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Osvald Demuth

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A NOTION OF SEMIGENERICITY

Oswald DEMUTH

Abstract: The class of semigeneric sets, i.e. nonrecursive sets covered by any recursively enumerable set of strings that covers all recursive sets, is studied. Besides, a new characterization of weak 1-genericity is given.

Key words: Recursion theory, tt-reducibility, T-reducibility, minimal degrees, hyperimmune sets, weakly 1-generic sets.

Classification: 03D30

In this paper, the notion of semigeneric set of natural numbers (NNs) is introduced on the basis of the class of recursively enumerable (r.e.) sets of (binary) strings covering all recursive sets of NNs. The well-known correspondence between recursive real numbers and recursive sets of NNs makes the obtained results interesting also from the point of view of constructive mathematics. Semigenericity is a generalization of weak 1-genericity introduced and studied by Kurtz [4]. We give a new characterization of this type of sets.

We now consider the notation. The sign $\doteq$ denotes graphical equality. For relations and operations on sets of NNs (or strings), a standard notation is used, $A\setminus B$ signifies the difference and $A\Delta B$ the symmetric difference of sets A and B. The set of (all) NNs is denoted by N, the symbols s, t, u, v, w, x, y, z are variables for NNs. $\lambda xy\ \langle x,y\rangle$ denotes a primitive recursive pairing function which is 1-1 and onto and π_1 and π_2 two primitive recursive functions such that $\langle \pi_1(m), \pi_2(m)\rangle = m$ for any NN m. For every sets A and B of NNs and every NN k $A\oplus B$ is the set $\{x: \exists y(x=2y \,\&\, y\in A \vee x=2y+1 \,\&\, y\in B)\}$ and $(A)_k \rightleftharpoons \{x:\langle k,x\rangle\in A\}$, where $\rightleftharpoons$ stands for "denotes". $\mu x(\dots)$ means: the least NN x fulfilling $(\dots)$.

A string is a finite sequence of 0's and 1's (i.e. a word in the alphabet $\{0,1\}$). In the sequel the symbols ρ, σ, τ play the role of variables for strings, Λ is the empty string, $lh(\sigma)$ is the length of σ. Further, $\sigma * \tau$ denotes the concatenation of σ and τ, $\rho \subseteq \sigma$ means σ extends ρ and $\rho \prec \sigma$ means ρ lexicographically precedes σ. For any NN n, let σ_n be a string with the number n in the linear ordering $\prec$ of all strings. Thus, for any NN k, strings of length k have just numbers 2^k-1, $2^k, \dots,$ $2^{k+1}-2$. Strings are often taken for (finite codes of) functions from finite initial segments of N into $\{0,1\}$ and signs denoting sets of NNs also stand for their characteristic functions. Thus, $\sigma(x)$ is defined if and only if $x < lh(\sigma)$ holds and $\lambda x\, A(x)$ is the characteristic function of A for any set A of NNs.

For any sets $\mathcal{S}_0$ and $\mathcal{S}_1$ of strings, any string σ, any set A of NNs and any NNs m and n, $m \leq n$,

a) $\langle A \rangle \rightleftharpoons \{\sigma_x : x \in A\}$; $A[m,n]$ denotes the string corresponding to the restriction of the function $\lambda x\, A(x+m)$ to the initial segment $\{x : x \leq n-m\}$;

b) $\sigma \subseteq A$ ("A extends σ" or "σ covers A") denotes $\sigma \subseteq A[0,lh(\sigma)]$; "$\mathcal{S}_0$ covers A" means: there is a string $\tau \in \mathcal{S}_0$ covering A; "$\mathcal{S}_0$ covers σ" means: any set of NNs covered by σ is also covered by $\mathcal{S}_0$; "$\mathcal{S}_1$ overlaps $\mathcal{S}_0$" means: there is a set of NNs covered by both $\mathcal{S}_0$ and $\mathcal{S}_1$.

Let us notice that by König's lemma ($\mathcal{S}_0$ covers σ) $\Longleftrightarrow$ $\Longleftrightarrow \exists x \forall \rho\, (lh(\rho)=x \Rightarrow \exists \tau(\tau \in \mathcal{S}_0 \,\&\, \tau \subseteq \sigma * \rho))$ holds and, consequently, for any r.e. set $\mathcal{S}$ of strings the predicate "$\mathcal{S}$ covers σ" of a variable σ is recursively enumerable.

