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ON THE RÉNYI DIMENSION

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Abstract: The concept of dimension (upper, lower and exact) is introduced for probability spaces equipped with a measurable semimetric, and its relation to A. Rényi's dimension of a vector-valued random variable is established. Under certain assumptions, the exact dimension function behaves like a "specific weight", and the dimension of the product of two spaces is equal to the sum of their dimensions.

Key words: Semimetrized measure space, Rényi weight, Rényi dimension.

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In 1956, the dimension $d(\xi)$ of an R^n-valued random variable ξ was introduced in a joint paper by J. Balatoni and A. Rényi. In 1959, A. Rényi introduced the upper and lower dimension, $\bar{d}(\xi)$ and $\underline{d}(\xi)$. Following Rényi's ideas, we introduce, for any extended Shannon semientropy φ (see [2]), three dimension functions, φ-ud, φ-ld and φ-Rd, which we will call, respectively, the upper, lower and exact Rényi φ-dimension. The dimensions φ-ud(P) and φ-ld(P) are defined for any W-space P, i.e. for any $P=\langle Q,\rho,\mu\rangle$, where μ is a finite measure and ρ is a measurable semimetric; φ-Rd(P) is defined iff φ-ud(P)$=\varphi$-ld(P), and is equal to their common value.

The case of φ equal to E, the largest extended Shannon entropy of the form C_τ (see [2]), is considered in some detail. It turns out that, for any R^n-valued random variable ξ on a probability space $\langle Q,\mu\rangle$, $\bar{d}(\xi)$ and $\underline{d}(\xi)$ are equal, respectively, to E-ud$\langle R^n,\rho,\mu\circ\xi^{-1}\rangle$ and E-ld$\langle R^n,\rho,\mu\circ\xi^{-1}\rangle$; if, in addition, ξ is bounded, then E can be replaced by any φ from a certain fairly large class of extended entropies.

In general, the behavior of the dimension functions E-ud, etc., is not very nice. If, however, E-Rd(S) exists for all $S\lessapprox P$ and the set of all E-Rd(S), $S\lessapprox P$, is bounded, then E-Rd(S) behaves as a "specific weight": there is a function f such that, for any $S \subseteq P$, E-Rd(S) is equal to the mean value of f on S. We also show that, under certain, not too restrictive, conditions, the exact Rényi E-dimension of $P_1 \times P_2$ is equal to the sum of dimensions of P_1 and P_2. 1

1.1. We use the terminology and notation of [3]. In particular, (1) if $x=(x_k : k \in K)$, $K \neq \emptyset$, $x_k \in R_+$, $\Sigma x_k < \infty$, then we put $H(x) = \Sigma L(x_k) - L(\Sigma x_k)$, where $L(0)=0$, $L(a) = -a \log a$ if $a > 0$, (2) if $P = \langle Q, \rho, \mu \rangle$ is a W-space and $\varepsilon \in R$ is positive, then $\varepsilon * P$ denotes the W-space $\langle Q, \varepsilon * \rho, \mu \rangle$, where $(\varepsilon * \rho)(x,y)=0$ if $\rho(x,y) \leqq \varepsilon$, $(\varepsilon * \rho)(x,y)=1$ if $\rho(x,y) > \varepsilon$.

1.2. Recall that $P = \langle Q, \rho, \mu \rangle$ is called a semimetrized measure space or a W-space ("weighted space") if μ is a measure on Q and ρ is a $[\mu \times \mu]$-measurable semimetric. If ρ is a metric and every Borel set is in dom $\bar{\mu}$, then P is called a weakly Borel metric W-space. If $P = \langle Q, \rho, \mu \rangle$ is a W-space, we put $wP = \mu Q$. - If $wP=0$, then P is called a null space. If P is a W-space, then exp P (respectively, $\exp^* P$) denotes the collection of all subspaces (all pure subspaces) of P, equipped by the order relation "to be a subspace".

1.3. <u>Proposition</u>. If P is a W-space, then exp P is a complete lattice, $\exp^* P$ is a complete Boolean algebra and if $\mathcal{M} \subset \exp P$, then there is a countable $\mathcal{M}' \subset \mathcal{M}$ such that $\sup \mathcal{M}' = \sup \mathcal{M}$.

We omit the proof, since the proposition is a direct consequence of well-known analogous propositions concerning e.g. the lattice of $\bar{\mu}$-measurable [0,1]-valued functions modulo those which are equal to zero $\bar{\mu}$-almost everywhere, etc.

