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ENDOMORPHIC CUTS AND TAILS

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Abstract: We consider two special kinds of cuts arisen in the Alternative Set Theory, namely a) the traces of transitive endomorphic universes on N (called here "endomorphic cuts") and b) cuts of the form $E_A(FN)$ (which we call "tails") part of which are the standard extensions of FN. We find conditions under which a pair of numbers can be separated by such cuts utilizing some old relations due to C. Puritz.

Key words: Alternative Set Theory, endomorphic universe, standard extension, cut (initial segment) of the class of natural numbers.

Classification: 03E70, 03H99

We assume familiarity of the reader with the basic Alternative Set Theory and the theory of cuts as exposed in [V] and [K-P] respectively. Knowledge of the fundamental paper [S-V] is also presupposed.

The paper is divided into three sections. In Section 0 we prove or just recall some technical results used in the following sections. In Section 1 we consider endomorphic cuts, while in Section 2 we study tails.

As usual, N, FN, are the classes of natural and finite natural numbers respectively, m, n,... denote elements of FN and $\alpha, \beta, \gamma, \dots$ denote elements of N. Letters I, J represent cuts of N and A, B represent endomorphic universe. The letters F, G, f, g are reserved for functions, the kind of which (endomorphisms, set-definable etc.) is specified in each case $\square$ signals the end of a proof.

§ 0. Preliminaries. In this section we prove or recall some technical results needed in the sequel.

Lemma 0.1. A class A is an endomorphic universe (e.u.) iff it has the following two properties:

i) For any set-formula $\varphi(x)$ of the language FL_A, $(\exists x)\varphi(x) \rightarrow (\exists x \in A)\varphi(x)$.

ii) For every countable $F \subseteq A$ there is an $f \in A$ such that $F \subseteq f$.

Proof. Cf. [S-V]. □

Let A be an e.u. An operation $Ex_A(X)$ defined for all classes $X \subseteq A$ is called a standard extension on A, if for any normal formula $\varphi(Z_1,\dots,Z_n)$ of FL_A and each sequence $X_1,\dots,X_n \subseteq A$,

$$\varphi^A(X_1,\dots,X_n) \equiv \varphi(Ex_A(X_1),\dots,Ex_A(X_n)).$$

If such an operation exists on A, then it is unique. We usually write Ex(X) instead of $Ex_A(X)$.

It is shown in [S-V] that an e.u. A has a standard extension iff

$(\forall x)(\exists \text{ countable } X \subseteq A)(\forall y \in A)(X \subseteq y \rightarrow x \in y)$.

We shall write e.u.s. to abbreviate the expression "endomorphic universe with standard extension".

Recall that an e.u. A is maximal (see [T]) if there is no e.u. B such that $A \subsetneq B \subsetneq V$.

Lemma 0.2. 1) If F is an automorphism then:

i) A is an e.u. $\rightarrow$ F"A is an e.u.

ii) A is an e.u.s. $\rightarrow$ F"A is an e.u.s.

iii) A is a maximal e.u. $\rightarrow$ F"A is a maximal e.u.

2) If A is an e.u.s. and $F \subseteq A$ is an automorphism in the sense of A, then Ex(F) is an automorphism.

Proof. 1) Check that the properties in the r.h.s. of the implications are preserved by automorphisms.

2) Similarly check that the property to be an automorphism is preserved by the operation Ex. (For details cf. [Ve] or [Č-V_j]). □

Given an e.u.A, we put for every countable $X \subseteq A$, $E_A(X) = \bigcap\{x \in A; X \subseteq x\}$. In particular, $E_A(FN) = \bigcap\{\alpha \in N \cap A; \alpha > FN\}$. If A is an e.u.s., then $E_A(X) = Ex(X)$.

Lemma 0.3. For any e.u. A and for any x there is an automorphism F such that $x \in F''A$.