We assume a standard indexing of all partial recursive functions of one variable and indexing as well as enumeration of all r.e. sets of NNs and, consequently, also of all r.e. sets of strings. Let φ_x be the partial recursive function with index x, W_x the domain of φ_x and W_x^s the finite subset of W_x enumerated after s steps. (Hence, $\langle W_x \rangle$ is the r.e. set of strings with index x and $\langle W_x^s \rangle$ its finite part obtained after s steps.)

Analogically, for any set A of NNs and any NN x φ_x^A denotes the partial A-recursive function of one variable with A-index x, W_x^A denotes the domain of φ_x^A, and A' the jump of A (i.e. the set $\{x : \varphi_x^A(x)$ is defined$\}$). The notation φ_x^{τ} has the usual meaning: if the evaluation of $\varphi_x^B(y)$, where $B \rightleftharpoons \{z : z < lh(\tau) \,\&\, \tau(z)=1\}$ (and thus, $\tau \subseteq B$), finishes within $lh(\tau)$ steps and all oracle information needed is coded in τ, then $\varphi_x^{\tau}(y)$ is defined and

$\varphi_x^{\tau}(y)$ denote the value of $\varphi_x^{B}(y)$; otherwise $\varphi_x^{\tau}(y)$ is not defined. Clearly, the predicate "$\varphi_x^{\tau}(y)$ is defined" of variables τ, x and y is recursive.

We use the notation on tt-reducibility and T-reducibility introduced in Rogers [8], $\deg_{tt}(A)$ and $\deg_T(A)$ denote tt-degree and T-degree of the set A, respectively. For any property P of sets of NNs a degree is called P (resp. P-free) if it contains some (resp. no) set with property P. (Thus, for some P a degree can be both P and non-P.)

For any strings ρ and σ, set $\mathcal{S}$ of strings and recursive function f

a) "$\rho \leq_{tt} \sigma$ via f" means: for any NN m, $m < lh(\rho)$, the associated set of the tt-condition f(m) is a subset of the set $\{x : x < lh(\sigma)\}$ and $(\rho(m)=1 \Longleftrightarrow (f(m)$ is satisfied by the set $\{y : y < lh(\sigma) \mathbin{\&} \sigma(y)=1\}))$ holds;

b) $\mathcal{I}\mathcal{I}(\rho, f)$ ($\mathcal{I}\mathcal{I}$ stands for inverse image) denotes a (finite and possibly empty) list of all strings being the shortest ones fulfilling the predicate ($\rho \leq_{tt} \tau$ via f) of a variable τ;

c) $\mathcal{I}\mathcal{I}(\mathcal{S}, f) \rightleftharpoons \bigcup_{\tau \in \mathcal{S}} \mathcal{I}\mathcal{I}(\tau, f)$.

Let us notice that for any recursive function f the predicates ($\rho \leq_{tt} \sigma$ via f) of variables ρ and σ and $\exists \tau(\rho \leq_{tt} \tau$ via f) of a variable ρ are, obviously, recursive.

The (Lebesgue) measure on the class of all sets of NNs introduced by Sacks [9] gives us in a natural way a measure on the class of all sets of strings. For any set $\mathcal{S}$ of strings and any string σ, the class of all sets of NNs covered by both $\mathcal{S}$ and σ is obviously measurable and we denote the measure of it by $\mu(\mathcal{S}, \sigma)$. Let $\mu(\mathcal{S})$ denote $\mu(\mathcal{S}, \Lambda)$. Thus, $\mu(\{\tau\}) = 2^{-lh(\tau)}$ and the predicate $\mu(\langle W_x \rangle, \rho) > (1-2^{-y}) \cdot \mu(\{\rho\})$ of variables x, y and ρ is recursively enumerable.

For any NN m and recursive function f we say "$\langle W_m \rangle$ is effectively measurable via f" if

$\forall xy(f(x) \leq y \Longrightarrow |\mu(\langle W_m^{f(x)} \rangle) - \mu(\langle W_m^{y} \rangle)| \leq 2^{-x})$ holds.

A set $\mathcal{S}$ of strings is called a <u>covering</u> if $\mathcal{S}$ is a r.e. set which covers all recursive sets of NNs; a covering is said to be <u>proper</u> if it does not cover the empty string (i.e. if none of its finite subsets is a covering).

<u>Remark 1</u>. The sets $\{x : \langle W_x \rangle$ is a covering$\}$ and $\{x : \langle W_x \rangle$ is

a proper covering} are $\Pi^{\emptyset}_{3}$-complete, every covering is a dense r.e. set of strings and $\{x: \langle W_x \rangle$ is dense$\}$ is a $\Pi^{\emptyset}_{2}$-complete set. Further, if the set E of NNs is in $\Pi^{\emptyset}_{n}$, $1 \leq n$, then the class of all sets of NNs covered by $\langle W_x \rangle$ for any $x \in E$ is a Π^{0}_{n+1}-class of sets.

