Osvald Demuth; Antonín Kučera

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REMARKS ON 1-GENERICITY, SEMIGENERICITY AND RELATED CONCEPTS

O. DEMUTH, A. KUČERA

Abstract: Properties of recursive enumerable sets of strings covering all recursive sets of natural numbers or, equivalently, Π_1^0 classes of a special kind are studied especially in a connection with modification of the notion of 1-genericity.

Key words: Recursion theory, tt-reducibility, T-reducibility, 1-genericity, coverings, semigenericity, Π_1^0 classes, NAP-sets, FPF-functions.

Classification: 03D30

The aim of the paper is to study modifications of the notion of 1-genericity and their relation to Π_1^0 classes. Especially, we show that for nonrecursive sets non-semigenericity (introduced by Demuth [4]) is equivalent to strong undecidability (introduced by Ceĭtin [2]). We also give some results on the structure of T-degrees.

We use the notation and terminology of [4].

The following notion was introduced by Ceĭtin [2].

Definition ([2]). A set A of NNs is said to be strongly undecidable if there exists a partial recursive function ψ such that for any recursive set M of NNs and any index v of the characteristic function of M $\psi(v)$ is defined and $A \cap \{0,1,\dots,\psi(v)\} \neq M \cap \{0,1,\dots,\psi(v)\}$.

The important fact is that the class of all strongly undecidable sets of NNs can be characterized by special Π_1^0 classes or, equivalently, by coverings.

Theorem 1. For any set A of NNs there exists a covering which does not cover A if and only if A is strongly undecidable.

Proof. The implication $\Rightarrow$ is obvious. The opposite implication is an immediate corollary of a result of Kušner [7, Theorem 1]. Let us also note that a weaker form of the mentioned Kušner's result is in Moschovakis [9, Theorem 11] and implicitly also in Ceĭtin [1].

We have an immediate corollary.

Corollary 2. A set of NNs is semigeneric if and only if it is neither strongly undecidable nor recursive.

Ceĭtin [2] studied the notion of strong undecidability only for r.e. sets. Nevertheless, for some of his results this restriction is not necessary. Now we give briefly a list of results on strong undecidable sets of NNs proved by Ceĭtin [2] where we omit the assumption of recursive enumerability whenever possible.

Theorem 3 ([2]). No strongly undecidable set of NNs has a hyperimmune complement.

Theorem 4 ([2]). Any of the following properties of a set A of NNs implies its strong undecidability.

1) A is a r.e. set and there exists a r.e. set B such that A, B are disjoint and form a recursively inseparable pair.

2) A is a creative set.

3) A is a simple set which is not hypersimple.

4) Some strongly undecidable set is tt-reducible to A.

Remark 1. 1) According to Theorem 3 and to parts 3 and 4 of Theorem 4 any simple set tt-reducible to some hypersimple set must be hypersimple, too ([2]).

2) On the basis of Corollary 2 we see that Theorem 3 and part 4 of Theorem 4 give us both part 1 of Theorem 9 and Corollary 12 from [4].

As we saw in Example 18 [4], the fact that for any recursive set M of NNs the set $M \Delta B$ is infinite but not hyperimmune, does not imply strong undecidability of B. On the other hand, we will show that a kind of uniformity of non-hyperimmunity (of such symmetric differences) does imply it.

Theorem 5. Let A be a set of NNs. Then A is strongly undecidable if and only if there is a recursive function f such that for any recursive set M of NNs and for any index v of the characteristic function of M the symmetric difference $M \Delta A$ is infinite and majorized by the recursive function with index f(v).

Proof. The theorem can be easily proved by the method used in the proof of Lemma 10 from [4].

Now we turn to questions concerning T-degrees of members of some Π^0_1 classes and also to their connection with 1-generic T-degrees.