Proof. It suffices to prove that there is some $y \in A$ such that $x \stackrel{\circ}{=} y$ (see [V], chapter V). Let $\varphi_n(t)$ be the sequence of all set formulas without parameters such that $\varphi_n(x)$. By Lemma 0.1 there is a sequence $\{a_i; i \in FN\}$ such that $\varphi_1(a_i) \& \ldots \& \varphi_i(a_i)$ for all $i \in FN$. Then by Lemma 0.1 again we easily get that there is $y \in A$ such that $\varphi_i(y)$ for every $i \in FN$, that is, $y \stackrel{\circ}{=} x$. □

Lemma 0.4. If A is an e.u. such that $E_A(FN) \neq FN$ and α, γ are such that $\gamma \in A$ and $\alpha \in \bigcap((Def(\{\gamma\}) \cap N)-FN)$, then there is an automorphism F such that $F(\gamma)=\gamma$ and $\alpha \in E_{F''A}(FN)$. (We use the convention $\bigcap \emptyset = V$.)

Proof. It suffices to find $\beta \in E_A(FN)$ such that $\alpha \doteq_{\{\gamma\}} \beta$. If $\alpha \in FN$, the property holds. Since $E_A(FN) \neq FN$ and $E_A(FN)$ is a cut, it suffices to show that the monad μ of α with respect to $\doteq_{\{\gamma\}}$ is coinitial to FN. Suppose not and σ is such that $FN < \sigma <$ $< \mu$. Then we could choose this σ from $Def(\{\gamma\})$ since μ is the intersection of classes definable with the parameter γ. Thus $\alpha \in \sigma$, a contradiction. □

Remark. Since there is a set-definable bijection $F: N \longrightarrow V$ (see [V]) and $F''FN=FV$ (=hereditarily finite sets), the preceding lemma holds as well with FV in the place of FN.

Theorem 0.5. Let $\alpha \in \bigcap((Def(\{\gamma\}) \cap N)-FN)$ and $\gamma \in (\bigcap((Def \cap N)-FN))-FN$. If A is an e.u., B is an e.u.s. and $A \subsetneqq B$, then there is an automorphism F such that $\alpha, \gamma \in E_{F''A}(FN)$, $\gamma \in F''B$ and $\alpha \in Ex_{F''B}(FN)$.

Proof. First we "shift" γ by a suitable automorphism (see Lemma 0.3) into the larger e.u., then we "shift" γ to the extension of FN of the smaller e.u. by working in the framework of the larger e.u. and using Lemma 0.4. We obtain the same shift if we use the standard extension of the relative automorphism. Finally, we "shift" α into the extension of FN of the larger e.u. The required automorphism F is, then, taken by a suitable composition of the automorphisms used in the previous steps. □

§ 1. Endomorphic cuts. A cut is said to be endomorphic if $I=A\cap N$ for some transitive endomorphic universe A.

From Lemma 0.1 as well as from the fact that every transitive e.u. is revealed we get easily the following:

Proposition 1.1. A cut I is endomorphic iff it is revealed and for every nonempty $X\subseteq N$ such that $X\in Sd_I$ (set-definable with parameters in I) $X\cap I\neq\emptyset$. □

Theorem 1.2. For $\alpha,\beta\in N$ the following properties are equivalent:

1) $Def(\{\alpha\})\cap N\subseteq\beta$
2) $(\forall F\in Sd_0$ strictly increasing$)(F''\alpha\subseteq\beta)$
3) $(\forall F\in Sd_0)(F''\alpha\subseteq\beta)$
4) $(\forall n\in FN)(\forall F\in Sd_0,$ F n-ary$)(F''\alpha^n\subseteq\beta)$
5) $(\exists$ transitive e.u. $A)(\alpha\in A\cap N\subseteq\beta)$

Proof. The implications 5) $\rightarrow$ 4) $\rightarrow$ 3) $\rightarrow$ 2) $\rightarrow$ 1) are obvious. We show 1) $\rightarrow$ 5). Let $Def(\{\alpha\})\cap N\subseteq\beta$. By Lemma 0.3 we can find e.u.s. B such that $\langle\alpha,\beta\rangle\in B$. Then $Ex(Def(\{\alpha\}))$ is a fully revealed class closed on the definitions by normal formulas, hence it is an e.u.. The transitive closure of this universe is, clearly, the required universe A. □

Definition 1.3. Let $\alpha,\beta\in N$. We say that β is much greater than α and write $\alpha<<\beta$ if α,β have some, and hence all, of the properties of Theorem 1.2.

We write also $\alpha\sim\beta$ for the fact that neither $\alpha<<\beta$ nor $\beta<<\alpha$. $\sim$ is an equivalence relation and we call skies sk( ), the equivalence classes of $\sim$.