Using the results of Jockusch and Soare [1,2] we shall remember that for any NN m the class of all (characteristic functions of) sets not covered by $\langle W_m \rangle$ is a recursively bounded (r.b.) Π^0_1-class or even a r.b. special Π^0_1-class, when $\langle W_m \rangle$ is a proper covering.

In Kučera [3] the class of NAP-sets which corresponds to notions from constructive mathematics (Martin-Löf [5], Demuth [10]) is studied. A set A of NNs is called a <u>NAP-set</u> if there is no recursive function f such that for any NN m $\mu(\langle W_{f(m)} \rangle) \leq 2^{-m}$ holds and $\langle W_{f(m)} \rangle$ covers A. It follows from [11, Remark 1] that there is a recursive function e such that, for any NN m, $\langle W_{e(m)} \rangle$ is a proper covering and $\mu(\langle W_{e(m)} \rangle) < 2^{-m}$ holds and, for any set A of NNs, (A is a NAP-set) $\Leftrightarrow \exists x$ (A is not covered by $\langle W_{e(x)} \rangle$).

For our purposes we need to know the following definition and results quoted from Kurtz [4]:

<u>Definition 2</u> ([4]). A set of NNs is said to be <u>weakly 1-generic</u> if it is covered by any dense r.e. set of strings.

<u>Theorem 3</u> ([4]).1)A T-degree is weakly 1-generic if and only if it is hyperimmune. So, by [6] the class of all weakly 1-generic T-degrees is closed upwards.

2) Every hyperimmune T-degree contains a hyperimmune set the complement of which is hyperimmune, too.

Further, the following result is known:

<u>Theorem 4</u>. For any weakly 1-generic set A of NNs and any NNs i and j, $i \neq j$, the sets $(A)_i$ and $(A)_j$ are tt-incomparable weakly 1-generic sets and, consequently, $(A)_i <_{tt} A$ holds. Thus, there is no minimal tt-degree being weakly 1-generic.

It is easy to prove the following statement.

<u>Theorem 5</u>. Let f and g be two increasing recursive functions with disjoint ranges and h a recursive function. Then there exists a NN m such that

a) $\mu(\langle W_m \rangle)=1$ and, consequently, $\langle W_m \rangle$ is dense;

b) for no set A of NNs covered by $\langle W_m \rangle$ $\{x : f(x) \in A\} \leq_{tt} \{x : g(x) \in A\}$ via h holds.

Definition 6. A set of NNs is called semigeneric if it is a nonrecursive set covered by any covering.

As we have seen, any NAP-set is non-semigeneric. Let us notice that any dense Π^0_2-class of sets of NNs contains all weakly 1-generic sets and also many of recursive sets. On account of this and regarding Remark 1 we have proved the following statement.

Theorem 7. 1) Any weakly 1-generic set is semigeneric.

2) The class of all semigeneric sets is a Π^0_4-class (of measure zero), the class of all weakly 1-generic sets is a Π^0_3-class (effectively measurable with measure zero [11]). These classes and their complements are dense and none of them is a Π^0_2-class.

Remark 8. Let f be a recursive function and m a NN. Then there is a NN n such that $\langle W_n \rangle = \mathcal{I}\mathcal{I}(\langle W_m \rangle, f)$ and for any set A of NNs there is a unique set B of NNs for which $B \leq_{tt} A$ via f holds and, consequently, (A recursive $\Rightarrow$ B recursive) and ($\langle W_m \rangle$ covers B) $\Longleftrightarrow$ ($\langle W_n \rangle$ covers A) hold. Thus, when $\langle W_m \rangle$ is a covering $\langle W_n \rangle$ is a covering, too.

Theorem 9. 1) Let the set A of NNs be semigeneric. Then any set B, for which $\emptyset <_{tt} B \leq_{tt} A$ holds, is semigeneric, too. Consequently, $\deg_{tt}(A)$ contains semigeneric sets only and for any NN i the set $(A)_i$ is either recursive or semigeneric. Thus, a) any tt-degree is either semigeneric-free or it contains only semigeneric sets;

b) the class of all semigeneric-free (i.e. non-semigeneric) nonrecursive tt-degrees is closed upwards, because its complement (i.e. the class of all semigeneric or recursive tt-degrees) is closed downwards.

