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AROUND A NEUTRAL ELEMENT IN A NEARLATTICE

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Abstract: Nearlattices, or lower semilattices in which any two elements have a supremum whenever they are bounded above, provide an interesting generalization of lattices. In this context, we define different types of elements in a nearlattice S and then for a fixed element n, using the ternary operation J_n, study the behaviour of $S_n=(S;\cap)$ where $x \cap y=(x \wedge y) \vee (x \wedge n) \vee \vee (y \wedge n); x,y \in S$.

Key words: Standard element, neutral element, nearlattice.

Classification: 06A12, 06A99, 06B10

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1. Introduction. A nearlattice is a lower semilattice which has the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman [1] called this the upper bound property. For detailed literature, we refer the reader to consult [1], [2] and [7].

A nearlattice-congruence Φ on a nearlattice S is a congruence of the underlying lower semilattice such that, whenever $a_1 \equiv b_1$, $a_2 \equiv b_2(\Phi)$ and $a_1 \vee a_2$, $b_1 \vee b_2$ exist, $a_1 \vee a_2 \equiv b_1 \vee b_2(\Phi)$. In the second section of [4], a fundamental contribution was made by Hickman. Defining a ternary operation j on a nearlattice S by $j(x,y,z)=(x \wedge y) \vee (y \wedge z)$, he showed that the resulting algebras of the type $(S;j)$ form a variety.

Standard and neutral elements, as well as standard ideals in a nearlattice were extensively studied in [2]. An element s in a nearlattice S is called standard if for all $x,y,t \in S$, $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$. An element n in a nearlattice S is called neutral if it is standard and for any $t,x,y \in S$, $n \wedge [(t \wedge x) \vee (t \wedge y)] = (n \wedge t \wedge x) \vee (n \wedge t \wedge y)$. Clearly, every element of a distributive nearlattice is neutral. An element n of a nearlattice S is called superstandard if it is standard and for any $x,y \in S$, $n \wedge [(x \wedge y) \vee (x \wedge n) \vee (y \wedge n)] = (x \wedge n) \vee \vee (y \wedge n)$, whenever $(x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ exists. Of course, every neutral element is superstandard. But in the pentagonal lattice $\{0,a,b,n,1\}$ where $0<a<n<1$; $0<b<1$: $a \wedge b = n \wedge b = 0$ and $a \vee b = n \vee b = 1$, n is superstandard but not neutral. [7] provides an example of a standard element in a lattice which is not superstandard.

An element n in a nearlattice S is called medial if $m(x,n,y)= =(x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ exists for all $x,y \in S$, while n is called sesquimedial if $J_n(x,y,z)=([(x \wedge n) \vee (y \wedge n)] \wedge [(y \wedge n) \vee (z \wedge n)]) \vee \vee j(x,y,z)$ exists for all $x,y,z \in S$ where $j(x,y,z)=(x \wedge y) \vee (y \wedge z)$. Since $J_n(x,y,x)=m(x,n,y)$ for all $x,y \in S$, any sesquimedial element is medial. A nearlattice S is called medial if $m(x,y,z)=(x \wedge y) \vee \vee (y \wedge z) \vee (z \wedge x)$ exists for all $x,y,z \in S$. Of course, every element of a medial nearlattice is sesquimedial (see Lemma 3.1).

Let n be a fixed element of a nearlattice S. By an n-ideal of S, we mean a convex subnearlattice of S containing n. The n-ideal generated by $a_1,\ldots,a_m$ is denoted by $\langle a_1,\ldots,a_m\rangle_n$. Clearly $\langle a_1,\ldots,a_m\rangle_n = \langle a_1\rangle_n \vee \ldots \vee \langle a_m\rangle_n$. When S is a lattice, $\langle a_1,\ldots,a_m\rangle_n = =\langle a_1 \wedge \ldots \wedge a_m \wedge n, a_1 \vee \ldots \vee a_m \vee n\rangle_n$. Thus, for a lattice S, the set of finitely generated n-ideals of S is a lattice and its members are simply the intervals $[a,b]$ such that $a \leq n \leq b$, and for such intervals, $[a,b] \vee [a_1,b_1]=[a \wedge a_1, b \vee b_1]$ and $[a,b] \cap [a_1,b_1]= =[a \vee a_1, b \wedge b_1]$. The n-ideal generated by a single element is called a Principal n-ideal and the set of Principal n-ideals of S is denoted by $P_n(S)$. When S is a lattice, it is not hard to see that $P_n(S)$ is a lattice if and only if n is complemented in each interval containing it.

