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ON BALANCING OF HYPERGRAPHS

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Abstract: Some inequalities concerning the number of edges of uniform unbalanced hypergraphs are given and also an easy construction of unbalanced uniform hypergraphs without short cycles is presented.

Key words: Uniform hypergraph, balancing.

Classification: 05C65, 05C45

In this note we will give several elementary results concerning balancing of hypergraphs. Under hypergraph $\mathcal{F}$ we understand a finite set of finite sets, elements of $\mathcal{F}$ are called edges, and elements of these edges are points. Two-coloring of $\mathcal{F}$ is said to be balancing if exactly half points of each edge have either color. Hence a hypergraph containing an edge of odd size cannot be balanced.

A hypergraph is n-uniform if every its edge has n points, and is said to be almost disjoint if any two edges have at most one point in common. For a given positive integer n we denote f(n) as the least number of edges of an n-uniform hypergraph which cannot be balanced, analogously $f_1(n)$ for almost disjoint hypergraphs. There was one basic question asked by P. Erdös and V.T. Sós whether the function f(n) is unbounded. A positive answer was given in [1]. P. Erdös [2] asked some other questions.

(i) Does there exist a positive number c_1 such that the inequality $f(n) \leq c_1 \log n$ holds for all n ?

(ii) And does there exist a positive c_2 such that $c_2 \log n \leq f(n)$ holds infinitely times ?

(iii) Is the function $f_1(n)/n$ unbounded ?

Generalizing an example from [1] we will present an easy proof of (i). Secondly we will obtain some inequalities involving the function $f_1(n)$. In the last paragraph we shall investigate balancing of hypergraphs without short cycles.

§ 1. Upper estimate for $f(n)$. In the proof of the following lemma we will show a recursive construction of unbalanced hypergraphs.

Lemma 1. Let $n=t\cdot K+q$, where $0<q<K$, then $f(n)\le K+f(q)$.

Proof: Let us consider $K+1$ blocks of t points and let $\mathcal{F}_0$ consist of all unions of K blocks of these $K+1$. Moreover let $\mathcal{F}_q$ be a q-uniform unbalanced hypergraph such that $|\mathcal{F}_q|=f(q)$. Now choosing some $h_0\in\mathcal{F}_0$ and $h_1\in\mathcal{F}_q$ we define an n-uniform hypergraph $\mathcal{F}$ as

$$\mathcal{F}=\{h_0\cup h \mid h\in\mathcal{F}_q\}\cup\{h\cup h_1 \mid h\in\mathcal{F}_0\}$$

consisting of $K+f(q)$ edges (we suppose $\bigcup\mathcal{F}_0\cap\bigcup\mathcal{F}_q=\emptyset$). Let us say that $\mathcal{F}$ is balanced i.e. we have a mapping $\varphi\colon\bigcup\mathcal{F}\to\{+1,-1\}$ such that $\sum_{x\in h}\varphi(x)=0$ for all $h\in\mathcal{F}$. It is clear that the sum on each block is the same, hence the sum on h_0 must be either zero or in absolute value greater than q. From this it follows that $\mathcal{F}_q$ has to be balanced by φ which leads to a contradiction. □

Let $s(n)$ denote the least number not dividing n; using our lemma, it is obvious that $f(n)\le 2s(n)$. From some n_0 there are among numbers $\{1,\dots,n\}$ more than $n/2\log n$ primes. Each such prime occurs in $[1,\dots,n]$ in a root greater than $n^{1/2}$ ($[1,\dots,n]$ denotes the least common multiple of numbers $1,\dots,n$). So from this n_0

$$e^{n/4}=(n^{1/2})^{n/2\log n}\le[1,\dots,n]$$

which yields $s(n)\le 4\log n$. This completes the proof of (i). □

§ 2. Balancing of graphs. Let G be a graph without loops. Evaluation φ of its edges by numbers $+1$ or -1 is called balancing of G if each vertex of G is incident with the same number of positive as well as negative edges.

Lemma 2. Let G be connected, then G is balanced iff the degrees of all its vertices are even and the number of its edges is even. Moreover let us have a balancing φ of G. Then there exists Euler path $v_0 e_1 v_1 \dots e_m v_m = v_0$ such that $\{e_1, \dots, e_m\} = E(G)$, $\varphi(e_i) \neq \varphi(e_{i+1})$ for $i=1,\dots,m-1$ and $\varphi(e_m) \neq \varphi(e_1)$.

<u>Proof</u>: Let each vertex G have even degree and number of edges in G be even. Then we can take some Euler path and following it alternatively evaluate edges of G. This evaluation has to be balancing of G. Necessity of this condition follows from the second part of the lemma.

