}(\tau_p))$. Thus, there exists an $n_0 = n_0(\varepsilon, p) \in \mathbb{N}$ such that for all positive integers $n \geq n_0$:

(2.41) $(A\hat{k}(\tau_p), \hat{k}(\tau_p)) \leq (Ak(\tau_p,x_n), k(\tau_p,x_n)) + \varepsilon$.

Furthermore, because $\lim_{n \to \infty} t_n = \infty$, a positive integer $n_1 = n_1(\varepsilon,p) \geq n_0(\varepsilon,p)$ such that $t_{n_1} > \tau_p$ has to exist. Obviously:

(2.42) $$(Ak(\tau_p,x_{n_1}), k(\tau_p,x_{n_1})) = (Ak(t_{n_1},x_{n_1}), k(t_{n_1},x_{n_1})) - 2\int_{\tau_p}^{t_{n_1}} \|k'(\tau,x_{n_1})\|^2 d\tau \leq (Ak(t_{n_1},x_{n_1}), k(t_{n_1},x_{n_1})).$$

Finally, the fact $k(t_{n_1},x_{n_1}) \in (\mathcal{L}\{u_{m-1}\} + H_1^{\perp}) \cap S(0,1)$ implies:

(2.43) $(Ak(t_{n_1},x_{n_1}), k(t_{n_1},x_{n_1})) \leq \lambda_{m-1}$.

From the relations (2.41) with $n=n_1$, (2.42) and (2.43) it follows:

(2.44) for all $\varepsilon > 0$ and all $p \in \mathbb{N}: (A\hat{k}(\tau_p), \hat{k}(\tau_p)) \leq \lambda_{m-1} + \varepsilon$.

Passing to the limit ($p \to \infty$ and $\varepsilon \to 0+$) in the last estimate we obtain (2.40). The proof is finished.

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