1.4. The (cartesian) product $P = P_1 \times P_2$ of semimetric spaces $P_i = \langle Q_i, \rho_i \rangle$ (of W-spaces $P_i = \langle Q_i, \rho_i, \mu_i \rangle$), i=1,2, is, by definition, the space $\langle Q_1 \times Q_2, \rho \rangle$ (respectively, $\langle Q_1 \times Q_2, \rho, \mu_1 \times \mu_2 \rangle$), where $\rho((x_1,x_2),(y_1,y_2)) = \max(\rho_1(x_1,y_1), \rho_2(x_2,y_2))$. In particular, R^n, n=1,2,..., and its subsets are always endowed with the metric $\rho((x_i),(y_i)) = \max |x_i - y_i|$.

1.5. <u>Notation</u>. If $\langle Q, \mu \rangle$ is a measure space, T is a set and $\xi : Q \to T$ is a mapping, then $\mu \circ \xi^{-1}$ denotes the measure ν on T defined as follows: dom ν consists of all $X \subset T$ such that $\xi^{-1} X \in$ $\in$ dom μ; if $\xi^{-1} X \in$ dom μ, then $\nu X = \mu(\xi^{-1} X)$.

1.6. <u>Definition</u>. If $\langle Q, \mu \rangle$ is a probability space, $\langle T, \rho \rangle$ is a metric space and $\xi : \langle Q, \mu \rangle \to \langle T, \rho \rangle$ is a random variable (i.e. $\mathcal{B}\langle T,\rho\rangle \subset \mathrm{dom}(\mu \circ \xi^{-1})$), then ξ will be called a metric random variable (more exactly, a $\langle T,\rho\rangle$-valued random variable on $\langle Q,\mu\rangle$).

1.7. Proposition. If $\xi : \langle Q,\mu\rangle \longrightarrow \langle T,\rho\rangle$ is a metric random variable and $\xi(Q) \subset \langle T,\rho\rangle$ is separable, then $\langle T,\rho,\mu\circ\xi^{-1}\rangle$ is a weakly Borel metric W-space. - This follows easily from [3], 1.8.

1.8. Remarks. A) In 1.7, the assumption that $\xi(Q)$ is separable can be replaced by a far weaker one, and it is consistent (relative to current axiomatic set theories) to assume that it can be omitted. - B) Clearly, if $\langle Q,\rho,\mu\rangle$ is a weakly Borel metric W-space, then the identity mapping $\xi : \langle Q,\bar{\mu}\rangle \longrightarrow \langle Q,\rho\rangle$ is a random variable.

1.9. In [1] (see also [6], which is, in fact, an abridged version of [1]), the concept of dimension of an R^n-valued random variable has been introduced. In [4] and [5], A. Rényi has introduced the upper (lower) dimension of ξ. The pertinent definitions (in a slightly more general form) will be stated below (1.11). First, we introduce some notation and conventions.

1.10. A) If $a \in \bar{R}$, $a>0$, we put $a/0=\infty$; if $b \in R_+$, we put $\infty/b=\infty$; we put $0/0=0$. - B) If a random variable $\xi : \langle Q,\mu\rangle \longrightarrow \langle T,\mathcal{A}\rangle$ assumes only countable many values, we put $H_0(\xi) = H(\mu(\xi^{-1}t) : t \in \xi(Q))$. - C) Z will denote the set of all integers. - D) If $x \in R$, then $[x] \in Z$, $[x] \leqq x < [x]+1$. If $x=(x_1,\dots,x_m) \in R^m$, then $[x] = ([x_1],\dots,[x_m])$. If ξ is an R^m-valued random variable on $\langle Q,\mu\rangle$, then $[\xi]$ is defined as follows: $[\xi](q) = [\xi(q)]$ for all $q \in Q$.

1.11. Let $\xi : \langle Q,\mu\rangle \longrightarrow R^n$, $n=1,2,\dots$, be a random variable. Then, by definition, $d(\xi)$, $\bar{d}(\xi)$ and $\underline{d}(\xi)$ are equal, respectively, to the limit (provided it exists), to the upper limit and to the lower limit of $H_0([m\xi])/\log m$ for $m \longrightarrow \infty$. - We will call $d(\xi)$, $\bar{d}(\xi)$ an d $\underline{d}(\xi)$, respectively, the (exact) Rényi dimension (upper dimension, lower dimension) of ξ.

1.12. Theorem (A. Rényi). Let $t=1,2,\dots$ and let $\xi : \langle Q,\mu\rangle \longrightarrow R^t$ be a random variable. Assume that $\mu\circ\xi^{-1}$ is absolutely continuous with respect to the Lebesgue measure on R^t and that $H_0([\xi]) < \infty$. Then $d(\xi)=t$. - See [4], Theorem 4.