Let us have a balancing φ of G. We can take a maximal trail $v_0 e_1 v_1 \dots e_k v_k$ with the property $\varphi(e_i) \neq \varphi(e_{i+1})$ for $i=1,\dots,k-1$. Without much effort we can show that this is the desired trail. □

We say that G is a transmitter on $e, e' \in E(G)$ if for each balancing φ of G it holds $\varphi(e) = \varphi(e')$. Let us have two connected balanced graphs H, Q (with $V(H) \cap V(Q) = \emptyset$) and

$$\{a,b\} = h \in E(H),\ \{a',b'\} = h' \in E(Q).$$

Then we can denote $e = \{a,a'\}$, $e' = \{b,b'\}$ and construct a graph G on the set of vertices $V(H) \cup V(Q)$ with the set of edges $E(G) = (E(H) \cup E(Q) \cup \{e,e'\}) \setminus \{h,h'\}$ (see fig. 1).

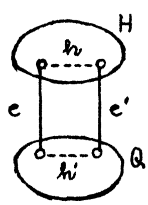

fig. 1

In addition if H and Q have no cycles of length ≤ 1, then G has the same property and distance of edges e, e′ is greater than $1-2$. From Lemma 2 it then follows that G is a transmitter on e, e′.

§ 3. <u>Correspondence between balancing of graphs and hypergraphs</u>. Family $R = (E_x, x \in X)$ of nonempty subsets E(G), where G is an n-regular graph, is called decomposition if

1. $\bigcup_{x \in X} E_x = E(G)$
2. $x \neq y \Rightarrow |\bigcup E_x \cap \bigcup E_y| \leq 1$
3. $\forall e_1, e_2 \in E_x : e_1 \neq e_2 \Rightarrow e_1 \cap e_2 = \emptyset$

Let us consider a mapping $h: V(G) \longrightarrow \mathcal{P}(X)$ such that

$$h_v = \{x \in X \mid v \in \bigcup E_x\}.$$

and denote $\overline{G}_R$ as the hypergraph $\{h_v \mid v \in V(G)\}$. We have assigned to each vertex $v \in V(G)$ an edge h_v of the hypergraph on the set X.

Using property 2 it is easy to see that $v_1 \neq v_2$ implies $|h_{v_1} \cap h_{v_2}| \leq 1$. From n-regularity and property 1 and 3 we get that $\overline{G}_R$ is n-uniform. So for $n \geq 2$ h is one-to-one correspondence between V(G) and $\overline{G}_R$. Moreover the degree of $x \in X$ in the hypergraph $\overline{G}_R$ is exactly $|\cup E_x|$ which is an even number. On the other hand, every even-degrees hypergraph has such a pattern.

Evaluation $\varphi: E(G) \to \{+1,-1\}$ is said to be balancing of a graph G_R with decomposition R if φ is balancing of G and $\varphi|E_x$ is identical for all $x \in X$. We see that G_R is balanced iff the hypergraph $\overline{G}_R$ is balanced.

We will be working only with simple decompositions of E(G) consisting of one-element sets except one which we denote T. Instead of G_R we can write G_T, if T has only one element we can write G.

§ 4. Function $f_1(n)$

Theorem 1. If $4|n$, then $n+1 < f_1(n) \leq 2n-1$. If $n=2(2t+1)$, then $f_1(n)=n+1$.

Proof: In order to find some upper bound for $f_1(n)$ we can use the method of dualization developed in § 3. Let n be even, x denote a set $\{a_1,\dots,a_n\}$, $y=\{b_1,\dots,b_{n-1}\}$ and $T=\{\{a_1,a_2\}, \{a_3,a_4\},\dots,\{a_{n-1},a_n\}\}$ be a pairing of x. Let

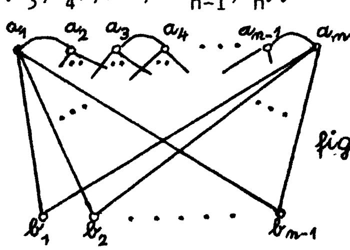

fig. 2

$K_{x,y}=(x \cup y, \{\{v_1,v_2\} | v_1 \in x \,\&\, v_2 \in y\})$ and $G=(x \cup y, T \cup E(K_{x,y}))$ (see fig. 2).

Let G_T be balanced by an evaluation φ such that $\varphi(e)$ is the same number for each $e \in T$. By Lemma 2 we can draw G by some trail on which the signs are alternating. Therefore each two edges from T have odd distance in this trail. It means that between some $x_1, x_2 \in x$ there has to be a trail with odd length in the graph $K_{x,y}$, but it is impossible because $K_{x,y}$ is bipartite. So the n-uniform almost disjoint hypergraph $\overline{G}_T$ with 2n-1 edges cannot be balanced and $f_1(n) \leq 2n-1$ for each even n.

If $n=2(2t+1)$, then the n-uniform almost disjoint hypergraph

$\overline{K}_{n+1}$ is unbalanced because the graph K_{n+1} (with a trivial decomposition) which has an odd number of edges, cannot be by Lemma 2 balanced. Hence $f_1(n) \leq n+1$.