2) For any weakly 1-generic set C the set $C \oplus C$ from $\deg_{tt}(C)$ is a semigeneric set which is not weakly 1-generic.

3) Let E be a non-semigeneric nonrecursive set. Then for any set B, $E \leq_T B$, the set $E \oplus B$ from $\deg_T(B)$ is non-semigeneric. Thus, the class of all non-semigeneric nonrecursive T-degrees is closed upwards.

4) Any T-degree (T-) comparable with $\deg_T(\emptyset')$ is non-semigeneric (i.e.it contains a set which is not semigeneric).

Proof. It follows immediately from Theorem 5, Remark 8 and [2, Corollary 1.1].

Let us notice that according to the last theorem NAP tt-degrees are semigeneric-free.

Now we shall study connections between hyperimmunity and weak 1-genericity and between hyperimmunity and semigenericity.

Lemma 10. Let m be a NN and M a recursive set of NNs such that for any set B of NNs and for infinitely many NNs n

(1) $\{x:n\leq x\}\cap M=\{x:n\leq x\}\cap B \Rightarrow (\langle W_m\rangle$ covers B) holds. Then

(2) $(M\Delta A$ hyperimmune$) \Rightarrow (\langle W_m\rangle$ covers A)

holds for any set A of NNs.

Proof. It is obvious that (1) must hold for any NN n and that, consequently, for any NN n there is a NN k such that $\forall\varphi(lh(\varphi)=n \Rightarrow (\langle W_m\rangle$ covers $\varphi * M[n,n+k]))$ is fulfilled. As we know, the predicate $(\langle W_m\rangle$ covers $\varphi * M[x,x+y])$ of variables φ, x and y is recursively enumerable and so we can construct an increasing recursive function f such that

$\forall x\varphi(lh(\varphi)=f(x) \Rightarrow (\langle W_m\rangle$ covers $\varphi * M[f(x),f(x+1)-1]))$

holds and the proof is completed.

Theorem 11. Let $\langle W_m\rangle$ be a covering. Then for any recursive set M and any set A (2) holds.

Proof. It is enough to use Lemma 10.

Corollary 12. Let A be a set of NNs. If there exists a recursive set M such that the set $M\Delta A$ is hyperimmune, then A is semigeneric.

In particular, all hyperimmune sets and all co-hyperimmune (e.g. hypersimple) sets are semigeneric.

By the results of Miller and Martin [6], Theorems 3, 7 and 9 and Corollary 12 the following theorem is proved.

Theorem 13. 1) For any set A, $\emptyset <_T A \leq_T \emptyset'$, there exists a hyperimmune and, consequently, semigeneric set B such that $A \leq_T B \leq_{tt} A$.

2) For any nonrecursive set A T-comparable with the set $\emptyset'$

a) the degree $\deg_T(A)$ is i) hyperimmune and, consequently, weakly

1-generic and thus also semigeneric;

ii) non-semigeneric;

b) if B is a semigeneric set and C a non-semigeneric set (both) from $deg_T(A)$, then $B <_{tt} B \oplus C \equiv_T A$ holds and $B \oplus C$ is non-semigeneric; thus, the class of all semigeneric tt-degrees (contained in $deg_T(A)$) is not closed upwards.

Any minimal tt-degree T-under $\emptyset'$ contains, according to 1) and Theorem 9, semigeneric sets only. In connection with this and 2) let us notice that Sacks has constructed a minimal T-degree under $deg_T(\emptyset')$ [9] and Degtev and Marchenkov a r.e. minimal tt-degree (see [7]).

Remark 14. Let $\langle W_m \rangle$ be a dense set of strings. Then for any string σ we can find a string ρ such that the string $\sigma * \rho$ is covered by $\langle W_m \rangle$. Iterating the process we can get a recursive function f for which $lh(\sigma_{f(n)}) \geq 1$ and $\forall \tau (lh(\tau)=n \Rightarrow (\langle W_m \rangle$ covers $\tau * \sigma_{f(n)}))$ hold for any NN n. Let g be a recursive function such that $g(0)=0$ and, for any NN k, $g(k+1)=g(k)+$ $+lh(\sigma_{f(g(k))})$. Then g is increasing and the set M, where $M \rightleftharpoons \{y: \exists x(g(x) \leq y < g(x+1) \,\&\, \sigma_{f(g(x))}(y-g(x))=1)\}$ is recursive. By Lemma 10, (2) is valid for any set A of NNs.