1.13. The following simple facts concerning the functional H are well known. - A) Let $x_{kj} \geqq 0$ for $k \in K$, $j \in J$ and let $\sum x_{kj} < \infty$.

Then $H(x_{kj}:(k,j)\in K\times J)=H(\sum(x_{kj}:k\in K):j\in J)+\sum(H(x_{kj}:k\in K):$ $:j\in J)$. - B) Let $x_k\geq 0$, $y_j\geq 0$ for $k\in K$, $j\in J$ and let $\sum x_k<\infty$, $\sum y_j<\infty$. Then $H(x_k y_j:(k,j)\in K\times J)=\sum x_k:H(y_j:j\in J)+\sum y_j\cdot$ $\cdot H(x_k:k\in K)$.

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2.1. Definition. Let $\varphi:\mathfrak{M}\longrightarrow\tilde{R}_+$ be an extended (in the broad sense) Shannon semientropy, as defined in [2], 2.26. Let P be a W-space. We put φ-uw(P)=$\overline{\lim}(\varphi(\sigma * P)/|\log\sigma|)$, φ-lw(P)= =$\underline{\lim}(\varphi(\sigma * P)/|\log\sigma|)$. If φ-uw(P)=φ-lw(P), then we put φ-Rw(P)= = φ-uw(P) and we say that φ-Rw(P) exists or that P is φ-dimension-exact; if not, then φ-Rw(P) is not defined. We call φ-uw(P), φ-lw(P) and φ-Rw(P), respectively, the upper (lower, exact) Rényi φ-weight of P. We put φ-ud(P)=φ-uw(P)/wP, φ-ld(P)= = φ-lw(P)/wP and φ-Rd(P)= φ-Rw(P)/wP (provided φ-Rw(P) exists). We call φ-ud(P), φ-ld(P) and φ-Rd(P), respectively, the upper (lower,exact) Rényi φ-dimension of P. - If φ=E (see [3], 1.13), we usually omit the prefix "φ".

Remark. It is possible (and sometimes useful) to consider, e.g., the "level 2" upper Rényi φ-weight of a W-space P, denoted by $(2,\varphi)$-uw(P) and defined as $\overline{\lim}(\varphi(\sigma * P)/|\log\sigma|^2)$; $(2,\varphi)$-lw(P), $(2,\varphi)$-Rw(P), $(2,\varphi)$-ud(P), $(3,\varphi)$-uw(P), etc., can be defined in a similar way. We will not go, however, into these matters here.

2.2. Conventions. A) Recall that if $P=\langle Q,\rho,\mu\rangle$ is a W-space, then $(P_k:k\in K)$, where $K\neq\emptyset$ is countable and $P_k\leq P$, is called an ω-partition of P whenever $\sum P_k=P$; a finite ω-partition of P is called simply a partition of P; an ε-partition of P, where $0<\varepsilon<\infty$, is, by definition (see [3], 1.19), a countable indexed collection $(X_k:k\in K)$ such that $X_k\in\operatorname{dom}\bar{\mu}$, diam $X_k\leq\varepsilon$, $X_i\cap X_j=\emptyset$ for $i\neq j$, $\sum\bar{\mu}X_i=\mu Q$. - B) An ε-partition $(X_k:k\in K)$ of P will be called an (ε,m)-partition, where $m\in N$, if, for any $Y\subset Q$ satisfying diam $Y\leq\varepsilon$, there is a set $M\subset K$ such that card $M\leq m$ and $\bar{\mu}(X_k\cap Y)=0$ for all $k\in K\setminus M$. - C) A covering of a semimetric space $\langle T,\rho\rangle$ is, by definition, an arbitrary (indexed) collection $(X_k:k\in K)$ such that $\bigcup X_k=T$; a covering $(X_k:k\in K)$ will be called (1) disjoint if $X_i\cap X_j=\emptyset$ for $i,j\in K$, $i\neq j$, (2) an ε-covering if diam $X_k\leq\varepsilon$ for all $k\in K$, (3) an (ε,m)-covering, where $m\in N$, if diam $X_k\leq\varepsilon$ for all $k\in K$ and each set $Y\subset T$ of diameter $\leq\varepsilon$ intersects m sets X_k at most.

2.3. Proposition. Let P be a metric W-space. Then, for all positive reals ε, (1) $E(\varepsilon * P)=E^*(\varepsilon * P)=\eta(\varepsilon * P)=\eta^*(\varepsilon * P)$, (2) $\bar{\eta}(\varepsilon * P)=E(\varepsilon * P)$ unless both $\bar{\eta}(\varepsilon * P)$ and $E(\varepsilon * P)$ are infinite for all sufficiently small ε. - See [3], 2.18. - For the definition of E, η, etc., see [3], 1.9, 1.13 and 1.20.

