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Pavel Pták

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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,1 (1987)

„HIDDEN VARIABLES" ON CONCRETE LOGICS (EXTENSIONS)

Pavel PTÁK

Abstract: We call a concrete logic smooth if all its hidden variables (= all its two-valued measures) admit extensions over larger logics. We show as the main result that every Boolean algebra is smooth and that every logic has a smooth representation. This seems to match the hidden variables hypotheses.

Key-words: Concrete quantum logic, hidden variables hypotheses, two-valued measure on a logic.

Classification: Primary 06C15, Secondary 81B10

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1. Introduction and preliminaries. Results. In axiomatic formulations of the foundations of quantum theories one often postulates that the "event structure" of a quantum experiment be a quantum logic, that is, an orthomodular partially ordered set. One sometimes speculates that the stochastic behavior of the experiment could be gone over, and the problem then approached by the tools of classical mechanics, if all the "hidden variables" could be discovered (see e. g. [1], [2], [3], [4], [9]). Suppose that we enlarge the experiment and ask whether the hidden variables remain preserved. As the hidden variables usually correspond to two-valued measures on the respective logic, our question translates as follows: Do two-valued measures admit extensions from sublogics over the entire logics? In this note we bring certain results along this line. The character of the problem obviously requires that the logics have "enough" two-valued measures. As known (see [3], [6]), these are exactly the logics which have a set representation. We call them concrete and, in view of the above remark, we restrict our consideration to concrete logics.

Let us first review basic notions as we shall use them in the sequel. Let S be a non-empty set and let Δ be a collection of subsets of S. Partially order Δ by set inclusion and, for each $A \in \Delta$, let A' be the set $S - A$. Then the couple (S, Δ) is called a <u>concrete logic</u> if the following three conditions are satisfied:

(i) $\emptyset \in \Delta$,

(ii) If $A \in \Delta$ then $A' \in \Delta$,

(iii) If A and B are in Δ and $A \cap B = \emptyset$ then $A \cup B \in \Delta$.

In other words, a concrete logic is a logic (= an orthomodular poset) which has a set representation. We shall sometimes write Δ instead of (S, Δ) if we do not need deal with the domain S. Obviously, each Boolean algebra may be viewed as a concrete logic, and a concrete logic is a Boolean algebra (Boolean logic) if and only if $A \cap B \in \Delta$ for each $A, B \in \Delta$.

Let (S, Δ) and (S, Δ_1) be logics. Then (S, Δ) is called a <u>sublogic</u> of (S, Δ_1) if $\Delta \subset \Delta_1$ and, for each $A, B \in \Delta$, $A \cap B \in \Delta$ if and only if $A \cap B \in \Delta_1$. Thus, for instance, if $A \in \Delta_1$ then $\{\emptyset, S, A, A'\}$ is a sublogic of (S, Δ_1). Observe also that a sublogic of a Boolean logic has to be Boolean.

When (S, Δ) is a logic we call a mapping $h : \Delta \to \{0, 1\}$ a <u>hidden variable</u> if $h(S) = 1$ and $h(A \cup B) = h(A) + h(B)$ for all $A, B \in \Delta$ with $A \cap B = \emptyset$. Let us denote by $\mathrm{Hid}(\Delta)$ the set of all hidden variables on (S, Δ). In what follows we shall be interested in the extensions of hidden variables. To simplify the setup of the results, let us call a concrete logic (S, Δ) <u>smooth</u> if the following condition is satisfied: If (S, Δ) is a sublogic of (S, Δ_1) and if $h \in \mathrm{Hid}(\Delta)$ then there exists $h_1 \in \mathrm{Hid}(\Delta_1)$ such that h_1 restricted to Δ equals h.

We are going to show that the class of smooth logics is relatively large. Let us start with the following observation. (Recall that a hidden variable $h \in \mathrm{Hid}(\Delta)$ is said to be concentrated at a point if there is a point $p \in S$ such that $h(A) = 1$ if, and only if, $p \in A$. If $h \in \mathrm{Hid}(\Delta)$ is not concentrated at any point we call it free.)

Proposition 1: Let (S, Δ) be a logic. If each hidden variable on Δ is concentrated at a point then (S, Δ) is smooth.

Proof is evident.

Let us first consider finite logics. Let $n \in N$ be an even number. Put $S_n = \{1, 2, 3, \ldots, n\}$ and denote by Δ_{even} the collection of all subsets of S_n with an even number of elements. Obviously, (S_n, Δ_{even}) is a logic.

Proposition 2:

(i) The logic (S_4, Δ_{even}) possesses a free hidden variable.

(ii) If $n \in N$ is an even number and $n \geq 6$, then each hidden variable on (S_n, Δ_{even}) is concentrated at a point.

(iii) The logic (S_n, Δ_{even}) is smooth for each even number $n \in N$.

Proof: (i) Put $h\{1, 2\} = h\{2, 3\} = h\{1, 3\} = 1$. It is easy to see that h uniquely extends to a free hidden variable on (S_4, Δ_{even}).