Remark 15. Let M be a recursive set of NNs and A a set of NNs such that the set $M \Delta A$ is not hyperimmune. Then there exist an increasing recursive function f and $\hat{M}$, $\hat{M}=M \vee \hat{M}=N \setminus M$, fulfilling $\{x: f(n) \leq x < f(n+1)\} \cap (\hat{M} \Delta A) \neq \emptyset$ for any NN n. Let $\mathcal{S}$ be the set $\{\sigma : \exists \tau\, x(lh(\tau)=f(x) \,\&\, \sigma \doteq \tau * \hat{M}[f(x), f(x+1)-1])\}$. Then $\mathcal{S}$ is a dense r.e. set of strings which does not cover A.

Remarks 14 and 15 give us a characterization of weakly 1-generic sets.

Theorem 16. A set A of NNs is weakly 1-generic if and only if for any recursive set M the set $M \Delta A$ is hyperimmune.

Corollary 17. For any set A of NNs and any recursive set M of NNs we have $A \equiv_{tt} M \Delta A$, (A semigeneric) $\Longleftrightarrow$ ($M \Delta A$ semigeneric) and (A weakly 1-generic) $\Longleftrightarrow$ ($M \Delta A$ weakly 1-generic).

Proof. Follows immediately from Theorems 9 and 16 and validity of $B \Delta (C \Delta D) = (B \Delta C) \Delta D$ for any sets B, C and D.

As we shall see, in general, semigenericity is not connected with hyperimmunity.

Example 18. Let A be a set of NNs. Then, for the set B, where $B \rightleftharpoons \{x : \pi_2(x) \in A\}$, $B \equiv_{tt} A$ and (thus) (A semigeneric) $\Leftrightarrow$ $\Leftrightarrow$ (B semigeneric) hold, but for no recursive set M the set $M \Delta B$ is hyperimmune.

Now we turn to tt-reducibility.

Lemma 19. 1) For any recursive function f there are recursive functions g and h such that for any NN m $\langle W_{g(m)} \rangle$ is a dense set, $\mathcal{J}\mathcal{J}(\langle W_{g(m)} \rangle, f)$ is a r.e. set of strings effectively measurable via $\lambda x\ h(\langle m,x \rangle)$ and $\mu(\mathcal{J}\mathcal{J}(\langle W_{g(m)} \rangle, f)) \leq 2^{-m}$ holds.

2) $(A \leq_{tt} B)$ & (A weakly 1-generic) & (B NAP-set) holds for no sets A and B of NNs.

Proof. Using the s-m-n-theorem we get a recursive function g such that for any NN m $W_{g(m)} = \{x : \exists y(lh(\sigma_y) > m \;\&\; x = \mu z(lh(\sigma_z) = 3.lh(\sigma_y)+1 \;\&\; \sigma_y \subseteq \sigma_z \;\&\; \mu(\mathcal{J}\mathcal{J}(\sigma_z, f)) \leq 2^{-2.lh(\sigma_y)}))\}$.

The last equality shows how to construct a recursive function h having together with g the properties described in 1). 2) follows immediately from 1) and Remark 8.

In the terminology used in Demuth [11] $\emptyset'$-almost every set of NNs is a NAP-set. The class of all sets B of NNs fulfilling $B \leq_T \emptyset'$ has $\emptyset'$-measure 1. Thus, most of sets of this class are NAP-sets and, as we have seen in Theorem 13 and in Lemma 19, tt-under any such set there are semigeneric sets but no weakly 1-generic sets.

Remark 20. Let $\langle W_m \rangle$ not cover the empty string and let A be the least set (in the lexicographic ordering) not covered by $\langle W_m \rangle$. Then the r.e. set $\{x : (\forall y)_{y \leq x}(lh(\sigma_y) = lh(\sigma_x) \Rightarrow (\langle W_m \rangle$ covers $\sigma_y))\}$ and A are tt-equivalent. Thus, according to the quoted results on NAP-sets, there is a r.e. NAP tt-degree.

Theorem 21. There is a hypersimple and thus semigeneric set E of NNs such that no set A of NNs fulfilling $A \leq_{tt} E$ is weakly 1-generic.

Proof. By Remark 20, Dekker's theorem and part 2 of Lemma 19 there is the desired set E.

Theorem 22. There is a weakly 1-generic r.e. tt-degree.

Proof. Let f be an increasing recursive function such that $f(0)=0$ and for any NN p there is a NN q fulfilling $2^{f(p)}-1 \leq q \leq$ $\leq 2^{f(p+1)}-2$ and ($\langle W_q \rangle$ covers Λ).