2.4. Fact. For any W-space $P=\langle Q,\rho,\mu\rangle$ and any (ε,m)-partition $(X_k:k\in K)$ of P, $\bar{\eta}(\varepsilon * P)\leqq H(\bar{\mu}X_k:k\in K)\leqq \bar{\eta}(\varepsilon * P)+wP\cdot\log m$.

Proof. The first inequality is evident. Assume that $\bar{\eta}(\varepsilon * P)<\infty$ and choose a number $b>\bar{\eta}(\varepsilon * P)$. Put $\nu=\bar{\mu}$. Clearly, there is an ε-partition $(Y_j:j\in J)$ of P such that diam $Y_j\leqq\varepsilon$ for all $j\in J$ and $H(\nu Y_j:j\in J)<b$. For $k\in K$, $j\in J$, put $V_{kj}=X_k\cap Y$. By 1.13 A, we have $H(\nu X_k:k\in K)\leqq H(\nu V_{kj}:(k,j)\in K\times J)=H(\nu Y_j:j\in J)+$ $+\Sigma(H(\nu V_{kj}:k\in K):j\in J)$. Since $(X_k:k\in K)$ is an (ε,m)-partition and diam $Y_j\leqq\varepsilon$ for each j, we get $H(\nu V_{kj}:k\in K)\leqq\nu X_k$ log m for all $j\in J$. Hence we obtain $H(\nu X_k:k\in K)\leqq H(\nu Y_j:j\in J)+\mu Q.\log m<b+$ $+\mu Q.\log m$, which proves the assertion.

2.5. Fact. Let $a>0$. Let f and g be non-increasing positive functions on $(0,a)$. Let $(\sigma_n:n\in N)$ be a decreasing sequence, $\sigma_n\to 0$. Let $g(\sigma_n)/g(\sigma_{n+1})\to 1$. Then the upper (lower) limit of $f(\sigma_n)/g(\sigma_n)$ for $n\to\infty$ is equal to that of $f(\varepsilon)/g(\varepsilon)$ for $\varepsilon\to 0$.

2.6. Proposition. Let $P=\langle Q,\rho,\mu\rangle$ be a metric W-space. For $n\in N$ let $(X_{nk}:k\in K_n)$ be an (ε_n,p_n)-partition of P. Assume that $\log p_n/|\log\varepsilon_n|\to 0$ and $|\log\varepsilon_n|/|\log\varepsilon_{n+1}|\to 1$ for $n\to\infty$. Then the upper (lower) limit of $H(\bar{\mu}X_{nk}:k\in K)/|\log\varepsilon_n|$ is equal to uw(P) (to lw(P), respectively).

Proof. By 2.4, we have $\bar{\eta}(\varepsilon_n * P)\leqq H(\bar{\mu}X_{nk}:k\in K_n)\leqq$ $\leqq\bar{\eta}(\varepsilon_n * P)+wP.\log p_n$ for each $n\in N$. Hence, due to $(\log p_n)/$ $/|\log\varepsilon_n|\to 0$, the upper (lower) limit of $H(\bar{\mu}X_{nk}:k\in K)/|\log\varepsilon_n|$ coincides with that of $\bar{\eta}(\varepsilon_n * P)/|\log\varepsilon_n|$. By 2.3 and 2.5, this implies the proposition.

2.7. Proposition. Let $\langle Q,\rho\rangle$ be a bounded subspace of R^n, n=1,2,..., and let $P=\langle Q,\rho,\mu\rangle$ be a W-space. Let τ be a normal gauge functional (see [3], 1.10), $\tau\geqq r$, and let $\varphi=C^*_\tau$ or $\varphi=$ $=C_\tau$. Then φ-ud(P)=E-ud(P), φ-ld(P)=E-ld(P).

This follows at once from [3], 3.7. - For the definition of C_τ, etc., see [3], 1.10-1.13.

2.8. Theorem. Let $\xi:\langle Q,\mu\rangle\to R^t$, t=1,2,..., be a random variable. Put $P=\langle R^t, \rho, \mu \circ \xi^{-1}\rangle$. Then $\bar{d}(\xi)=ud(P)$, $\underline{d}(\xi)=ld(P)$ and hence either both $d(\xi)$ and $Rd(P)$ exist (and are equal) or neither $d(\xi)$ nor $Rd(P)$ exists. If, in addition, ξ is bounded, then the assertion holds with ud, ld and Rd replaced, respectively, by φ-ud, φ-ld and φ-Rd, where $\varphi=C_\tau$ or $\varphi=C_\tau^*$, τ being a normal gauge functional, $\tau \geqq r$.