For a fixed element n, the binary operation $x \cap y=m(x,n,y)= =(x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ has been studied by several authors including Jakubík and Kolibiar [5] for distributive lattices, Sholander [8] for distributive medial near lattices and Kolibiar [6] for an arbitrary lattice with n as a neutral element in it. Sholander [8] showed that for a distributive medial nearlattice S, $(S;\cap)$ is a semilattice. On the other hand Kolibiar [6] showed that if n is a neutral element in an arbitrary lattice S, $(S;\cap)$ is a semilattice. Recently, Noor [7] extended their work and showed that for a neutral and sesquimedial element n of a nearlattice S, $S_n=(S;\cap)$ is not only a semilattice, it is a nearlattice. Moreover, the n-ideals of S are precisely the ideals of S_n. According to [7], we refer to S_n as an <u>isotope</u> of S.

In Section 2, we introduce the notion of a <u>nearly neutral</u> element in a nearlattice and then generalize and extend some of the results in [7]. We show that for a medial superstandard element n of a nearlattice S, S_n is a nearlattice wherein $J_n(x,y,z)=$ $=j^{S_n}(x,y,z)$ if and only if n is nearly neutral and sesquimedial in S. We also show that for a nearly neutral and sesquimedial element of a nearlattice S, n is neutral if and only if the nearlattice congruences of S are precisely the nearlattice congruences of S_n.

In Section 3, introducing the ternary operation $M_n(x,y,z)$ we show that for a sesquimedial neutral element n of a nearlattice S, S is medial if and only if S_n is so.

2. <u>Nearly neutral element of a near lattice</u>. An element n of a nearlattice is called nearly neutral if it is standard and has the property $n\wedge((t\wedge x\wedge n)\vee(t\wedge y))=(t\wedge x\wedge n)\vee(t\wedge y\wedge n)$ for all $x,y,t\in S$. Of course, a neutral element is always nearly neutral. Observe that in Figure 1, n is nearly neutral but $n\wedge(a\vee b)>$ $>(n\wedge a)\vee(n\wedge b)$ shows that it is not neutral there.

The following result shows that every nearly neutral element is superstandard, but in the pentagonal lattice $\{0,a,b,n,1\}$ where $0<a<n<1$; $0<b<1$; $a\wedge b=n\wedge b=0$; $a\vee b=n\vee b=1$, n is superstandard but not nearly neutral.

<u>Proposition 2.1</u>. For an element n of a nearlattice S, the following conditions are equivalent.

(i) For all $x,y,t\in S$,
$n\wedge((t\wedge x\wedge n)\vee(t\wedge y))=(t\wedge x\wedge n)\vee(t\wedge y\wedge n)$.

(ii) For all $x,y\in S$,
$n\wedge((x\wedge n)\vee y)=(x\wedge n)\vee(y\wedge n)$, whenever $(x\wedge n)\vee y$ exists.

Moreover, if n is sesquimedial, (i) and (ii) are also equivalent to each of the next two conditions.

(iii) For all $x,y,z\in S$, $(x\cap y)\wedge n=(x\wedge n)\vee(y\wedge n)$ and $J_n(x,y,z)\wedge n=(x\cap y)\wedge(y\cap z)\wedge n$, where $x\cap y=(x\wedge y)\vee$ $\vee(x\wedge n)\vee(y\wedge n)$.

(iv) For all $x,y,z \in S$, $(x \cap y) \wedge n = (x \wedge n) \vee (y \wedge n)$ and $J_n(x,y,z) \wedge n \leq x \cap y$.

Proof. (i) $\Rightarrow$ (ii). Suppose $(x \wedge n) \vee y$ exists. Then $n \wedge ((x\wedge n)\vee y) = n\wedge[(((x\wedge n)\vee y)\wedge x\wedge n)\vee(((x\wedge n)\vee y)\wedge y)] = (x\wedge n)\vee(y\wedge n)$.