First, let us recall that a set A of NNs is called <u>1-generic</u> if $\forall z \exists \sigma [\sigma \subseteq A \& ((\varphi^{\sigma}_z(z)$ is defined$) \vee \forall \tau(\tau \supseteq \sigma \Rightarrow \varphi^{\tau}_z(z)$ is undefined$))]$

or, equivalently,

for any r.e. set $\mathcal{S}$ of strings there is a string σ such that $\sigma \subseteq A$ and either $\sigma \in \mathcal{S}$ or no set of NNs is covered by both σ and $\mathcal{S}$.

Any 1-generic set of NNs is, obviously, semigeneric.

As we saw in [4], there are weakly 1-generic T-degrees which contain NAP-sets or, more generally, FPF-functions, i.e. which are NAP T-degrees or FPF T-degrees. Let us recall that a function f is called a <u>FPF-function</u> if $\forall x(f(x) \neq \varphi_x(x))$ holds. On the other hand, we shall show that the classes of 1-generic T-degrees and of FPF T-degrees are disjoint and that even below any 1-generic T-degree there is no FPF T-degree. Since some other classes of T-degrees also possess an analogical property, we present a more general statement.

First, we introduce a notation. By Red(σ, τ, z) we denote $(\forall x < lh(\sigma))((\varphi^{\tau}_z(x)$ is defined$) \& (\varphi^{\tau}_z(x) = \sigma(x)))$. The predicate Red is obviously recursive. Further, for any sets A and B of NNs and for any NN z we have $(A \leq_T B$ via $z) \Longleftrightarrow \forall \sigma [(A$ is covered by $\sigma) \Rightarrow$

$$(B \text{ is covered by } \{\tau : Red(\sigma, \tau, z)\})].$$

<u>Theorem 6</u>. Let A be a 1-generic set of NNs. Then any set B of NNs, $B \leq_T A$, is covered by any simple set of strings.

Proof. Suppose $B \leq_T A$ via z. Let $\mathcal{S}$ be a simple set of strings. We denote the set

$\{\tau : \exists \sigma (Red(\sigma, \tau, z) \& (\sigma$ is covered by $\mathcal{S}))\}$

by $\mathcal{R}$. Obviously, $\mathcal{R}$ is recursively enumerable.

Suppose that B is not covered by $\mathcal{S}$. Then A cannot be covered by $\mathcal{R}$. Since A is 1-generic, there exists a string σ such that $\sigma \subseteq A \& \forall \tau(\tau \supseteq \sigma \Rightarrow \tau \notin \mathcal{R})$. Consequently, the set of strings $\{\rho : \exists \tau(\tau \supseteq \sigma \& Red(\rho, \tau, z))\}$ is disjoint with $\mathcal{S}$.

Further, it is obviously recursively enumerable and, according to the supposed $B \leq_T A$ via z, also infinite. This contradiction to the simplicity of $\mathcal{S}$ shows that B must be covered by $\mathcal{S}$.

Remark 2. Let us note that any simple set of strings is necessarily a covering (may be, not a proper one). Further, for any pair A, B of disjoint r.e. recursively inseparable sets of NNs the set of strings

$\{\tau : (\exists x < lh(\tau))(x \in A \,\&\, \tau(x) = 0 \vee x \in B \,\&\, \tau(x) = 1)\}$

is simple and does not cover any set of NNs separating A and B (consequently, it is a proper covering). Later we shall study coverings $\mathcal{S}$ such that neither $\mathcal{S}$ nor $\{\tau : \tau$ is covered by $\mathcal{S}\}$ is simple.

The class of all sets of NNs not covered by a given r.e. set of strings forms a Π^0_1-class. Since any Π^0_1class can be obtained in this way, Theorem 6 can be reformulated as follows.

Corollary 7. Let A be a 1-generic set of NNs and let $\mathcal{A}$ be a Π^0_1 class of sets of NNs such that the set of all $\mathcal{A}$-extendible strings (i.e. strings extendible to elements of $\mathcal{A}$) is immune. Then there is no set B of NNs such that $B \leq_T A \,\&\, B \in \mathcal{A}$.

