Stephen D. Comer

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DIMENSION STABLE POSETS

Stephen D. COMER [1]

Abstract: The notion of a dimension stable poset is introduced and the minimal members of this class are investigated. The minimal stable posets of dimension 2 are completely described and the general crowns which are minimal stable are determined. In particular, there are an infinite number of minimal stable posets for each dimension greater than 1.

Key words: Poset, linear extension, dimension, crown, greedy, stable.

Classification: Primary 06A10
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1. Introduction. Throughout we assume that P is a finite poset. The underlying set of a poset P will also be denoted by P while the order relation is written as $\leq_P$ (or, as $\leq$ if there is no confusion). A collection $\mathcal{C}$ of linear extensions of P whose intersection is the order relation on P is called a realizer of P. The dimension of P, introduced by Dushnik and Miller [1] and written as dim(P), is defined as the minimum size of a realizer of P.

The class of general crowns S_n^k was introduced in Trotter [3]. These posets will be considered in section 4. For $n,k \geq 0$ the crown S_n^k is defined as a poset of height 1 with n+k maximal elements $a_1,\dots,a_{n+k}$ and n+k minimal elements $b_1,\dots,b_{n+k}$. The ordering in S_n^k is defined by $b_i < a_j$ iff $j \notin \{i,i+1,\dots,i+k\}$. (Subscripts are added modulo n+k.) The set of maximal elements is denoted by A and the set of all minimal elements is denoted by B. For $b \in B$, let I(b) denote the set of all $a \in A$ incomparable to b. For $a \in A$ the set I(a) is defined dually. Note that $|I(a)|=|I(b)|=k+1$ for

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all $a \in A$ and $b \in B$.

A point x in a poset P is unstable if $\dim(P-\{x\}) < \dim(P)$. A poset is called irreducible if every point in it is unstable. Irreducible posets have been extensively studied; in particular, the crowns that are irreducible are described in [3]. Posets with a "small" amount of unstability seem to have been neglected. We call a poset P (dimension) stable if it has no unstable points. A stable poset is d-stable if it has dimension d. The class of d-stable posets is large. Section 2 contains some simple observations about the class of d-stable posets. In particular, the class is determined by its minimal members, that is, d-stable posets for which the removal of some element produces a poset that is not d-stable. We say that a poset is minimal stable if it is a stable poset such that removing some pair of elements lowers the dimension. In sections 3 and 4 we describe the minimal 2-stable posets and determine the crowns S_n^k that are minimal stable.

2. Stable posets. In this section we initiate a study of d-stable posets. The first result follows immediately from the definitions. It says that the class of stable posets is a filter (that is an upward closed subset) in the poset of all isomorphism types of dimension d posets and that this filter is generated by the minimal stable posets.

Proposition 1. (1) A poset of dimension d that extends a d-stable poset is d-stable.

(2) Every d-stable poset contains a minimal d-stable poset.

The next goal is to show that every poset is embeddable in a stable poset. The following notation is needed for the construction. For $x \in P$, let $L(x)$ denote the set of all elements in P covered by x and let $U(x)$ denote the set of elements in P which cover x. The lemma below gives properties of an extension of P obtained by adding a new element to act like an old one.

Lemma 1. Suppose x is a point in a poset P and x' is a new symbol not in P. Form a poset P(x) with universe $P \cup \{x'\}$ and order relation generated by $\leq_P \cup (L(x) \times \{x'\}) \cup (\{x'\} \times U(x))$. Then

(1) P(x) is a "conservative" extension of P, i.e., for $a, b \in P$, $a \leq_{P(x)} b$ iff $a \leq_P b$.

(2) if $\dim(P) \geq 2$, $\dim P = \dim P(x)$.

Proof. (1) is clear. (2) Removing the new element x' from each linear extension in a realizer for $P(x)$ produces a realizer for P by (1). Thus, $\dim P \leq \dim P(x)$. Now, suppose $\{L_1,\ldots,L_d\}$ is a minimal realizer for P where $d=\dim(P)\geq 2$. Form L_i' from L_i by replacing x in L_i by either x, x' or x', x making sure that each pair is used at least once. (This is possible since $d\geq 2$.) Clearly each L_i' is a linear extension of $\leq_{P(x)}$. Now suppose $a,b \in P(x)$ and $(a,b) \notin \leq_{P(x)}$. If $a,b \in P$, then a is over b in some L_i, hence in some L_i'. If $\{a,b\}=\{x,x'\}$, then, by the definition of the extensions, a is over b in some L_i'. If $a=x'$ and $x\neq b \in P$, it follows that $(x,b)\notin \leq_P$. So, x is over b in some L_i. Hence, x and x' are over b in τ_i'. The case of $b=x'$ is similar, so $\{L_1',\ldots,L_d'\}$ is a realizer for $P(x)$ and $\dim P=\dim P(x)$. □

Lemma 2. If x is an unstable element in P, $\dim(P)\geq 2$ and y is unstable in $P(x)$, then y is unstable in P and $y\neq x$.