(ii) $\Rightarrow$ (i) is trivial.

Suppose now that n is sesquimedial and (i) and (ii) hold. Then $n\wedge(x\cap y) = n\wedge((x\wedge n)\vee(y\wedge n)\vee(x\wedge y)) = n\wedge[(((x\wedge n)\vee(y\wedge n))\wedge n)\vee\vee(x\wedge y)] = (x\wedge n)\vee(y\wedge n)\vee(x\wedge y\wedge n) = (x\wedge n)\vee(y\wedge n)$. Also,

$J_n(x,y,z)\wedge n = n\wedge[(((x\wedge n)\vee(y\wedge n))\wedge((y\wedge n)\vee(z\wedge n)))\vee(x\wedge y)\vee(y\wedge z)] =$

$= n\wedge[((x\cap y)\wedge(y\cap z)\wedge n)\vee(x\wedge y)\vee(y\wedge z)] =$

$= ((x\cap y)\wedge(y\cap z)\wedge n)\vee(n\wedge((x\wedge y)\vee(y\wedge z))) = (x\cap y)\wedge(y\cap z)\wedge n$.

Thus (iii) holds.

Clearly (iii) implies (iv).

Finally suppose (iv) holds. Let $x,y \in S$ be such that $(x\wedge n)\vee y$ exists. Then

$J_n(x,y,(x\wedge n)\vee y) = [((x\wedge n)\vee(y\wedge n))\wedge(y\wedge n)\vee(n\wedge((x\wedge n)\vee y)))]\vee(x\wedge y)\vee y =$

$= (x\wedge n)\vee(y\wedge n)\vee y = (x\wedge n)\vee y$, and so by (iv) $n\wedge((x\wedge n)\vee y) \leq x\cap y$. Thus, $n\wedge((x\wedge n)\vee y) \leq n\wedge(x\cap y) = (x\wedge n)\vee(y\wedge n)$; it follows that $n\wedge((x\wedge n)\vee y) = (x\wedge n)\vee(y\wedge n)$ and (ii) holds. □

The following result is found in [7, Th. 2.1].

Proposition 2.2. If n is a standard element of a nearlattice S, then $(S;\subseteq)$ is a partially ordered set and the map $x \to \langle x\rangle_n$ is an isomorphism of $(S;\subseteq)$ onto $P_n(S)$, where on S, $x \subseteq y$ if and only if $(x\wedge y)\vee(x\wedge n)\vee(y\wedge n)$ exists and is equal to x. □

Let n be a medial element of a nearlattice S. For any $x,y \in S$ define the binary operation $x\cap y = m(x,n,y) = (x\wedge y)\vee(x\wedge n)\vee(y\wedge n)$. Recently Noor in [7] proved the following result.

Theorem 2.3. If n is a medial and standard element of a nearlattice S, then S_n is a semilattice if and only if n is superstandard in S.

Moreover, when n is neutral and sesquimedial then S_n is in fact a nearlattice and the n-ideals of S are precisely the ideals of S_n. □

Our next theorem generalizes and extends the above Theorem. To obtain this, we need the following lemma. (i) is found in

[7; Lemma 2.4], and the proof of (ii) is similar to the proof of (ii) in [7; Lemma 2.4].

Lemma 2.4. In a nearlattice S,

(i) a subset K of S is an ideal of S if and only if for all $x,y \in K$ and $a \in S$, $j(x,a,y) \in K$.

(ii) If n is a superstandard element of S such that S_n is a nearlattice wherein

$$J_n(x,y,z) = j^{S_n}(x,y,z) = (x \cap y) \cup (y \cap z),$$

then a subset K of S is an n-ideal of S if and only if it contains n and $J_n(x,a,y) \in K$ for any $x,y \in K$ and $a \in S$. □

Corollary 2.5. Suppose n is a superstandard element of a nearlattice S such that the isotope S_n of S is itself a nearlattice wherein $J_n(x,y,z) = j^{S_n}(x,y,z)$. Then the ideals of S_n are precisely the n-ideals of S. □

Theorem 2.6. If S is a nearlattice and $n \in S$ is medial and superstandard, then the following conditions are equivalent.