For Π^0_1 classes which are not necessarily recursively bounded, we need an additional care. The following notions will be useful.

Definition. 1) By an F-string we mean a finite sequence of NNs.

2) A set $\mathcal{S}$ of strings V-covers (i.e., covers in the sense of Vitali) a set A of NNs if for every NN k there is a string $\sigma \in \mathcal{S}$ such that $lh(\sigma) \geq k \,\&\, \sigma \subseteq A$. Analogically, it is defined that a set of F-strings V-covers a function.

Theorem 8. Let A be a 1-generic set of NNs and let $\mathcal{A}$ be a nonempty Π^0_1 class such that there is no r.e. set of $\mathcal{A}$-extendible F-strings which V-covers some function. Then $\mathcal{A}$ contains no A-recursive function.

Proof. The statement can be proved by the method used in the proof of Theorem 6.

Corollary 9. No FPF-function is recursive in a 1-generic set.

Proof. It is easy to see that the class of all FPF-functions, say the class $\mathcal{F}$, is a Π^0_1 class containing no recursive function. Suppose $\mathcal{S}$ is a r.e. set of $\mathcal{F}$-extendible F-strings which V-covers some function. Observe that if σ and $\tau * \rho$ are $\mathcal{F}$-extendible F-strings and $lh(\sigma)=lh(\tau)$ holds then the F-string $\sigma * \rho$ is also $\mathcal{F}$-extendible. Now, by enumerating $\mathcal{S}$ and applying the method just described, we can construct a recursive function being an element of $\mathcal{F}$. We have a contradiction.

Corollary 10. No NAP-set is recursive in a 1-generic set.

Proof. It follows immediately from the above Corollary 9 and from Corollary 1 of Theorem 6 of [6].

Remark 3. 1) Let $\mathcal{F}_0$ be the class of all $\{0,1\}$-valued FPF-functions. Obviously, $\mathcal{F}_0$ is a recursively bounded Π^0_1class. We claim that the set of all strings which are not $\mathcal{F}_0$-extendible is an effectively simple set of strings.

First, there is a recursive function h such that for every NNs x and y $\varphi_{h(x)}(y)$ is

a) defined and equal to $\sigma(y)$, where σ is the first string of the length $\geq y$ which appears in $\langle W_x \rangle$ (under the standard enumeration) - if there is such a string;

b) undefined - otherwise.

Suppose that $\langle W_x \rangle$ contains only $\mathcal{F}_0$-extendible strings. Then ($\varphi_{h(x)}(y)$ is defined) $\Longrightarrow \varphi_{h(x)}(y) \neq \varphi_y(y)$ holds for any y. Thus, $\varphi_{h(x)}(h(x))$ is necessarily undefined and the set $\langle W_x \rangle$ contains no string of length $\geq h(x)$.

2) Let $\mathcal{F}$ be the class of all FPF-functions. Since at most one F-string of the length 1 is not $\mathcal{F}$-extendible, we see that the set of all $\mathcal{F}$-extendible F-strings is not immune. On the other hand, we can prove, by the method used in part 1, the following statement. There is a recursive function f such that for any NN x for which the r.e. set of F-strings with index x, say set $\mathcal{S}$, contains only $\mathcal{F}$-extendible F-strings we have: $\mathcal{S}$ contains no F-string of length $\geq f(x)$.

As we saw, there are proper coverings which are simple or even effectively simple. Now we shall be interested in proper coverings $\mathcal{S}$ for which the set of all strings not covered by $\mathcal{S}$ is not immune, i.e. the set $\{\tau : \mathcal{S} \text{ covers } \tau\}$ (which is again a covering and covers the same sets of NNs as $\mathcal{S}$ does) is not simple. The existence of such proper coverings follows from Theorem

6,[5, Corollary 1.1] and the fact that there are 1-generic sets recursive in $\emptyset'$.

Definition. For any set $\mathcal{S}$ of strings let $Cl(\mathcal{S})$ denote the set $\{\tau : \tau$ is covered by $\mathcal{S}\}$.