Proof. By Lemma 1, if y is unstable in $P(x)$, $y\neq x$ and $y\neq x'$; so y is unstable in P. □

Proposition 2. If a poset is not stable, it is embeddable in a minimal stable poset.

Proof. The result is clear for P with $\dim(P)=1$ since such a poset is stable if $|P|\geq 2$. For $\dim(P)\geq 2$ and P not stable, the result follows by iterating the construction in Lemma 1. Induction on the number of unstable elements in P is justified by Lemma 2. □

Note that the construction in Lemma 1 can also be used to show that every finite poset has an infinite number of stable extensions.

3. Minimal 2-stable posets.

In this section we describe the minimal stable posets of dimension 2. We begin the classification by identifying special posets. A poset is called absolute minimal stable if it is minimal stable and no proper subposet is stable. For example, all of the posets in Fig. 1 are minimal 2-stable; however, Q and R are not absolute since they contain P_3 as a proper subposet.

The next result implies that P_1, P_2, P_3 and P_4 are the only absolute minimal 2-stable posets.

Proposition 3. Every minimal 2-stable poset contains one of P_1, P_2, P_3 or P_4.

Proof. Suppose P is a minimal 2-stable poset. If P contains an antichain of size ≥ 3, then P contains P_1. Otherwise, every antichain in P has size 2. If P contains only one antichain, deleting one of its elements reduces the dimension. So P must have at least 2 antichains (of size 2), call one A and another B. Every element in A is comparable with some element in B. (Otherwise, adding it to B creates an antichain of size 3.) If each element in A is comparable with exactly one element of B, then P contains P_2. If some element of A is comparable with both elements of B, then P contains P_3 or P_4. □

The classification of all minimal 2-stable posets is obtained by combining an absolute minimal stable poset with a chain in various ways. Six infinite families result. They can be defined using the notion of an ordinal sum of posets (see [2]). In particular, let $\underline{n}$ denote an n-element chain, $A \oplus B$ denote the linear sum of A and B, and A+B denote the disjoint sum of A and B. Thus, for example, $(\underline{k} \oplus (\underline{n+1}) \oplus \underline{m})+\underline{1}$ (which is A(0,k,n,m,0) below) is the poset Q_0 in Fig. 2. The posets Q_1, Q_2, and Q_3 are ordinal sums of Q, P_3 and R, respectively.

We now define several infinite families of posets:

(i) $A(r,k,n,m,s)=\underline{r} \oplus (\underline{k} \oplus (\underline{n+1}) \oplus \underline{m})+\underline{1}) \oplus \underline{s}$ where $n \geq 1$ and $r,s,m,k \geq 0$.

(ii) $B(r,k,n,m,s)=\underline{r} \oplus Q_1(k,n,m) \oplus \underline{s}$ where $k,m \geq 1$ and $n,r,s \geq 0$

(iii) $C(k,n,m)=\underline{k} \oplus (\underline{n+2}) \oplus \underline{m}$ where $n \geq 2$ and $k,m \geq 0$.

(iv) $D(r,n,m,s)=\underline{r} \oplus Q_2(n,m) \oplus \underline{s}$ where $n,m \geq 1$ and $r,s \geq 0$

(v) $E(r,k,n,m,s)=\underline{r} \oplus (\underline{k+1}) \oplus \underline{n} \oplus (\underline{m+1}) \oplus \underline{s}$ where $k,n,m \geq 1$ and $r,s \geq 0$

(vi) $F(r,k,n,m,s)=\underline{r} \oplus Q_3(k,n,m) \oplus \underline{s}$ where $n,r,s \geq 0$ and $k,m \geq 1$.

Notice that each class of posets, except D, is closed under duals. The posets of type A, B, C, D, E and F listed above are all minimal 2-stable. The main result of this section is that the list above is complete.

Proposition 4. Suppose P is a minimal 2-stable poset.

(1) If P contains P_1, it is isomorphic to a poset of type A or type B with $n > 0$.

(2) If P contains P_2, but not P_3, it is isomorphic to a poset

of type C.

(3) If P contains P_3, but neither P_1 nor P_4, it is isomorphic to a poset of type D (or its dual), a poset of type E, or a poset of type B with n=0.