Proof. For $n=1,2,\ldots$, $z=(z_1,\ldots,z_t)\in Z^t$, put $X_{nz}=\{x=(x_1,\ldots,x_t)\in R_t : z_i \leqq nx_i < z_i+1$ for $i=1,\ldots,t\}$. Then $(X_{nz}:z\in Z^t)$ is a $(1/n,2^t)$-partition of P. Hence, by 2.6, the upper (lower) limit of $H(\bar{\mu}X_{nz}:z\in Z^t)/\log n$ is equal to $uw(P)=ud(P)$ (respectively, to $lw(P)=ld(P)$). On the other hand, by the definition of $\bar{d}(\xi)$, $\underline{d}(\xi)$ and $d(\xi)$, see 1.11, the upper (lower) limit of $H(\bar{\mu}X_{nz}:z\in Z^t)$ is equal to $\bar{d}(\xi)$ (respectively, to $\underline{d}(\xi)$). - The second assertion follows from 2.7.

2.9. Theorem. Let $P=\langle R^t, \rho, \mu\rangle$ be a W-space and let μ be absolutely continuous with respect to the Lebesgue measure λ; let $wP>0$. For any $z=(z_1,\ldots,z_t)\in Z^t$ put $A_z=\{x=(x_1,\ldots,x_t)\in R^t : z_i \leqq x_i < z_i+1$ for $i=1,2,\ldots,t\}$. If $H(\bar{\mu}A_z:z\in Z^t)<\infty$, then $Rd(P)=t$; if $H(\bar{\mu}A_z:z\in Z^t)=\infty$, then $Rd(P)=\infty$.

Proof. For $x\in R^t$ put $\xi(x)=x$. We can assume that $wP=1$. Clearly, $\xi:\langle R^t,\bar{\mu}\rangle \longrightarrow \langle R^t,\rho\rangle$ is a metric random variable. By 2.8, $ud(P)=\bar{d}(\xi)$, $ld(P)=\underline{d}(\xi)$. By 1.12, $\bar{d}(\xi)=\underline{d}(\xi)=t$ if $H(\bar{\mu}A_z:z\in Z^t)<\infty$, and it is easy to see that $\bar{d}(\xi)=\underline{d}(\xi)=\infty$ if $H(\bar{\mu}A_z:z\in Z^t)=\infty$.

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3.1. Fact. If (S,T) is a partition of a W-space P, then $lw(S)+lw(T)\leqq lw(P)\leqq lw(S)+uw(T)\leqq uw(P)\leqq uw(S)+uw(T)$.

This follows at once from 2.3 and [3], 2.5.

3.2. Proposition. Let (S,T) be a partition of a W-space P. If both S and T are dimension-exact, then P is dimension-exact and $Rw(P)=Rw(S)+Rw(T)$. If $Rw(P)<\infty$ and both P and S are dimension-exact, then P-S is dimension-exact, too, and $Rw(P-S)=Rw(P)-Rw(S)$.

Proof. The first assertion follows easily from 3.1. To prove the assertion concerning P-S, observe that, with T=P-S, we have $lw(S)+uw(T)\leqq uw(P)$, $lw(P)\leqq lw(T)+uw(S)$, hence $Rw(S)+uw(T)\leqq Rw(P)\leqq lw(T)+Rw(S)$.

3.3. Definition. A W-space P will be called (1) dimension-bounded if $\sup\{ud(S):S\leqq P\}<\infty$, (2) hereditarily dimension-exact (abbreviation: h.d.e.) if every $S\leqq P$ is dimension-exact.

3.4. Proposition. Let P be a dimension-bounded W-space. If $(P_k:k\in K)$ is an ω-partition of P, then $uw(P)\leqq\Sigma(uw(P_k):k\in K)$.

Proof. Since P is dimension-bounded, there is a $b\in R_+$ such that $uw(S)\leqq b.wS$ for each $S\leqq P$. We can assume that K=N. For any $n\in N$, put $T_n=\Sigma(P_k:k\leqq n)$. By 3.1, we have $uw(P)\leqq\Sigma(uw(P_k):$ $:k\leqq n)+uw(P-T_n)$, hence $uw(P)\leqq\Sigma(uw(P_k):k\in N)+b.w(P-T_n)$, which implies the proposition.