We construct a partial recursive function $\varkappa$ of two variables with the following properties. Let m and n be NNs and let $k \rightleftharpoons \mu x(n \leq 2^{f(x)}-1)$ and $t \rightleftharpoons (2^{f(k+1)+1}-2)$. Let us remember that $\forall y(\mathrm{lh}(\sigma_y)=f(k+1) \Longleftrightarrow 2^{f(k+1)}-1 \leq y \leq t)$. For any NN x $\varkappa(\langle m,n\rangle,x)$ is defined if and only if $n \leq x \leq 2^{f(k+1)}-2$ and $\forall z(\mathrm{lh}(\sigma_z)=f(k+1) \Rightarrow (\langle W_x \rangle$ overlaps $\sigma_m * \sigma_z))$ hold. If $\varkappa(\langle m,n\rangle,x)$ is defined then there exists a NN w such that $\langle W_x \rangle$ covers σ_w, $\sigma_m * \sigma_{t-x} \subseteq \sigma_w$ and $\varkappa(\langle m,n\rangle,x)=\langle w,x+1\rangle$.

There are recursive functions g_0 and g and $\emptyset'$-recursive functions h_0 and h fulfilling: $W_{g_0(\langle m,n\rangle)}$ is the range of $\lambda x\, \varkappa(\langle m,n\rangle,x)$, $h_0(\langle m,n\rangle)=\varkappa(\langle m,n\rangle,\mu x(\varkappa(\langle m,n\rangle,x)$ is defined$))$, $h(0)=\langle 0,0\rangle$, $h(p+1)=h_0(h(p))$, $W_{g(0)}=\{\langle 0,0\rangle\}$ and $W_{g(p+1)}=$ $=\bigcup_{x \in W_{g(p)}} W_{g_0(x)}$ hold for any NNs m, n and p.

Let c be a NN for which $W_c=\{x: \exists vyz(\langle v,y\rangle \in W_{g(z)} \,\&\, \mathrm{lh}(\sigma_x)=$ $=\mathrm{lh}(\sigma_v) \,\&\, \sigma_x \prec \sigma_v)\}$.

Then the increasing sequence $\{\pi_2(h(x))-1\}_{x=1}^{\infty}$ contains all NNs p for which $\langle W_p \rangle$ is a dense set. Further, for any positive NN x the string $\sigma_{\pi_1(h(x))}$ is covered by $\langle W_{(\pi_2(h(x))-1)} \rangle$ and extended by the string $\sigma_{\pi_1(h(x+1))}$. Let A be a (unique) set of NNs covered by $\sigma_{\pi_1(h(x))}$ for any NN x. Then A is obviously weakly 1-generic set being the least set (in the lexicographic ordering) not covered by the set $\langle W_c \rangle$ and according to Remark 20 the proof is completed.

Now we shall present some results on the structure of semigeneric T-degrees and tt-degrees.

<u>Theorem 23</u>. There is a hyperimmune set E of NNs such that $E <_T \emptyset''$ and $(\emptyset <_T C \leq_T E \Rightarrow C$ semigeneric) holds for any set C of NNs.

Proof. The construction of E will proceed in stages. For each NN n at the end of stage n we shall have a string τ_{n+1} (covering the set under construction) such that a) $\mathrm{lh}(\tau_{n+1}) \geq 2n$ and any covering with index m (i.e. of the form $\langle W_m \rangle$), where

$m \leq n$, covers any set C of NNs for which the function $\lambda x\, C(x)$ is the extension of the function $\varphi_n^{\tau_{n+1}}$,

b) no recursive function with index m, $m \leq n$, majorizes any set of NNs covered by τ_{n+1}.

Let $\tau_0 \rightleftharpoons \Lambda$.

Stage n. We have a string τ_n. Let A_n be the set $\{x : x < lh(\tau_n) \,\&\, \tau_n(x)=1\}$.

Substage (a). See whether φ_n is a recursive function.

If so, find a NN p_n such that $lh(\tau_n) \leq p_n$ and no set B, $A_n[0,p_n]*1 \subseteq B$, is majorized by φ_n.

If not, define $p_n \rightleftharpoons lh(\tau_n)$.

Let $\overline{\tau}_n \rightleftharpoons A_n[0,p_n]*1$ (thus, $\overline{\tau}_n$ extends τ_n).

Substage (b). Let $\mathcal{S}_{n,w} \rightleftharpoons \{\rho : \overline{\tau}_n \subseteq \rho \,\&\, (\varphi_n^{\rho}(w)$ is defined and $\varphi_n^{\rho}(w) \leq 1)\}$ for any NN w.