These notions are due to C. Puritz and we keep his suggestive terminology here.

Let us note that for any $m,n\in FN$, $FN\subseteq sk(n)=sk(m)$ and $FN\subsetneq sk(n)$ iff $Def\neq FV$ (Def contains infinite natural numbers).

By Theorem 1.2.1) we have that $\alpha\sim\beta$ iff $\bigcup(Def(\{\alpha\})=$ $=\bigcup(Def(\{\beta\})$ and this characterization gives the following.

Proposition 1.4. $\alpha\sim\beta$ is a Σ-equivalence hence $sk(\alpha)$ is a Σ-class. □

Lemma 1.5 (Puritz). Let $F\in Sd_0$ such that for every α,

$F^{-1}(\alpha)$ is a set. Then $F(\alpha) \sim \alpha$.

Proof. If $\alpha \leqq F(\alpha)$, the claim is obvious. If $F(\alpha) < \alpha$, put $G(x)=\max(F^{-1}"\{x\})$ or $G(x)=0$ if $F^{-1}"\{x\} = \emptyset$. Then, we see that $GF(\alpha) \geqq \alpha$. □

Lemma 1.6. Let $\beta > \alpha$. If $\beta \sim \alpha$, then there is a G such that $G^{-1}(x)$ are sets and $G(\alpha) \leqq \beta$.

Proof. Since $\beta \sim \alpha$, there is $F \in Sd_0$ strictly increasing such that $F(\beta) \geqq \alpha$ (see Theorem 1.2.2). Now put

$$G(x) = \text{the least } y \text{ such that } F(y) \geqq x. \ \square$$

Corollary 1. Let $\mathcal{F}_0$ be the class of functions $F \in Sd_0$ such that $F: N \to N$ and all the fibres of F are sets. Then, for any $\alpha \in N$ the countable class $\{F(\alpha); F \in \mathcal{F}_0\}$ is cofinal and coinitial in $Sk(\alpha)$. □

Corollary 2. i) Every sky is a Σ-class of the form $\bigcup\{[\alpha_n, \beta_n]; n \in FN\}$ where $\{\alpha_n; n \in FN\}$ is decreasing and $\{\beta_n; n \in FN\}$ is strictly decreasing.

ii) The natural ordering of Skies is dense with first element Sk(n) but without last element.

iii) There are arbitrarily long endomorphic cuts. On the other hand, there are arbitrarily short endomorphic cuts iff Def=FV.

Proof. i) See Corollary 1 ii). Two disjoint Σ-classes can be separated by a set-definable class. iii) See Theorem 1.1.1), 5). □

Proposition 1.7. Given $\alpha \in N$, let $I_\alpha = \{\beta; \beta << \alpha\}$. If $I_\alpha \neq \emptyset$ then I_α is the greatest endomorphic cut not containing α and it is a Π-cut. On the other hand, there is no least endomorphic cut containing α.

Proof. If $I_\alpha \neq \emptyset$ then I_α is a Π-cut closed with respect to set-definitions, hence endomorphic. On the other hand $Sk(\alpha)$ is a Σ-class and $\bigcap\{I \text{ endomorphic}; \alpha \in I\}$ is cofinal to $sk(\alpha)$. □

Proposition 1.8. Let $\mathcal{J}$ be a bounded (with respect to $\subseteq$)

class of endomorphic cuts not containing any countable cofinal subclass. Then, $\bigcup \mathcal{J}$ is an endomorphic cut.

Proof. $\bigcup \mathcal{J}$ is revealed, closed w.r.t. set-definitions and transitive.

§ 2. Tails. Given an e.u. A let us call tail of A the class $E_A(FN) = \bigcap\{\alpha > FN; \alpha \in A\}$. Obviously every tail is a cut. If the universe A has a standard extension then the operation E_A is identical to Ex_A and the tail of A is just the standard extension of FN. For every A we have $E_A(FN) \cap A = FN$. There are universes such that $E_A(FN) = FN$ (e.g. the transitive ones). It is evident that we are interested in A such that $FN \subsetneq E_A(FN)$. Such tails will be called proper (e.g. every standard extension of FN is a proper tail).