3.5. Example. Let $(a_n:n\in N)$ be a decreasing sequence of reals, $a_n\to 0$. Let b_n, $n\in N$, be positive reals, $\Sigma b_n<\infty$. Consider the W-space $P=\langle N,\rho,\mu\rangle$, where $\rho(i,j)=a_i+a_j$ for $i\neq j$, dom $\mu=\exp N$, $\mu\{i\}=b_i$. It is easy to prove that uw(P) (respectively, lw(P)) is equal to the upper limit of $\Sigma(Lb_i:i\leqq n)/$ $/|\log a_n|$ (to the lower limit of $\Sigma(Lb_i:i\leqq n)/|\log a_{n+1}|$). Put $X_m=$ $=\{i\in N:i\geqq m\}$. Clearly, $uw(X_n\cdot P)=uw(P)$, $lw(X_n\cdot P)=lw(P)$. - Assume that $uw(P)>0$. Then $ud(X_n\cdot P)\to\infty$ and therefore P is not dimension-bounded. Since, evidently, $uw(\{n\}\cdot P)=0$ for each $n\in N$, the conclusion of 3.4 does not hold. - It is easy to find a set $X\subset N$ such that, with $y_n=\Sigma(Lb_i:i\leqq n,i\in X)$, $\overline{\lim}(y_n/|\log a_n|)=uw(P)$, $\underline{\lim}(y_n/$ $/|\log a_{n+1}|)=0$. Hence P is not h.d.e.

3.6. Proposition. Let P be a dimension-bounded W-space. If $(P_k:k\in K)$ is an ω-partition of P and all P_k are dimension-exact, then P is dimension-exact and $Rw(P)=\Sigma(Rw(P_k):k\in K)$.

Proof. We can assume that K=N. Put $T_n=\Sigma(P_k:k\leqq n)$. By 3.2, all T_n are dimension-exact and $Rw(T_n)=\Sigma(Rw(P_k):k\leqq n)$. Since $T_n\leqq P$, we have $Rw(T_n)\leqq uw(P)$ for all $n\in N$, hence $\Sigma(Rw(P_k):k\in N)\leqq$ $\leqq uw(P)$. By 3.4, $uw(P)\leqq\Sigma(Rw(P_k):k\in N)$, which proves the proposition.

3.7. Proposition. Let $(P_k:k\in K)$ be an ω-partition of a dimension-bounded W-space P. If all P_k are hereditarily dimension-exact, then so is P. - This is an easy consequence of 3.6.

3.8. Proposition. Let P be a dimension-bounded W-space. Let $\mathcal{M}\subset\exp P$. If all $S\in\mathcal{M}$ are hereditarily dimension-exact, then so is $\sup\mathcal{M}$.

Proof. By 1.3, we can assume that $\mathcal{M}$ is countable. If $\mathcal{M}=$ $=\{S_0,S_1\}$, then, clearly, $\{S_0-S_0\wedge S_1,\ S_0\wedge S_1, S_1-S_0\wedge S_1\}$ is s partition of $S_0\vee S_1$, consisting of h.d.e. subspaces and therefore, by 3.7, $S_0\vee S_1$ is h.d.e. If $\mathcal{M}=\{S_0,S_1,\dots\}$, put, for $n=0,1,2,\dots$,

$T_n = \bigvee(S_i : i \leqq n)$, $U_0 = T_0$, $U_{n+1} = T_{n+1} - T_n$. Then U_n are h.d.e., $(U_k : k \in N)$ is an ω-partition of $\sup \mathcal{M}$. Hence, again by 3.7, $\sup \mathcal{M}$ is h.d.e.

3.9. Proposition. Let P be a dimension-bounded W-space. Then there exists exactly one maximal hereditarily dimension-exact subspace $S \leqq P$. The subspace S is pure and no non-null $T \leqq P-S$ is hereditarily dimension-exact.

Proof. Let $\mathcal{M}$ be the collection of all h.d.e. subspaces $U \leqq P$. Put $S = \sup \mathcal{M}$. By 3.8, S is h.d.e. Clearly, if $T \leqq P-S$ is h.d.e., then, by 3.7, S+T is h.d.e., hence $S+T \leqq S$, $wT=0$. To prove that S is pure, let $S = f \cdot P$, let $0 < \varepsilon < 1/2$ and let $X = \{q \in Q : \varepsilon < f(q) < 1-\varepsilon\}$. Then $\varepsilon \cdot (X \cdot S)$ is h.d.e., hence $S + \varepsilon \cdot (X \cdot S)$ is h.d.e. and therefore $w(X \cdot S) = 0$. This implies $\bar{\mu} X = 0$.

3.10. We present an example of a dimension-bounded W-space P such that no non-null $S \leqq P$ is dimension-exact. The example is closely related to A. Rényi's example (see [4]) of a real-valued random variable ξ such that $\bar{d}(\xi) \neq \underline{d}(\xi)$.