(i) n is nearly neutral and sesquimedial in S.

(ii) The isotope $S_n = (S;\cap)$ is a nearlattice wherein $J^{S_n}(x,y,z) = J_n(x,y,z)$.

(iii) S_n has the upper bound property and n-ideals of S are precisely the ideals of S_n.

(iv) Any finitely generated n-ideal contained in a principal n-ideal is a principal n-ideal.

Proof. (i) $\Rightarrow$ (ii). Suppose n is nearly neutral and sesquimedial in S. Then, clearly

$$J_n(x,y,z) = ((x \cap y) \wedge (y \cap z) \wedge n) \vee j(x,y,z),$$

and so by [2;Th. 2.4], $J_n(x,y,z) \equiv j(x,y,z)(\Theta_n)$ and $x \cap y \equiv x \wedge y (\Theta_n)$. Hence $(x \cap y) \wedge J_n(x,y,z) \equiv x \wedge y (\Theta_n)$ and similarly $(y \cap z) \wedge J_n(x,y,z) \equiv y \wedge z (\Theta_n)$. Therefore,

$$[(x \cap y) \wedge J_n(x,y,z)] \vee [(y \cap z) \wedge J_n(x,y,z)] \vee [n \wedge J_n(x,y,z)] \equiv$$
$$\equiv (x \wedge y) \vee (y \wedge z) \vee (n \wedge j(x,y,z)) = j(x,y,z)(\Theta_n).$$

Since the left hand side of this congruence exceeds the right hand side, by [2;Th. 2.4],

left hand expression

= j(x,y,z)∨(n∧(left hand expression))

$= j(x,y,z)\vee(n\wedge J_n(x,y,z)) = J_n(x,y,z)$.

Thus, $J_n(x,y,z)\in\langle x\cap y, y\cap z\rangle_n$. On the other hand, $(x\cap y)\wedge J_n(x,y,z)\equiv x\wedge y(\Theta_n)$ implies $(x\cap y)\wedge J_n(x,y,z) = (x\wedge y)\vee(n\wedge(x\cap y)\wedge J_n(x,y,z))$, and so $((x\cap y)\wedge J_n(x,y,z))\vee((x\cap y)\wedge n) = (x\wedge y)\vee(x\wedge n)\vee(y\wedge n) = x\cap y$. Hence, $x\cap y\in\langle J_n(x,y,z)\rangle_n$ and similarly $y\cap z\in\langle J_n(x,y,z)\rangle_n$. Thus, $\langle x\cap y, y\cap z\rangle_n = \langle J_n(x,y,z)\rangle_n$ and so by Proposition 2.2, $(x\cap y)\cup(y\cap z) = J_n(x,y,z)$.

(ii) $\Longrightarrow$ (iii) follows immediately from Corollary 2.5.

(iii) $\Longrightarrow$ (iv) is an easy consequence of the isomorphism of $(S_n;\subseteq)$ and $(P_n(S);\subseteq)$, and the upperbound property of S_n.

(iv) $\Longrightarrow$ (i). Let $a,b,c\in S$. Since $a\cap b, b\cap c\subseteq b$, $\langle a\cap b, b\cap c\rangle_n\subseteq\langle b\rangle_n$ by Proposition 2.2. Thus, by (iv), there exists $t\in S$ such that $\langle a\cap b, b\cap c\rangle_n = \langle t\rangle_n$, and so $(a\cap b)\wedge(b\cap c)\wedge n = t\wedge n$. Now, $a\cap b\subseteq t$ implies $a\cap b = ((a\cap b)\wedge t)\vee(a\cap b)\wedge n)\vee(t\wedge n) = ((a\cap b)\wedge t)\vee((a\cap b)\wedge n)$, and so $a\cap b\equiv(a\cap b)\wedge t(\Theta_n)$. Hence $a\wedge b\equiv a\cap b\equiv(a\cap b)\wedge t\equiv a\wedge b\wedge t(\Theta_n)$.