Remark 4. Let $\langle W_p \rangle$ be a proper covering. Then the set $\mathcal{T}=\{\tau : \exists s(lh(\tau)=s \,\&\, (\tau$ is not covered by $\langle W_p^s \rangle))\}$ is an infinite r.e. set of strings such that for any set A of NNs A is not covered by $\langle W_p \rangle$ if and only if A is V-covered by $\mathcal{T}$.

Remark 5. A class $\mathcal{K}$ of sets of NNs is a Π_2^0 class if and only if there exists a NN t such that $\mathcal{K}$ is the class of all sets of NNs V-covered by $\langle W_t \rangle$.

The following result is a modification of [5, Corollary 1.3].

Theorem 11. Let t be a NN such that $\langle W_t \rangle$ V-covers no recursive set. Then there exists a proper covering $\mathcal{S}$ such that $Cl(\mathcal{S})$ is not simple and for any set A of NNs V-covered by $\langle W_t \rangle$ there is a set B of NNs, $B \equiv_T A$, not covered by $\mathcal{S}$.

Proof. Let us take a proper covering $\mathcal{T}$ such that $Cl(\mathcal{T})$ is not simple and $\mathcal{T}$ is a set of incomparable (with respect to $\subseteq$) strings. Since $\mathcal{T}$ is infinite, let us fix a recursive enumeration $\{\tau_x\}_{x\in N}$ of $\mathcal{T}$ such that $\tau_x \neq \tau_y$ for $x \neq y$.

Suppose $W_t^0=\emptyset$ and $W_t^{i+1} \setminus W_t^i$ contains at most one element. We enumerate $\mathcal{S}$ in steps. At the beginning of step i we have two lists of strings $\{\sigma_x\}_{x=0}^{\varkappa_i}$, $\{\rho_x\}_{x=0}^{\varkappa_i}$. Let $\varkappa_0=0$, $\sigma_0=\rho_0=\Lambda$ (an empty string).

Step i. Case 1. $W_t^{i+1} \setminus W_t^i=\emptyset$. Then $\varkappa_{i+1}=\varkappa_i$ and we enumerate into $\mathcal{S}$ all strings of the form $\sigma_x * \tau_i$ for $x \leq \varkappa_i$.

Case 2. Let $n \in W_t^{i+1} \setminus W_t^i$.

Subcase 2a. $(\exists x \leq \varkappa_i)(\sigma_n \subseteq \rho_x)$. Proceed as in case 1.

Subcase 2b. Subcase 2a does not apply. Find k for which ρ_k is the longest string ρ_x, $x \leq \varkappa_i$, satisfying $\rho_x \subseteq \sigma_n$. Observe that $\rho_k \neq \sigma_n$. Let η be a string such that $\rho_k * \eta = \sigma_n$ and P_η the list of all strings of length $lh(\eta)$ and different from η. Enumerate into $\mathcal{S}$ all strings of the form

i) $\sigma_x * \tau_i$ for $x \leq \varkappa_i \,\&\, x \neq k$,

ii) $\sigma_k * \tau_i * \alpha$, where $\alpha \in P_\eta$,

iii) $\sigma_k * \tau_i * \eta * \tau_y$ for $y \leq i$.

Let $\varkappa_{i+1} = \varkappa_i + 1$, $\rho_{\varkappa_{i+1}} = \sigma_n$ and $\sigma_{\varkappa_{i+1}} = \sigma_k * \tau_i * \eta$.
This concludes the construction. Informally, the idea is as follows. Let $\mathfrak{M} = \{x: \exists y(x \leq \varkappa_y)\}$. The set of strings $\{\rho_x\}_{x \in \mathfrak{M}}$ V-covers the same class of sets as $\langle W_t \rangle$ does. For any NNs x, y contained in $\mathfrak{M}$ the string ρ_x is coded by σ_x, σ_x is not covered by $\mathcal{S}$ and $\sigma_x \subseteq \sigma_y \Longleftrightarrow \rho_x \subseteq \rho_y$ holds.