(4) If P contains P_4, it is isomorphic to a poset of type F.

Proof. (1) If P is minimal stable and contains an antichain of size 3, two elements from this antichain must be removed to drop the dimension. The result will be a chain. Thus, P can be constructed from a chain L by adjoining a two element antichain $\{x,y\}$ in such a way that both x and y are incomparable to some element in L. There are various possibilities depending upon whether or not each of x and y is incomparable from all elements in L, below some element in L, above some element in L, or both. The table below enumerates the joint possibilities where the entry corresponding to a row and column is the type of poset specified by the conditions. We write $x \parallel L$ to mean that x is incomparable with all elements of L, $x<L$ to mean that $x<c$ for some $c \in L$, etc. In all cases $n>0$.

	$x \parallel L$	$x<L$	$x>L$	$L<x<L$
$y \parallel L$	A(0,0,n,0,0)	A(0,0,n,m,0)	A(0,k,n,0,0)	A(0,k,n,m,0)
$y<L$	A(0,0,n,m,0)	A(0,0,n,m,s)	B(0,k,n,0,0)	A(0,k,n,m,s) B(0,k,n,m,s)
$L<y$	A(0,k,n,0,0)	B(0,k,n,m,0)	A(r,k,n,0,0)	A(r,k,n,m,0) B(r,k,n,m,0)
$L<y<L$	A(0,k,n,m,0)	A(0,k,n,m,s) B(0,k,n,m,s)	A(r,k,n,m,0) B(r,k,n,m,0)	A(r,k,n,m,s) B(r,k,n,m,s)

The proof of parts (2), (3) and (4) is similar. □

4. Minimal stable crowns. In [3] conditions on n and k are given which determine when the crown S_n^k is irreducible. If S_n^k is not irreducible, it is stable! (This follows from the observation that $\dim(S_n^k-\{x\})=\dim(S_n^k)$ for all x whenever it holds for some x; a result which is a consequence of the fact that the automorphism group of S_n^k is transitive on the minimal (maximal) elements.) In this section we determine which crowns are minimal stable.

Proposition 5. A crown S_n^k is a minimal d-stable poset if and only if n and k satisfy one of the following conditions:

(1) $k=1$ and $n+1=3q$ (so $d=2q$),

(2) $n+k=q(k+2)+2$ (so $d=2q+1$),

(3) $n+k=q(k+2)+[(k+2)/2]+1$ where k is an even positive integer (so $d=2q+2$).

Proof. The arguments are only sketched since the techniques are very similar to those used in [3] to characterize the irreducible crowns.

If S_n^k is a minimal d-stable poset, $\dim(S_n^k-\{x,y\})=\dim(S_n^k)-1$ for some x, y. Using the observation that, for crowns, stable is the same as not irreducible, and comparing the weights of the posets involved with the weights of linear extensions (as in Theorem 5.8 of [3]) it follows that one of the following four conditions must hold:

(i) $n+k=q(k+2)$ where $k=1$ or $k=2$,

(ii) $n+k=q(k+2)+2$,

(iii) $n+k=q(k+2)+[(k+2)/2]+1$ where k is a positive even integer,

(iv) $n+k=q(k+2)+[(k+2)/2]+2$

We next observe that in case (i) $k=2$ is impossible and case (iv) is also impossible. The argument for $k=2$ in (i) and for k even and positive in (iv) is similar to the proof of Theorem 5.6 of [3] in the k even and positive case. This works because if $S_n^k-\{x,y\}$ lowers the dimension in these cases, each linear extension in a minimal realizer must have maximal possible weight. This is not the case when k is odd and positive in (iv), but a modification of the argument still works. There are four cases to be considered depending upon whether x, y are both minimal (maximal) in S_n^k or one of each and whether $|I(x)\cap I(y)|$ is 0 or 1. For sake of this sketch we assume $x,y\in B$. Assuming that $S_n^k-\{x,y\}$ has a realizer $\{L_1,\dots,L_{2q+1}\}$ it is possible to show (along the lines of the argument in Theorem 5.6 of [3]) there exists another realizer $L_1',\dots,L_{2q}',L_{2q+1}$ where each L_i' has maximal possible weight and L_{2q+1} must place $t+1=[(k+2)/2]$ elements of B over $k+1$ elements of A. This is impossible since each $b\in B$ is incomparable with a different subset of A of size $k+1$.