Similarly, $b\wedge c\equiv b\wedge c\wedge t(\Theta_n)$. This implies

$$j(a,b,c)\equiv(a\wedge b\wedge t)\vee(b\wedge c\wedge t)\ (\Theta_n)$$

and so $j(a,b,c) = (a\wedge b\wedge t)\vee(b\wedge c\wedge t)\vee(n\wedge j(a,b,c))$. Also, $j(a,b,c)\wedge t\equiv(a\wedge b\wedge t)\vee(b\wedge c\wedge t)(\Theta_n)$, and so

$$j(a,b,c)\wedge t = (a\wedge b\wedge t)\vee(b\wedge c\wedge t)\vee(n\wedge t\wedge j(a,b,c)).$$

Thus, $j(a,b,c)\cap t = (j(a,b,c)\wedge t)\vee(j(a,b,c)\wedge n)\vee(t\wedge n) = j(a,b,c)\vee(t\wedge n)$.

Again, $a\cap b\equiv a\wedge b(\Theta_n)$. So $(a\cap b)\wedge j(a,b,c)\equiv a\wedge b\wedge j(a,b,c) = a\wedge b(\Theta_n)$, and hence $(a\cap b)\wedge j(a,b,c) = (a\wedge b)\vee((a\cap b)\wedge j(a,b,c)\wedge n)$. This implies $(a\cap b)\cap j(a,b,c) = a\cap b$; that is, $a\cap b\subseteq j(a,b,c)$. Similarly, $b\cap c\subseteq j(a,b,c)$. Hence, $t\subseteq j(a,b,c)$, and so $t = t\wedge j(a,b,c) = j(a,b,c)\vee(t\wedge n) = j(a,b,c)\vee((a\cap b)\wedge(b\cap c)\wedge n) = J_n(a,b,c)$, as n is superstandard. Hence n is sesquimedial, and $J_n(a,b,c)\wedge n = t\wedge n = (a\cap b)\wedge(b\cap c)\wedge n$. Also $(x\cap y)\wedge n = (x\wedge n)\vee(y\wedge n)$, as n is superstandard. Therefore, by 2.1(iii), n is nearly neutral. □

The following lemma is due to Hickman [4; Proposition 2.2].

<u>Lemma 2.7</u>. In a nearlattice S, an equivalence relation is a nearlattice congruence if and only if it is a congruence for the algebra $(S;j)$. □

Now we consider the influence of J_n on congruences. The following theorem is an extension of [7; Lemma 2.6(ii)].

Theorem 2.8. Let n be a sesquimedial, nearly neutral element of a nearlattice S. Then the following conditions are equivalent.

(i) n is neutral in S;

(ii) an equivalence relation on S is a congruence for the algebra $(S;J_n)$ if and only if it is a n earlattice-congruence of S.

Proof. (i) $\Rightarrow$ (ii) is proved in [7; Lemma 2.6(ii)].

(ii) $\Rightarrow$ (i). Define a relation Θ on the nearlattice S by $x \equiv y(\Theta)$ if and only if $x\wedge n = y\wedge n$. This is clearly an equivalence relation on S.

Now suppose $x \equiv y(\Theta)$. Then $x\wedge n = y\wedge n$, and so by 2.1, for any $s,t \in S$, $n\wedge J_n(x,s,t) = (x\wedge s)\wedge(s\wedge t)\wedge n = ((x\wedge n)\vee(s\wedge n))\wedge((s\wedge n)\vee(t\wedge n)) = ((y\wedge n)\vee(s\wedge n))\wedge((s\wedge n)\vee(t\wedge n)) = n\wedge J_n(y,s,t)$. Thus, $J_n(x,s,t) \equiv J_n(y,s,t)(\Theta)$. Similarly, $J_n(s,x,t) \equiv J_n(s,y,t)(\Theta)$ and $J_n(s,t,x) \equiv J_n(s,t,y)(\Theta)$, and so Θ is a congruence for the algebra $(S;J_n)$. Thus, by (ii), Θ is a nearlattice congruence on S. Now, clearly $x \equiv x\wedge n(\Theta)$ and $y \equiv y\wedge n(\Theta)$ for all $x,y \in S$. So for any $t \in S$, $(t\wedge x)\vee(t\wedge y) \equiv (t\wedge x\wedge n)\vee(t\wedge y\wedge n)(\Theta)$, and hence, $n\wedge[(t\wedge x)\vee(t\wedge y)] = n\wedge[(t\wedge x\wedge n)\vee(t\wedge y\wedge n)] = (t\wedge x\wedge n)\vee(t\wedge y\wedge n)$, which implies n is neutral in S. □

Combining Theorem 2.6, Lemma 2.7 and the above theorem, we have the following extension of [7,Th. 2.7].