Observe that any string which is not covered by $\mathcal{T}$ is not covered by $\mathcal{S}$, too.

Let A be a set of NNs V-covered by $\langle W_t \rangle$. Then $\mathfrak{M} = N$ and there are an increasing A-recursive function h such that $\forall x(\rho_{h(x)} \subseteq A)$ and a unique set B of NNs satisfying $\forall x(\sigma_{h(x)} \subseteq B)$. Thus, $\forall y(\rho_y \subseteq A \Longleftrightarrow \sigma_y \subseteq B)$, $B \equiv_T A$ and B is not covered by $\mathcal{S}$

On the other hand, if a set B is not covered by $\mathcal{S}$ then there are two possibilities:

a) there are a NN $k \in \mathfrak{M}$ and a set C of NNs not covered by $\mathcal{T}$ such that $B = \sigma_k * C$,

b) there is a unique set A of NNs V-covered by $\langle W_t \rangle$ for which $\forall x(\rho_x \subseteq A \Longleftrightarrow \sigma_x \subseteq B)$, and thus $B \equiv_T A$.

We omit further details.

On the basis of Remark 4 we obtain the following corollary.

Corollary 12. If A is a nonrecursive non-semigeneric (i.e. strongly undecidable) set of NNs then there are a set B of NNs, $B \equiv_T A$, and a proper covering $\mathcal{S}$ such that $Cl(\mathcal{S})$ is not simple and $\mathcal{S}$ does not cover B.

We would like to characterize nonrecursive non-semigeneric T-degrees.

Lemma 13. For any NNs t, z if we take a NN p such that $W_p = \{y: \exists x(x \in W_t \,\&\, Red(\sigma_x, \sigma_y, z) \,\&\, \neg(\exists v < y)(Red(\sigma_x, \sigma_v, z) \,\&\, \sigma_v \subseteq \sigma_y))\}$, then

a) if $\langle W_t \rangle$ V-covers no recursive set, then so does $\langle W_p \rangle$,

b) for any sets A and B, $\emptyset <_T A \,\&\, (A \leq_T B$ via z$)$, $\langle W_t \rangle$ V-covers A if and only if $\langle W_p \rangle$ V-covers B.

Proof. Immediate.

Theorem 14. For any set C of NNs $deg_T(C)$ contains a nonrecursive non-semigeneric set if and only if there is a NN t such that $\langle W_t \rangle$ V-covers no recursive set but it does V-cover C.

Proof. The implication $\Leftarrow$ follows from Theorem 11. The opposite implication follows from Remark 4 and Lemma 13.

In a connection with Theorem 6 we will show that T-degrees of sets of NNs not covered by some simple set of strings form an upper cone.

Lemma 15. Let f be a recursive function, $\mathcal{S}$ a simple set of strings. Then there exists a r.e. set $\mathcal{T}$ of strings such that

1) $\mathcal{T}=Cl(\mathcal{T})$ and either $\mathcal{T}$ contains all strings or $\mathcal{T}$ is simple,

2) for any sets A, B of NNs such that $A \leq_{tt} B$ via f, $\mathcal{S}$ covers A if and only if $\mathcal{T}$ covers B.

Proof. We take a r.e. set $\mathcal{T}_0$ of strings,
$\mathcal{T}_0=\{\sigma : \exists \rho\, (\rho \in \mathcal{S} \,\&\, (\rho \leq_{tt} \sigma \text{ via } f))\}$ (cf.[4, Remark 8]).
Let $\mathcal{T}=Cl(\mathcal{T}_0)$.

Suppose $\langle W_t \rangle$ is infinite and disjoint with $\mathcal{T}$. Then, obviously, the set $\{\rho : \exists \sigma\, ((\sigma \text{ belongs to } \langle W_t \rangle) \,\&\, (\rho \leq_{tt} \sigma \text{ via } f))\}$ is r.e. disjoint with $\mathcal{S}$ and, as it can be easily verified, infinite. It contradicts the simplicity of $\mathcal{S}$. The proof of 2) is immediate.