Case 1. The set $\bigcap_w \mathcal{S}_{n,w}$ of strings is dense in $\{\rho : \overline{\tau}_n \subseteq \rho\}$. Construct recursive sets B_n and C_n such that $\overline{\tau}_n$ covers B_n and $\varphi_n^{B_n}$ is the characteristic function of C_n. Denote P_n the set $\{x : x \leq n \,\&\, (\langle W_x \rangle$ covers $C_n)\}$.

If $P_n \neq \emptyset$, find NNs s_n and t_n such that for any NN $m \in P_n$ the string $C_n[0,t_n]$ is covered by $\langle W_m \rangle$, $lh(\overline{\tau}_n) \leq s_n$ and the function $\varphi_n^{B_n[0,s_n]}$ is defined at each NN $x \leq t_n$. Define $\tau_{n+1} \rightleftharpoons B_n[0,s_n]$.

When $P_n = \emptyset$ define $\tau_{n+1} \rightleftharpoons \overline{\tau}_n$.

Case 2. There are a NN w and a string σ such that $\overline{\tau}_n \subseteq \sigma$ and no extension of σ belongs to $\mathcal{S}_{n,w}$. Find such pair w, σ and define $\tau_{n+1} \rightleftharpoons \sigma$.

This completes our description of stage n.

Observe that τ_n is defined for any NN n and let $E \rightleftharpoons \{x : \exists y (x < lh(\tau_y) \,\&\, \tau_y(x)=1)\}$. The described construction is obviously recursive in $\emptyset''$, E is the desired set ($E \equiv_T \emptyset''$ is excluded by Theorem 9) and the proof is completed.

Corollary 24. There is a hyperimmune T-degree under $deg_T(\emptyset'')$ which contains semigeneric sets only.

Remark 25. Let A and B be sets of NNs such that $deg_T(A)$ is hyperimmune-free and $B \leq_T A$. Then, according to [7], we have $B \leq_{tt} A$ and $deg_T(B)$ is hyperimmune-free (consequently, $deg_T(A) =$

$= \deg_{tt}(A)$ and by Theorem 9 this degree is either semigeneric-free or containing semigeneric sets only).

<u>Theorem 26</u>. There is a set A of NNs such that $A'' \equiv_T \emptyset''$ and $\deg_T(A)$ is a hyperimmune-free minimal T-degree which contains semigeneric sets only.

Proof. It is sufficient to modify slightly the proof of Theorem XVII [8, pp. 276-279]. Let us remember that in the proof the construction of (the characteristic function of) nonrecursive set A, the T-degree of which is minimal, proceeds in stages. For each positive NN n at the end of stage n we have two characteristic functions of (different) recursive sets, say A_1^n and A_2^n, and an increasing recursive function h^n for which (a.o.) $h^n(0) > 0$ and $A_i^n[0,h^n(0)-1] \doteq A[0,h^n(0)-1]$, where $1 \leq i \leq 2$, hold.

For any NN p we modify

a) stage 2p+1 so, that we choose $h^{2p+1}(0)$ so great that not only the functions φ_n and $\lambda x\, A(x)$ differ on the segment $\{x : x < h^{2p+1}(0)\}$ but we also have: when $\langle W_p \rangle$ covers both sets A_1^{2p} and A_2^{2p}, then it covers the string $A_1^{2p+1}[0,h^{2p+1}(0)-1]$ and, consequently, the set A;

b) stage 2p+2 using the hint from Exercise 13,34 [8,p. 298] to ensure the T-degree of A to be a hyperimmune-free one and $A'' \equiv_T \emptyset''$ to hold.

According to Remark 25 the described modification of the quoted proof ensures the existence of the desired T-degree.

We shall need the following result of Jockusch and Simpson (quoted here in our terminology).

<u>Theorem 27</u>. There is a proper covering $\langle W_b \rangle$ such that any pair B, C of different sets of NNs not covered by $\langle W_b \rangle$ forms a tt-minimal pair, i.e. B and C are nonrecursive and $A \leq_{tt} B\ \&$ $\&\ A \leq_{tt} C \Longrightarrow$ (A recursive) holds for any set A of NNs.

<u>Theorem 28</u>. There is a set C of NNs such that $C'' \equiv_T \emptyset''$ and $\deg_T(C)$ is a hyperimmune-free minimal T-degree which is semigeneric-free.