Using induction we prove that for every even n each almost disjoint n-uniform hypergraph with n edges can be balanced. Let $\mathcal{F}$ be such hypergraph. A point in some hypergraph is said to be free resp. bounded if it has the degree just one resp. ≥ 3. Let h_1, h_2 be in $\mathcal{F}$, $\mathcal{F}' = \mathcal{F} \setminus \{h_1, h_2\}$. In each edge from $\mathcal{F}'$ there are at least three free points. Now we cut from every edge $h \in \mathcal{F}'$ two free points preferring such points which lie in h_1 or h_2. By this way we get an (n-2)-uniform hypergraph $\mathcal{F}''$. From induction hypothesis it follows that we can balance $\mathcal{F}''$. If b_1 denotes the number of bounded points in h_1, then the number of free points in h_1 is $\geq b_1+1$, similarly for h_2. Consequently even in the worse case when h_1, h_2 have a common point, the points which are not in $\bigcup \mathcal{F}''$ can be evaluated so that we get a balancing of $\mathcal{F}$.

This proves $f_1(n) > n$ for even n. The inequality $f_1(n) > n+1$ for n=4t can be proved analogously also proving that for n=2(2t+1) the only n-uniform almost disjoint unbalanced hypergraph with n+1 edges is $\overline{K}_{n+1}$. □

§ 5. Balancing of hypergraphs without short cycles. By a cycle in a hypergraph $\mathcal{F}$ of length l we mean a sequence $(x_1 h_1 x_2 \ldots x_l h_l x_{l+1} = x_1)$, where $x_1, \ldots, x_l$ are pairwise different points of $\mathcal{F}$, $h_1, \ldots, h_l$ pairwise different edges of $\mathcal{F}$ and $x_i, x_{i+1} \in h_i$ for i=1,...,l. Now we define $f^l(n)$ as the least number of edges of an unbalanced n-uniform hypergraph from $\mathcal{P}_l$, where $\mathcal{P}_l$ denotes the class of all hypergraphs without cycles of length $\leq l$.

Unbalanced hypergraph is said to be a condenser if every $\mathcal{F}' \subsetneq \mathcal{F}$ is balanced. By an easy observation we firstly establish the following lower estimates.

Theorem 2. (1) If k=2l+1, then $f^k(n) > 2(\frac{n}{2})^l$,

(2) if k=2l, then $f^k(n) > (\frac{n}{2})^l$,

where n is even.

Proof: Let $\mathcal{F} \in \mathcal{P}_k$ be an n-uniform condenser. Every its edge contains more than $\frac{n}{2}$ points with degree > 1. Let us suppose for example k=2l and take some $h_0 \in \mathcal{F}$. Then every two paths of length l starting in some points of h_0 and not using the edge h_0 cannot have an edge in common. From that, the inequality (2) follows. Similarly in the case $k=2l+1$. □

There is no simple construction of 3-chromatic hypergraphs without short cycles. We shall show an easy construction of unbalanced hypergraphs from $\mathcal{P}_k$. Let $V(n,l)$ denote the least number of vertices of an n-regular graph without cycles of length $\leq l$.

<u>Theorem 3</u>. For every even n it holds

$$f^l(n) \leq nV(n,l)+1 \tag{3}$$

<u>Proof</u>: Let H be an n-regular graph with $V(n,l)$ edges without cycles of length $\leq l$. If $V(n,l)$ is odd, then (3) immediately holds. Otherwise let G be a transmitter on e, e' if we use two copies of H in the construction mentioned in § 2. We denote $\mathcal{P}=\overline{G}$ and by $x_1, x_2 \in \cup\mathcal{P}$ the points which are dual to the edges e, e'. Obviously $\mathcal{P}$ with $2V(n,l)$ edges is $\in \mathcal{P}_l$, the distance of x_1, x_2 is $\geq l-1$ and for each balancing φ of $\mathcal{P}$ it holds $\varphi(x_1)=\varphi(x_2)$. Now we can take $\frac{n}{2}$ copies $\mathcal{P}^{(i)}$ of $\mathcal{P}$ with points $x_1^{(i)}, x_2^{(i)}$ $(i=1,\dots,\frac{n}{2})$ and connect these copies just as it is done on the fig. 3. Then to the resulted system we add n-points edge

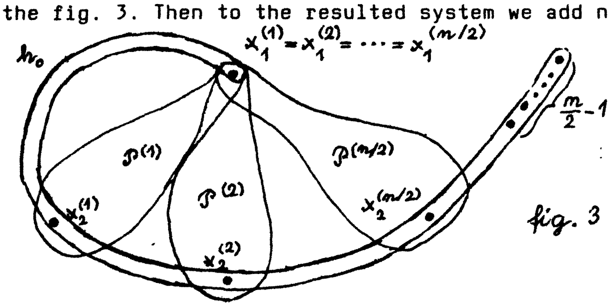

fig. 3

h_0. We get an n-uniform hypergraph with $nV(n,l)+1$ edges from $\mathcal{P}_l$ which cannot be balanced. □

In [3] there is a construction which yields $V(n,l) \leq n^{cl}$. Hence we can conclude that there exist two constants c_1, c_2 such that $n^{c_1 l} \leq f^l(n) \leq n^{c_2 l}$ for all even n and $l > 1$.

P.S. In a later version of [1], using another method, the upper bound for f(n) was improved and consequently the presumption (ii) was rejected.

R e f e r e n c e s

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