Let $(a_n : n \in N)$ be a decreasing sequence of positive reals, $a_n \longrightarrow 0$. Let $\langle Q, \nu \rangle$ be the product of ω copies of $\langle \{0,1\}, \nu_0 \rangle$, where $\nu_0\{0\} = \nu_0\{1\} = 1/2$. Put $\mu = \bar{\nu}$. For $(x_i),(y_i) \in Q$ put $\rho((x_i),(y_i)) = \sup(a_i |x_i - y_i| : i \in N)$. Clearly, $P = \langle Q, \rho, \mu \rangle$ is a W-space.

We are going to give an outline of the proof of (1) $ud(X \cdot P) = \overline{\lim}(n/|\log a_n|)$, $ld(X \cdot P) = \underline{\lim}(n/|\log a_n|)$ for each $X \in \operatorname{dom} \mu$ of positive measure. The following simple fact will be used: (2) if $m \in N$, $m \geqq 1$, $a > 0$, $b > 0$, $ma \geqq b$, $0 \leqq x_i \leqq a$, $\sum x_i = b$, then $H(x_1, \dots, x_n) \geqq b \log(b/a)$. The proof of this fact is easy and can be omitted. - Let $n \in N$, $a_n > \sigma \geqq a_{n+1}$. It is easy to see that $E(\sigma * (X \cdot P)) = H(\mu(X \cap B(u_0, \dots, u_n)) : (u_0, \dots, u_n) \in \{0,1\}^{n+1})$, where $B(u_0, \dots, u_n)$ consists of all $(x_i) \in Q$ such that $x_i = u_i$ for $i = 0, \dots, n$. This implies that (3) $E(\sigma * (X \cdot P)) \leqq (n+1) \cdot \mu X$. On the other hand, by (2), we have (4) $E(\sigma * (X \cdot P)) \geqq \mu X \cdot \log(\mu X \cdot 2^{n+1}) = (n+1) \cdot \mu X - L(\mu X)$. - For any positive $\sigma < a_0$, let $f(\sigma)$ be the largest n such that $a_n > \sigma$. Then, by (3) and (4), we have $|E(\sigma * (X \cdot P)) - \mu X \cdot (f(\sigma)+1)| / |\log \sigma| \longrightarrow 0$ for $\sigma \longrightarrow 0$, and therefore $ud(X \cdot P) = \overline{\lim}(f(\sigma)/|\log \sigma|)$, $ld(X \cdot P) = \underline{\lim}(f(\sigma)/|\log \sigma|)$. It is easy to see that the upper (lower) limit of $f(\sigma)/|\log \sigma|$ for $\sigma \longrightarrow 0$ is equal to that of $n/|\log a_n|$ for $n \longrightarrow \infty$. This proves the assertion (1).

Clearly, it is possible to choose a sequence $(a_n : n \in N)$ such that the upper (lower) limit of $n/|\log a_n|$ is equal to 1 (to 0). Then, by (1), we have ud(S)=1, ld(S)=0 for each pure non-null $S \leqq P$. If $S \leqq P$ is not pure, $wS > 0$, then there exists a non-null pure $T \leqq S$, hence ud(S) > 0 (in fact, it is easy to see that ud(S)= =1). Clearly, ld(S)=0. Hence, no non-null $S \leqq P$ is dimension-exact.

3.11. Theorem. Let P be a dimension-bounded W-space and let S be its maximal hereditarily dimension-exact subspace. Then $X \mapsto Rw(X.S)$, defined for $X \in \mathrm{dom}\,\bar{\mu}$, is a measure on Q, absolutely continuous with respect to $\bar{\mu}$.

Proof. If $X_n \in \mathrm{dom}\,\bar{\mu}$, $n \in N$, are mutually disjoint, $X = \cup X_n$, then $(X_n . S : n \in N)$ is an ω-partition of X.S and therefore, by 3.6, $Rw(X.S) = \sum Rw(X_n . S)$. Hence $X \mapsto Rw(X.S)$ is a measure on Q, which is abvolutely continuous, since there is a number b such that, for any $X \in \mathrm{dom}\,\bar{\mu}$, $uw(X.P) \leqq b.w(X.P) = b.\bar{\mu}X$.

3.12. Definition. Let $P = \langle Q, \rho, \mu \rangle$ be a W-space. A $\bar{\mu}$-measurable function $f : Q \to R_+$ will be called an Rw-density function (or simply an Rw-density) for P if, for any $S = g.P \leqq P$, Rw(S)= $= \int fg\,d\mu$ (hence $Rd(S) = \int fg\,d\mu / wS$).