(ii) Let us suppose that $h \in \mathrm{Hid}(\Delta_{even})$. Write $S_n = \{1, 2\} \cup \{3, 4\} \cup \ldots \cup \{n - 1, n\}$. The additivity of h gives $h\{k, k + 1\} = 1$ for some k ($k \leq n - 1$). We may suppose that $k = 1$ (otherwise we simply permute the numbers). So we have $h\{1, 2\} = 1$ and this yields that either $h\{1, 3\}$ or $h\{2, 4\}$ equals 1. Let us assume that $h\{1, 3\} = 1$ (the other case argues similarly). Then we claim that h is concentrated at 1. Indeed, if there is a set $A \in \Delta_{even}$ such that $h(A) = 1$ and $1 \notin A$, then $\{2, 3\} \in A$ and moreover, $h\{2, 3\} = 1$. Since $n \geq 6$, we can write $S_n = \{1, 4\} \cup \{2, 5\} \cup \{3, 6\} \cup (S_n - S_6)$ and therefore $h(S_n) = 0$ - a contradiction. This completes the proof.

(iii) The case of $n = 2$ is trivial. Suppose that $n = 4$. By the definition of a sublogic, if (S_4, Δ_{even}) is a sublogic of (S_4, Δ) then Δ has to be Δ_{even}. Finally, if $n \geq 6$ then we use Prop. 2 (ii).

As we see, many finite logics are smooth. Yet not all as the following example shows.

Example 3: Take the logics (S_4, Δ_{even}) and (S_2, Δ_{even}) and form a new logic (S, Δ) as follows: The set S is the (disjoint) union of S_4 and S_2 and $A \in \Delta$ if, and only if, both $A \cap S_4$ and $A \cap S_2$ have even cardinalities. Then (S, Δ) is not a smooth logic.

To show that (S, Δ) is not smooth, let us observe that (S, Δ) may be viewed as a sublogic of (S_6, Δ_{even}). By Proposition 2 (i), (ii), the logic (S, Δ) possesses a free hidden variable whereas (S_6, Δ_{even}) does not. It follows that (S, Δ) is not smooth.

Let us now consider Boolean logics. We have the following result. (It should be noted that this result complements the results of the papers [5], [7] and [8].)

Theorem 4: Suppose that (S, Δ) is a Boolean sublogic of (S, Δ_1) and suppose that $h \in \mathrm{Hid}(\Delta)$. Suppose that A_1 is an element of Δ_1 with the following property: If $A \in \Delta$ such that $A \subset A_1$ then $h(A) = 0$. Then there is a hidden variable $h_1 \in \mathrm{Hid}(\Delta_1)$ such that $h_1/\Delta = h$ and $h_1(A_1) = 0$. A corollary: If B is a Boolean algebra then each its set representation is smooth.

Proof: Observe first that $\mathrm{Hid}(\Delta_1)$ is a compact set when understood as a subset of the topological product $\langle 0, 1\rangle^{\Delta_1}$. Indeed, since the product $\langle 0, 1\rangle^{\Delta_1}$ is compact, and so is also $\{0, 1\}^{\Delta_1}$, we only have to verify that "the pointwise limit" of the elements of $\mathrm{Hid}(\Delta_1)$ belongs to $\mathrm{Hid}(\Delta_1)$, which is easy. Put now $I = \{A \in \Delta \mid h(A) = 1\}$ and set, for each $A \in I$, $C(A) = \{h_1 \in \mathrm{Hid}(\Delta_1) \mid h_1(A) = 1$ and $h_1(A_1) = 0\}$. Obviously, each set $C(A)$ is closed in $\mathrm{Hid}(\Delta_1)$. We are going to show that the family $\mathcal{F} = \{C(A) \mid A \in I\}$ is centered in $\mathrm{Hid}(\Delta_1)$. Let $D_1, D_2, \ldots, D_n$ be a finite family in I. Then $h(D_1) = h(D_2) = \ldots = h(D_n) = 1$ and since Δ is Boolean, we have $h_1(D_1 \cap D_2 \cap \ldots \cap D_n) = 1$. Therefore $D_1 \cap D_2 \cap \ldots \cap D_n \in$

$\in I$ and we obtain the equality $C(D_1) \cap C(D_2) \cap \ldots \cap C(D_n) = C(D_1 \cap D_2 \cap \ldots \cap D_n)$. We need to show that $C(D_1 \cap D_2 \cap \ldots \cap D_n) \neq \emptyset$. Since $h(D_1 \cap D_2 \cap \ldots \cap D_n) = 1$, we infer that the set $D_1 \cap D_2 \cap \ldots \cap D_n$ cannot be a subset of the given set $A_1 \in \Delta_1$. Therefore there is a point $p \in S$ such that $p \in (D_1 \cap D_2 \cap \ldots \cap D_n) - A_1$. Take now the element h_p of $\mathrm{Hid}(\Delta_1)$ concentrated at p. Then $h_p \in C(D_1 \cap D_2 \cap \ldots \cap D_n)$ and therefore $\mathcal{F}$ is a centered family.