It remains to see that S_n^k is minimal stable in case (i) with $k=1$ and in cases (ii) and (iii). Crowns S_n^k in (i) with $k=1$ have the form S_{3q+2}^1 where $q\geq 1$. Since S_{3q+2}^1 is $(2q+2)$-stable it suffices to see that the poset P obtained by removing a_{3q+2} and a_{3q+3} has dimension $2q+1$. If $\{L_1,L_2,\dots,L_{2q+2}\}$ is the realizer for S^1_{3q+2} constructed on pp. 90-91 of [3] and if L extends

$$[a_1, b_{3q+1}, b_{3q}, a_{3q+1}, b_{3q-1}]$$

(remember, in [3], larger elements are listed before smaller ones), the chains $\{L_1, L_3, L_4, \ldots, L_{2q}, L, L_{2q+2}\}$ restrict to a realizer of P with size 2q+1. It follows that S^1_{3q+2} is minimal stable.

Case (ii) is similar. It suffices to construct a realizer of size 2q for the poset obtained from S^k_n by removing a_{n+k-1} and a_{n+k}. Again, using the notation from pp. 90-91 of [3], such a realizer is $\{L_1, L_3, L_4, \ldots, L_{2q}, L_{2q+2}\}$.

To show that S^k_n is a minimal (2q+2)-stable poset where n, k are given in case (iii) the construction in Theorem 4.8 of [3] is employed. It suffices to construct 2q+1 linear extensions that realize $S^k_n - \{a_{n+k}, b_{n+t}\}$ where k=2t. Th is is done in the following way. Partition A into sets A_j and I_j as in the argument cited and form linear extensions $L_2, \ldots, L_{2q+1}$ corresponding to $I_2, \ldots, I_{2q+1}$. Now form L' by ordering I_1 by increasing subscripts, placing the last t+1 elements of A_{q+1} above these elements in decreasing subscript order, and finally inserting the elements of $I(a_1)$ in the list as high as allowed by the ordering on S^{2t}_n. The collection $\{L', L_2, \ldots, L_{2q+1}\}$ is the desired realizer. This completes the proof of Proposition 5.

From the number-theoretic conditions in Proposition 5 we obtain

<u>Corollary</u>. There exist an infinite number of minimal d-stable posets for each $d \geq 3$.

Other infinite families of minimal stable posets can be obtained from non-minimal stable S^k_n's by removing one, two,... ... elements. It may be worth classifying these clipped crowns.

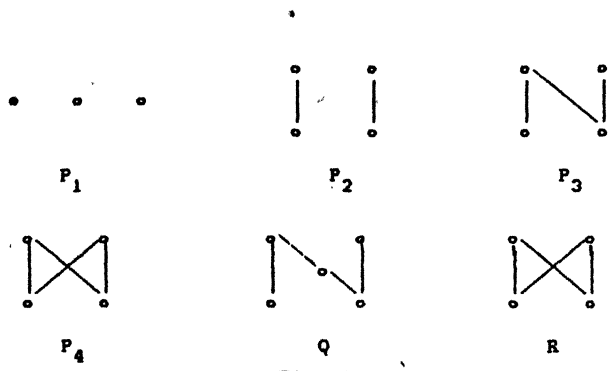

Fig. 1

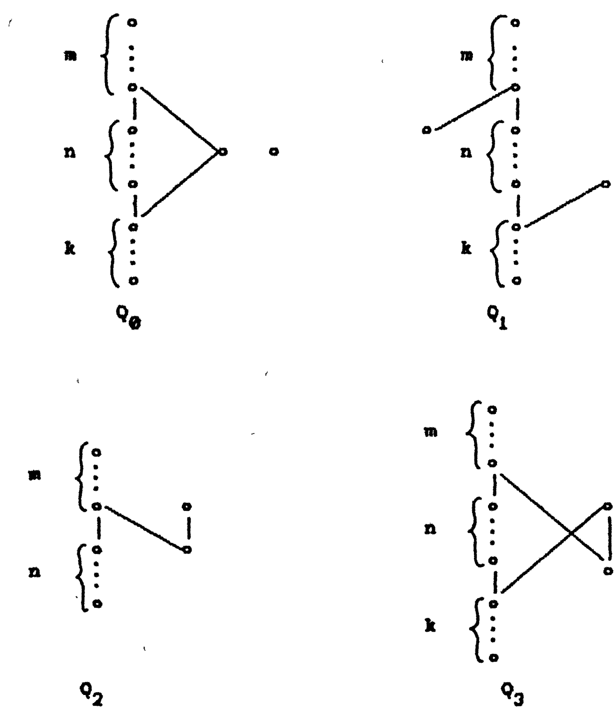

Fig.2

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The Citadel, Charleston, S.C. 29409, U.S.A.

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