Theorem 2.9. Let n be a nearly neutral sesquimedial element of a nearlattice S. Then n is neutral if and only if the nearlattice congruences of S are precisely the nearlattice congruences of S_n. □

The following proposition will be needed to prove one of our main results in Section 3. This was known by Kolibiar [6] in case of a bounded lattice with n as a central element.

Proposition 2.10. If n is a nearly neutral sesquimedial element of a nearlattice S with 0, then 0 is neutral and medial in S_n. Moreover, the double isotope $(S_n)_0$ is precisely S.

If, in addition, n is neutral in S, then 0 os sesquimedial in S_n and $J_0^{S_n}(x,y,z) = j_0^{(S_n)}(x,y,z) = j(x,y,z) = J_0(x,y,z)$ for all $x,y,z \in S$.

Proof. By 2.6, for all $r,x,y \in S$, $0 \cap ((r \cap x) \cup (r \cap y)) =$ $= 0 \cap J_n(x,r,y) = n \wedge J_n(x,r,y) = J_n(r \cap x, 0, r \cap y) = (0 \cap r \cap x) \cup (0 \cap r \cap y)$. Also, $r \cap ((x \cap y) \cup (x \cap 0)) = r \cap J_n(y,x,0) = r \cap ((x \wedge n) \vee (x \wedge y)) = (r \wedge n) \vee$ $\vee (x \wedge n) \vee (r \wedge x \wedge y)$ as n is nearly neutral and hence standard. On the other hand, $(r \cap x \cap y) \cup (r \cap x \cap 0) = J_n(y, r \cap x, 0) = ((r \cap x) \wedge n) \vee (y \wedge (r \cap x)) =$ $= (r \wedge n) \vee (x \wedge n) \vee [y \wedge (((r \cap x) \wedge r \wedge x) \vee ((r \cap x) \wedge n))] = (r \wedge n) \vee (x \wedge n) \vee (r \wedge x \wedge y)$. That is $r \cap ((x \cap y) \cup (x \cap 0)) = (r \cap x \cap y) \cup (r \cap x \cap 0)$; consequently 0 is neutral in S_n.

Now, clearly $x \cap y, x \cap 0, y \cap 0 \leq x \wedge y$, and so $(x \cap y) \cup (x \cap 0) \cup (y \cap 0)$ exists and it is $\leq x \wedge y$. Thus, 0 is medial in S_n, and so $((S_n)_0; \bar{\wedge})$ is a semilattice by Theorem 2.3, where

$$x \bar{\wedge} y = (x \cap y) \cup (x \cap 0) \cup (y \cap 0).$$

Suppose $x \cap y, x \cap 0, y \cap 0 \leq s$ for some $s \in S_n$. Then $s \wedge n \leq (x \cap 0) \wedge n =$ $= x \wedge n$. Similarly $s \wedge n \leq y \wedge n$, and so $s \wedge n \leq x \wedge y \wedge n$. Also,

$$x \cap y = (x \cap y) \cap s = ((x \cap y) \wedge s) \vee ((x \cap y) \wedge n) \vee (s \wedge n) =$$
$$= ((x \cap y) \wedge s) \vee ((x \cap y) \wedge n).$$

Then

$$x \wedge y = (x \wedge y) \wedge (x \cap y) = (x \wedge y \wedge s) \vee (x \wedge y \wedge n) = (x \wedge y) \cap s.$$

This implies $x \wedge y \leq s$, and hence

$$x \bar{\wedge} y = (x \cap y) \cup (x \cap 0) \cup (y \cap 0) = x \wedge y;$$

in other words, $(S_n)_0 = S$.