Recall that a cut is semi-regular if for every $\alpha \in I$ and every f, $f''\alpha$ is not cofinal in I. A cut is strong if it is semi-regular and for every f there is a $\beta > I$ such that $(\forall \alpha \in I)(f(\alpha) \in I \vee f(\alpha) > \beta)$. The importance of strong cuts lies in the fact that they are models of Peano arithmetic.

Proposition 2.1. Ex(FN) is a strong cut.

Proof. Suppose Ex(FN) is not semi-regular. Then

$$(\exists \alpha \in Ex(FN))(\exists f)(f''\alpha \text{ is cofinal in } Ex(FN)).$$

By the definition of the standard extension (see § 0) the following holds:

$$(\exists n \in FN)(\exists f \in A)(f''n \text{ is cofinal in } FN),$$

which is absurd.

To show that Ex(FN) is a strong cut, observe that the formula

$$(\forall f \in A)(\exists \beta \in A)(\beta > FN \;\&\; (\forall \alpha \in FN)(f(\alpha) \in FN \vee f(\alpha) > \beta)$$

is true as a consequence of the axiom of prolongation, hence the formula

$$(\forall f)(\exists \beta > Ex(FN)(\forall \alpha \in Ex(FN)(f(\alpha) \in Ex(FN) \vee f(\alpha) > \beta)$$

establishing strongness of Ex(FN) holds. □

Open questions. 1) Is $E_A(FN)$ semiregular, regular, strong?

2) Does the following hold:

$(\forall \text{e.u. } A)(\exists \text{e.u.s. } B)(E_A(FN) \neq FN \rightarrow E_A(FN) = Ex_B(FN))$?

<u>Theorem 2.2</u>. Given $\alpha < \beta$ the following are equivalent.

1) $(\exists \text{e.u. } A)(\alpha \in E_A(FN) < \beta)$

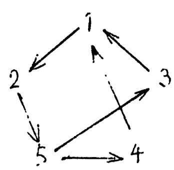
2) $(\exists \gamma)(\alpha < \gamma \leqq \beta \;\&\; \alpha \in \bigcap((\mathrm{Def}(\{\gamma\}) \cap N) - FN))$

3) $(\exists \text{e.u.s. } A)(\alpha \in Ex(FN) < \beta)$

4) $(\exists \text{e.u. } A)(\exists \gamma \leqq \beta)(\alpha, \gamma \in E_A(FN) \;\&\; \alpha \in E_{A[\gamma]}(FN))$

5) $(\exists \text{e.u.s. } A)(\exists \gamma \leqq \beta)(\alpha, \gamma \in Ex(FN) \;\&\; \alpha \in Ex_{A[\gamma]}(FN))$.

<u>Proof</u>. We proceed according to the diagram



5) $\rightarrow$ 4), 3) $\rightarrow$ 1), 5) $\rightarrow$ 3), 4) $\rightarrow$ 1) are obvious.

1) $\rightarrow$ 2): Take some $\gamma \in A$ such that $\alpha < \gamma \leqq \beta$. Then $\alpha \in \bigcap((\mathrm{Def}(\{\gamma\}) \cap N) - FN))$.

2) $\rightarrow$ 5): Suppose $\alpha \in \bigcap((\mathrm{Def}\{\gamma\}) \cap N) - FN)$. By Lemma 0.4 there is an e.u.s. B such that $\alpha \in Ex_B(FN)$ and $\gamma \in B$. In B we can relativize the construction of maximal e.u. (see [T]) and take $A \subseteq B$ which has standard extension and is maximal in the sense of B. Then A has an absolute standard extension (see [Ve] or [Č-Vj]) and $A[x] = B$ for every $x \in B - A$. If γ is not in the A-extension of FN relativized to B (i.e. γ is larger) then take instead of γ an arbitrary element of this extension. Hence we have $\alpha, \gamma \in Ex_A(FN)$ and $\alpha \in Ex_{A[\gamma]}(FN)$, which proves the implication. □

<u>Remarks</u>. 1) The clause 1) of the preceding theorem says that α, β can be separated by a tail, thus the theorem gives equivalent conditions for this fact.

2) The first author has found one more equivalent condition, namely the property

$(*)$ $\quad \alpha \in \bigcap((\mathrm{Def} \cap N) - FN) \;\&\; (\forall F \in Sd_0)(F''FN \subseteq FN \rightarrow F(\alpha) < \beta)$.