Since $\mathrm{Hid}(\Delta_1)$ is compact and each $C(A)$ is closed, there exists an element $h_1 \in \bigcap_{A \in I} \mathcal{F}$. By the construction, if $h(A) = 1$ then $h_1(A) = 1$ and therefore h_1 extends h. The proof is complete.

Before giving our next result, let us recall that a mapping $f : \Delta_1 \to \Delta_2$ is called a morphism (of two logics (S_1, Δ_1) and (S_2, Δ_2)) if the following conditions are satisfied: 1. $f(\emptyset) = \emptyset$, 2. $f(A') = f(A)'$ for each $A \in \Delta_1$, and 3. $f(A \cup B) = f(A) \cup f(B)$ for each pair of disjoint sets $A, B \in \Delta_1$. An injective morphism $f : \Delta_1 \to \Delta_2$ is called an isomorphism if f is surjective and f^{-1} is a morphism.

<u>Theorem 5:</u> Each concrete logic is isomorphic to a concrete logic whose all hidden variables are concentrated. A corollary: Each concrete logic is isomorphic to a smooth one.

Proof: Let (S, Δ) be a concrete logic. Put $S_1 = \mathrm{Hid}(\Delta)$ and, for each $A \in \Delta$, put $S_A = \{h \in \mathrm{Hid}(\Delta) \mid h(A) = 1\}$. Let Δ_1 be the collection $\{S_A \mid A \in \Delta\}$. By standard reasoning, the couple (S_1, Δ_1) becomes a logic and the mapping $f : \Delta \to \Delta_1$, defined so that $f(A) = S_A$, becomes an isomorphism. We need to show that each hidden variable on (S_1, Δ_1) is concentrated. Suppose that h belongs to $\mathrm{Hid}(\Delta_1)$. Then $hf \in \mathrm{Hid}(\Delta)$ and therefore hf may be viewed as a point of Δ_1. Let k be the hidden variable of $\mathrm{Hid}(\Delta_1)$ concentrated at hf. Suppose that $B \in \Delta_1$. Then $B = S_A$ for some $A \in \Delta$. Suppose now that $k(B) = 1$. This means that $k(S_A) = 1$ and therefore $hf \in S_A$. This yields that $hf(A) = 1$ which gives

$h(B) = 1$. We obtain that $k(B) = 1$ implies $h(B) = 1$ and therefore $h = k$. The proof is complete.

In our final results we further add to the examples of smooth logics. Let us recall that a morphism $f : \Delta_1 \to \Delta_2$ (of two logics (S_1, Δ_1) and (S_2, Δ_2)) is said to be carried by a mapping if there is a mapping $g : S_2 \to S_1$ such that $f(A) = g^{-1}(A)$ for each $A \in \Delta_1$.

Proposition 6: Let (S_1, Δ_1) and (S_2, Δ_2) be logics and let $f : \Delta_1 \to \Delta_2$ be a surjective morphism carried by a mapping. Then, if Δ_1 is smooth then so is also Δ_2.

Proof: Let (S_2, Δ_2) be a sublogic of a logic (S_2, Δ_3). Let f be carried by $g : S_2 \to S_1$. Put $\Delta_4 = \{A \subset S_1 \mid A = g^{-1}(B)$ for some set $B \in \Delta_3\}$. Since g^{-1} preserves the complements, unions and intersections, we see that (S_1, Δ_1) becomes a sublogic of (S_1, Δ_4). If $h \in \mathrm{Hid}(\Delta_2)$ then $hf \in \mathrm{Hid}(\Delta_1)$ and therefore hf can be extended to some hidden variable $k \in \mathrm{Hid}(\Delta_4)$. Since f is carried by g, we obtain that h_1 defined by putting $h_1(A) = k(g^{-1}(A))$ extends h, and this completes the proof.

When (S_1, Δ_1) and (S_2, Δ_2) are logics then by the direct product of (S_1, Δ_1) and (S_2, Δ_2) we mean the logic (S, Δ), where S is the disjoint union of S_1 and S_2 and Δ is taken such that $A \in \Delta$ if, and only if, $A \cap S_i$ belongs to Δ_i $(i = 1, 2)$.

Proposition 7: Let (S_1, Δ_1) be a smooth logic and let Δ_2 be the collection of all subsets of a set S_2. Then the direct product of (S_1, Δ_1) and (S_2, Δ_2) is a smooth logic.

The proof is straightforward. Observe in conclusion that, by a simple consequence of Prop. 7 and Theorems 4, 5, we can construct smooth logics with arbitrarily many free hidden variables and with an arbitrary degree of non-compatibility (see also [3] and [9]). This seems to accord with the hidden variables hypothesis.

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Department of Mathematics, Czech Technical University - El. Eng.
Suchbátarova 2, 166 27 Prague 6, Czechoslovakia

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