Finally, suppose that n is neutral in S. Since 0 is neutral in S_n,

$$((x \cap 0) \cup (y \cap 0)) \cap ((y \cap 0) \cup (z \cap 0)) = (x \bar{\wedge} y) \cap (y \bar{\wedge} z) \cap 0 =$$
$$= (x \wedge y) \cap (y \wedge z) \cap 0 = [(x \wedge y) \cap (y \wedge z)] \wedge n =$$
$$= (x \wedge y \wedge n) \vee (y \wedge z \wedge n) = n \wedge j(x,y,z)$$

as n is neutral. Also it can be easily shown that $x \cap y, y \cap z \leq$ $\leq j(x,y,z) = J_0(x,y,z)$. Therefore

$$[((x \cap 0) \cup (y \cap 0)) \cap ((y \cap 0) \cup (z \cap 0))] \cup (x \cap y) \cup (y \cap z)$$

exists in S_n; whence 0 is sesquimedial in S_n. The rest follows by 2.6. $\square$

It should be noted that the above proposition is not true when n is merely nearly neutral. For example, in Figure 2 which is the isotope of Figure 1, 0 is not sesquimedial.

3. Medial nearlattices. Recall that a nearlattice S is *medial* if for all $x,y \in S$, $m(x,y,z) = (x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$ exists. A nearlattice S is said to have the *three property* if, for any $x,y,z \in S$, $x\vee y\vee z$ exists whenever $x\vee y$, $y\vee z$ and $z\vee x$ exist. Nearlattices with the three property were discussed by Evans in [3], where he referred to them as strong conditional lattices. It is easy to see that a nearlattice S has the three property if and only if it is medial.

Lemma 3.1. Every element of a medial nearlattice is sesquimedial.

Proof. Suppose S is medial and n is any element of S. For any $x,y,z \in S$, $((x\wedge n)\vee(y\wedge n))\wedge((y\wedge n)\vee(z\wedge n))$, $x\wedge y \leq m(x,n,y)$ and $((x\wedge n)\vee(y\wedge n))\wedge((y\wedge n)\vee(z\wedge n))$, $y\wedge z \leq m(y,n,z)$. Thus using the upper bound property and the three property of S, $(((x\wedge n)\vee(y\wedge n))\wedge$ $\wedge((y\wedge n)\vee(z\wedge n)))\vee(x\wedge y)\vee(y\wedge z) = J_n(x,y,z)$ exists in S. □

Suppose S is a medial nearlattice and $a,b,c \in S$. If $a\vee b$, $b\vee c$, $c\vee a$ exists, we define $m^d(a,b,c) = (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$. Of course, when S is distributive, $m^d(a,b,c) = m(a,b,c)$. For a fixed element n of S, let us introduce a ternary operation M_n, defined by $M_n(x,y,z) = m^d(x\wedge n, y\wedge n, z\wedge n)\vee m(x,y,z)$; $x,y,z \in S$. Notice that $m^d(x\wedge n, y\wedge n, z\wedge n)$ always exists in S. But also we have:

Lemma 3.2. In a medial nearlattice S with $n \in S$, $M_n(x,y,z)$ always exists for all $x,y,z \in S$.

Proof. Notice that $m^d(x\wedge n,y\wedge n,z\wedge n)$, $x\wedge y \leq m(x,n,y)$, $m^d(x\wedge n,y\wedge n,z\wedge n)$, $y\wedge z \leq m(y,n,z)$ and $m^d(x\wedge n,y\wedge n,z\wedge n)$, $z\wedge x \leq m(z,n,x)$. Then by the upper bound property and the three property both $m^d(x\wedge n,y\wedge n,z\wedge n)\vee(z\wedge x)$ and $m^d(x\wedge n,y\wedge n,z\wedge n)\vee(x\wedge y)\vee(y\wedge z)$ exist. Thus a second application of the three property yields the existence of $M_n(x,y,z)$. □

Note that if n is nearly neutral in a nearlattice S, $M_n(x,y,z) = ((x\cap y)\wedge(y\cap z)\wedge(z\cap x)\wedge n)\vee m(x,y,z)$, and when n is neutral, $M_n(x,y,z)\wedge n = (x\cap y)\wedge(y\cap z)\wedge(z\cap x)\wedge n$. Also if S is a lattice and n is neutral, $M_n(x,y,z) = (m^d(x,y,z)\wedge n)\vee m(x,y,z) = m^d(x,y,z)\wedge$ $\wedge(n\vee m(x,y,z))$.