Proof. Let b be the NN from Theorem 27. By [1, Theorem 2.4] there is a set C not covered by $\langle W_b \rangle$ (and thus non-semigeneric) such that $C'' \equiv_T \emptyset''$ and $\deg_T(C)$ is hyperimmune-free. According to Remark 25 this degree is semigeneric-free and for any set E of

NNs we have $E \leq_T C \Rightarrow E \leq_{tt} C$. Using this fact and properties of $\langle W_b \rangle$ we can easily show (as in [12]) that $\deg_T(C)$ is a minimal degree.

As we have seen, there are both pure semigeneric and semigeneric-free minimal T-degrees (or tt-degrees). Now we shall show that the class of all semigeneric T-degrees is not closed upwards.

Lemma 29. Let $\langle W_a \rangle$ be a proper covering and C a set of NNs such that $\deg_T(C)$ is hyperimmune-free. Then there is a set A of NNs not covered by $\langle W_a \rangle$ and such that $(A \oplus C)'' \equiv_T C''$ and $\deg_T(A \oplus C)$ and $\deg_T(A)$ are both hyperimmune-free and semigeneric-free.

Proof. Under the supposed conditions any C-recursive function is majorized by some recursive function. Let $\mathcal{T}$ be the class of all sets of NNs of the type $B \oplus C$, where B is any set not covered by $\langle W_a \rangle$. Then $\mathcal{T}$ is a r.b. $\Pi_1^{0,C}$-class and by relativization of [1, Theorem 2.4] there is a set A of NNs not covered by $\langle W_a \rangle$ such that $(A \oplus C)'' \equiv_T C''$ and $\deg_T(A \oplus C)$ is C-hyperimmune-free and thus also hyperimmune-free. The set A is obviously nonsemigeneric. To complete the proof it is sufficient to use Theorem 9 and Remark 25.

By means of constructive mathematical analysis we can prove the following result:

Lemma 30. For any NAP-set A and any set B of NNs such that $\emptyset <_{tt} B \leq_{tt} A$ holds there is a NAP-set C fulfilling $C \leq_T B \leq_{tt} C$.

Example 31. By Remark 25, Theorem 26 and Lemma 29, where $a \not\equiv e(0)$, there are sets C and A of NNs such that among the following three hyperimmune-free T-degrees $\deg_T(C)$, $\deg_T(A)$ and $\deg_T(A \oplus C)$ the first one is semigeneric (hence NAP-free) and minimal, the second and third ones are semigeneric-free, the second one is a NAP T-degree and the third one is, according to Lemma 30, NAP-free. So, the class of all semigeneric (as well as that of NAP) T-degrees is not closed upwards (for NAP T-degrees this result was proved in Kučera [3] by a different method).

Theorem 32. Under any hyperimmune-free NAP T-degree there is no minimal T-degree.

Proof. As it follows from Kučera [3, Theorem 5], no NAP T-degree is a minimal one. By use of Remark 25 and Lemma 30, the proof is completed.

At the end, we present two results without proofs.

Theorem 33. Let $\mu(\langle W_a \rangle) < 1$ and let B be an A-recursively enumerable set. Then there are a set E not covered by $\langle W_a \rangle$ and an A-r.e. set C such that $B \leq_T C \equiv_{tt} E$ and, consequently, $E \leq_{tt} A'$.

Remark 34. In the last theorem, the condition $\mu(\langle W_a \rangle) < 1$ is substantial. Indeed, according to Jockusch and Soare [1, p. 48], for any nonrecursive set B there is a proper covering $\langle W_d \rangle$ which covers any set E of NNs fulfilling $B \not\leq_T E$. Thus, $\mu(\langle W_d \rangle) = 1$ must hold.

Theorem 35. Let A and C be sets of NNs. Then $C \leq_{tt} A'$ holds if and only if there are a recursive function f and an A-recursive function g of two variables such that for any NN m $C(m) = \lim_y g(m,y)$ and $\sum_{y=0}^{\infty} |g(m,y)-g(m,y+1)| \leq f(m)$ hold.

Addition. A. Kučera has informed me that in Ceĭtin's paper [13] there are results which can be, according to a result of Kušner([14, Theorem 1]), reformulated as follows and which, consequently, are a weaker form of part 1 of Theorem 9 and of Corollary 12:

a) (A,B r.e.)&$(\emptyset <_{tt} B \leq_{tt} A)$&(A semigeneric) $\Rightarrow$ B semigeneric,

b) any hypersimple set is semigeneric.

For details see Demuth, Kučera "Remarks on 1-genericity, semigenericity and related concepts", in this volume.

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Matematicko-fyzikální fakulta, Karlova Universita, Malostranské nám. 25, 118 00 Praha 1, Czechoslovakia

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