This property is apparently weaker than

$(**)$ $\quad (\exists A)(\forall f \in A)(f''FN \subseteq FN \rightarrow f(\alpha) < \beta)$

which is used in the sequel. In a forthcoming paper of K. Čuda, a proof of the equivalence of $(*)$, $(**)$ will appear.

<u>Definition 2.3</u>. We denote by $\alpha \lhd \beta$ the fact that the conditions of Theorem 2.2 hold.

This theorem has some interesting consequences. One of them is that α, β can be separated by a cut of the form $E_A(FN)$ iff they can be separated by a cut of the form $Ex_A(FN)$. Thus, these two kinds of cuts are symbiotic in the terminology of [K-P].

Corollary 1. If Def=FV then $\alpha \lhd \beta \rightarrow \alpha << \beta$.

Proof. Let $\alpha \lhd \beta$. By 2.2.3 there is a standard extension Ex(FN) such that $\alpha \in Ex(FN) < \beta$. It is well-known that Ex(Def) is an e.u. Put $A = \bigcup Ex(Def)$. Since Def=FV, $Ex(Def) \cap N = Ex(Def \cap N) = Ex(FN)$. Thus $A \cap N = Ex(FN)$. By Theorem 1.2.5) $\alpha << \beta$ □

Remark. Trivially if Def=FN then $\alpha << \beta \rightarrow (*)$.

Definition 2.4. We put

$$\alpha <<_A \beta \text{ iff } (\forall f \in A)(f''FN \subseteq FN \longrightarrow f(\alpha) < \beta).$$

Theorem 2.5. If Def=FV then

$$(\forall \text{e.u. } A)(\forall \alpha\beta)(\alpha <<_A \beta \rightarrow \alpha << \beta).$$

Proof. Easy consequence of the fact that for any $F \in Sd_0$ and every $\gamma \in A$, $F \upharpoonright_\gamma \in A$ and $F''FN \subseteq FN$. □

That the converse implication does not hold is shown by the following example.

Example 2.6. Let Def=FV and let A be a maximal e.u.s. Let $\alpha \in Ex(FN)-FN$. Then $Def(\{\alpha\}) \cap N \subseteq Ex(FN)$ because if $\delta \in Def(\{\alpha\})$ there is an $F \in Sd_0$ such that $F(\alpha) = \delta$ and we may suppose that $F''FN \subseteq FN$ since Def=FV. Ex(FN) is fully revealed and $Def(\{\alpha\})$ is countable, hence there is a $\beta \in Ex(FN)$ such that $Def\{\alpha\} \cap N < \beta$. Therefore $\alpha << \beta$. On the other hand, $\alpha \notin A$ and from the maximality of A we have $A[\alpha] = V$. Thus there is an f being 1-1 (see [T]) such that $f(\alpha) = \beta$. Then we may assume that $F''FN \subseteq FN$ as both $\alpha, \beta \in Ex(FN)$. Therefore $\alpha <\!\!|\!\!<_A \beta$.

We shall close this article by giving conditions under which two elements of V can be separated by an e.u.

Theorem 2.7. The following properties are equivalent.

1) $a \in Def(\{b\})$ & $b \in Def(\{a\})$.
2) There is a 1-1 $F \in Sd_0$ such that $F(a)=b$.

3) $(\forall \text{e.u. } A)(a\in A \equiv b\in A)$

4) $(\forall \text{e.u.s.} A)(a\in A \equiv b\in A)$

Proof. The theorem is an easy consequence of the following lemma. □

Lemma 2.8. If $b\notin \mathrm{Def}(\{a\})$, then there is an e.u.s. A such that $a\in A \,\&\, b\notin A$.

Proof. By Lemma 0.3 there is an e.u.s. A such that $a\in A$. As $b\notin \mathrm{Def}(\{a\})$ the monad μ of b with respect to $\overset{\circ}{=}_{\{a\}}$ is an infinite Π-class and clearly $\mu - A \neq \emptyset$ since A has only finite subsets (see [S-V]). If $b\in A$, take $c\in \mu - A$ and an automorphism F such that F(b)=c, F(a)=a. Then the e.u.s. F^{-1}"A is as required. □

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