Of course $m(x,y,z)$ and $M_n(x,y,z)$ are symmetric in x,y and z, whereas $j(x,y,z)$ and $J_n(x,y,z)$ are not. Thus, the operations m and M_n are better behaved and easier to handle than the operations j and J_n respectively.

The following proposition is easily verifiable and so is given without proof.

Proposition 3.3. For an element n of a medial nearlattice S, $M_n(x,y,z) = m(x,y,z)$ for all $x,y,z \in S$ if and only if $(n]$ is a distributive lattice.

Hence in a distributive medial nearlattice S, $M_n(x,y,z) = m(x,y,z)$ for all $x,y,z \in S$. $\square$

Now we present the following interesting result which extends Theorem 2.6.

Theorem 3.4. Suppose n is a neutral sesquimedial element of a nearlattice S. Then the following conditions are equivalent.

(i) S is medial;

(ii) S_n is a medial nearlattice and $m^{S_n}(x, y,z) = M_n(x,y,z)$ for all $x,y,z \in S$.

Moreover, (i) does not necessarily imply (ii) when n is merely nearly neutral.

Proof. (i) $\Longrightarrow$ (ii). Since n is neutral,

$$M_n(x,y,z) \wedge n = (x \cap y) \wedge (y \cap u) \wedge (z \cap x) \wedge n.$$

By [2,Th. 2.4],

$$M_n(x,y,z) \equiv m(x,y,z)(\Theta_n)$$

and $x \cap y \equiv x \wedge y(\Theta_n)$. Thus, $(x \cap y) \wedge (M_n(x,y,z) \equiv x \wedge y(\Theta_n)$. Similarly,

$$(y \cap z) \wedge M_n(x,y,z) \equiv y \wedge z(\Theta_n),$$

and

$$(z \cap x) \wedge M_n(x,y,z) \equiv z \wedge x(\Theta_n).$$

Then using the technique of the proof of (i) $\Longrightarrow$ (ii) in Theorem 2.6, we obtain $\langle x \cap y, y \cap z, z \cap x \rangle_n = \langle M_n(x,y,z) \rangle_n$, and (ii) follows from the isomorphism of $(S_n;\subseteq)$ and $(P_n(S);\subseteq)$.

(ii) $\Longrightarrow$ (i). Adjoint a new 0 in S and form $(S;0)_n$. Then by 2.10, 0 is neutral and medial in $(S;0)_n$. Thus $(S;0)_n$ is medial as S_n is medial. Hence, by (i) $\Longrightarrow$ (ii), $((S;0)_n)_0$ is medial. But $((S;0)_n)_0 = (S;0)$ by 2.10, and so S is medial as required.

For the final assertion consider the lattice of Figure 1, where n is nearly neutral but not neutral. But its isotope, given by Figure 2 is not medial. □

It is well-known by Kolibiar [6] that if L is a lattice with 0 and 1 and n is central in it, then L_n is also a bounded lattice with n and n' as the smallest and the largest elements respectively, where $x \cup y = m(x,n',y)$ for all $x,y \in L$.

An element n in a lattice L is called <u>central</u> if it is neutral and complemented in each interval containing it.

We conclude this paper with the following extension of Kolibiar's result.

<u>Proposition 3.5</u>. Suppose L is a lattice and $n \in L$ is standard. Then the isotope L_n is a lattice if and only if n is central in L.

<u>Proof</u>. Since n is standard, $(L;\subseteq)$ and $(P_n(L);\subseteq)$ are isomorphic by 2.2. Thus, L_n is a lattice if and only if $P_n(L)$ is a lattice, i.e. if and only if n is complemented in each interval containing it. Consequently, the result follows by [2,Th. 3.5].

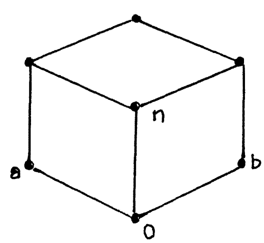

Figure 1

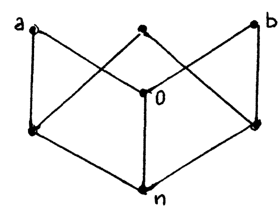

Figure 2

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