Corollary 16. For sets A, B of NNs such that $A \leq_{tt} B$ and A is not covered by some simple set of strings, there exists a simple set of strings which does not cover B.

Theorem 17. The class of all T-degrees containing a set of NNs which is not covered by some simple set of strings forms an upper cone.

Proof. For any sets A, B of NNs, $A \leq_T B$, we have $A \oplus B \equiv_T B$ and $A \leq_{tt} A \oplus B$. It remains to use Corollary 16.

At the end we return to NAP-sets. We shall study how r.e. sets of strings of a small measure cover sets to which a NAP-set is T-reducible. First, we need a more detailed information about the recursive function e mentioned in [4]. We can suppose that for any NN m $\langle W_{e(m)} \rangle = Cl(\langle W_{e_0(m)} \rangle)$, where

$$W_{e_0(m)} = \{x : \exists y\, z\, s(m < y < z \,\&\, (\varphi_y(z) \text{ is defined}) \,\&\, \mu(\langle W^s_{\varphi_y(z)} \rangle) \leq 2^{-z} \,\&\, x \in W^s_{\varphi_y(z)})\}.$$

Then, in addition to [4] we have the following. For any NNs m, p, q

$$m < p < q \,\&\, (\varphi_p(q) \text{ defined}) \,\&\, \mu(\langle W_{\varphi_p(q)} \rangle) \leq 2^{-q} \Rightarrow W_{\varphi_p(q)} \subseteq W_{e(m)}$$

holds. (Cf.[8],[3],[6].) Further, similarly as in Remark 3 we can prove that for any NN m $\langle W_{e(m)} \rangle$ is an effectively simple set of strings.

Theorem 18. For any NNs m, z there are a NN p and a recursive function f such that for any NN t

a) $\mu(\langle W_{f(t)} \rangle) \leq 2^p \cdot \mu(\langle W_t \rangle)$;

b) for any sets A and B of NNs for which $A \leq_T B$ via z holds and A is not covered by $\langle W_{e(m)} \rangle$ (thus, A is a NAP-set) we have

(A is covered by $\langle W_t \rangle$) $\Longleftrightarrow$ (B is covered by $\langle W_{f(t)} \rangle$).

Proof. Let m and z be NNs. By the s-m-n theorem we have recursive functions h and g such that for any NNs x and v

$W_{h(x)} = \{y : Red(\sigma_x, \sigma_y, z)\}$ and

$W_{g(v)} = \{w : \mu(\langle W_{h(w)} \rangle) > 2^v \cdot \mu(\{\sigma_w\})\}$.

(Observe, $\mu(\{\sigma_x\}) = 2^{-lh(\sigma_x)}$.) Obviously, $2^v \cdot \mu(\langle W_{g(v)} \rangle) \leq 1$ for any NN v. Let b be an index of g fulfilling m<b and let p=b+1. Then, as we know, $W_{g(p)} \subseteq W_{e(m)}$.

We can construct two recursive functions f_0 and f such that for any NN t

a) $\langle W_{f_0(t)} \rangle$ is a set of incomparable (with respect to $\subseteq$) strings and ${}^0\forall\tau((\tau$ covered by $\langle W_t \rangle) \Longleftrightarrow (\tau$ covered by $\langle W_{f_0(t)} \rangle))$ holds;

b) $W_{f(t)} = \{y : \exists x\, s(x \in W_{f_0(t)} \,\&\, (\mu(\langle W^s_{h(x)} \rangle) \leq 2^p \cdot \mu(\{\sigma_x\}))$
$\& \, y \in W^s_{h(x)})\}$.

Then, p and f satisfy all the required properties.

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Matematicko-fyzikální fakulta, Karlova Universita, Malostranské nám. 25, 118 00 Praha 1, Czechoslovakia

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