3.13. Theorem. If a W-space $P = \langle Q, \rho, \mu \rangle$ is dimension-bounded and hereditarily dimension-exact, then (1) there exists an Rw-density function for P, (2) if both f_1 and f_2 are Rw-density functions for P, then f_1 and F_2 coincide μ-almost everywhere.

Proof. I. Let ν denote the measure $X \mapsto Rw(X.P)$, see 3.11. Since, by 3.11, ν is absolutely continuous with respect to $\bar{\mu}$, there exists a function $f : Q \to R_+$ such that $\int_X f\,d\mu = \nu(X)$ for any $X \in \mathrm{dom}\,\bar{\mu}$. It is easy to prove that $\int fg\,d\mu = Rw(g.P)$ whenever $g.P \leqq P$. - II. If both f_1 and f_2 are Rw-density functions, then $\int_X f_1\,d\mu = \int_X f_2\,d\mu$ for all $X \in \mathrm{dom}\,\bar{\mu}$, hence f_1 and f_2 coincide μ-almost everywhere.

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4.1, Fact. For any non-null W-spaces P_1 and P_2, $d(P_1 \times P_2) = \max(d(P_1), d(P_2))$.

Proof. Put $P_i = \langle Q_i, \rho_i, \mu_i \rangle$, $P_1 \times P_2 = \langle Q, \rho, \mu \rangle$. For any $u \in R_+$, put $B(u) = \{(x_1,x_2),(y_1,y_2)) \in Q \times Q : \rho((x_1,x_2),(y_1,y_2)) > u\}$, $B_i(u) = \{(x_i,y_i) \in Q_i \times Q_i : \rho_i(x_i,y_i) > u\}$, i=1,2. If $X \subset Q_1 \times Q_1$, put

$M(X)=\{((x_1,x_2),(y_1,y_2))\in Q\times Q:(x_1,y_1)\in X\}$; if $Y\subset Q_2\times Q_2$, put $M(Y)=\{((x_1,x_2),(y_1,y_2))\in Q\times Q:(x_2,y_2)\in Y\}$. It is easy to see that (1) if $u_1,u_2\in R_+$, $u=\max(u_1,u_2)$, then $B(u)\subset M(B_1(u_1))\cup \cup M(B_2(u_2))$, (2) if $u\in R_+$, then $B(u)\supset M(B_1(u))\cup M(B_2(u))$. Put $A=\{u:[\mu\times\mu](B(u))=0\}$, $A_i=\{u:\bar{\mu}_i(B_i(u))=0\}$, $i=1,2$. By (1), we have (3) if $u_1\in A_1$, $u_2\in A_2$, then $\max(u_1,u_2)\in A$; by (2), we get (4) if $u\in A$, then $u\in A_1\cap A_2$. Clearly, (3) and (4) imply the assertion.

4.2. <u>Fact</u>. If P_i, $i=1,2$, are non-null W-spaces, then $E(\sigma*(P_1\times P_2))\leqq E(\sigma*P_1).wP_2+E(\sigma*P_2).wP_1$, for all positive reals σ.

<u>Proof</u>. We can assume that $E(\sigma*P_i)<\infty$. Let $b>E(\sigma*P_1).$ $.wP_2+E(\sigma*P_2).wP_1$. Choose b_1 and b_2 such that $E(\sigma*P_i)<b_i$, $b_1.wP_2+b_2.wP_1<b$. Put $P_i=\langle Q_i,\wp_i,\mu_i\rangle$. By 2.3, there are pure ω-partitions $(X_{ik}.P_i:k\in K_i)$ of P_i, $i=1,2$, such that $d(X_{ik})\leqq\sigma$ and $H(\bar{\mu}_iX_{ik}:k\in K_i)<b_i$. Put $K=K_1\times K_2$ and, for any $(k,j)\in K$, put $V_{kj}=X_{1k}\cap X_{2j}$. Clearly, $(V_{kj}.P:(k,j)\in K)$ is a pure ω-partition of $P=P_1\times P_2$. By 4.1, $d(V_{kj}.P)\leqq\sigma$ for all $(k,j)\in K$. Since $\bar{\mu}V_{kj}=$ $=\bar{\mu}_1X_{1k}.\bar{\mu}_2X_{2j}$, we get, by 1.13B, $H(\bar{\mu}V_{kj}:(k,j)\in K)=H(\bar{\mu}_1X_{1k}:$ $:k\in K_1).wP_2+H(\bar{\mu}_2X_{2j}:j\in K_2).wP_1<b_1.wP_2+b_2.wP_1<b$. Hence, by 2.3, $E(\sigma*P)<b$, which proves the assertion.

4.3. <u>Fact</u>. Let $P_i=\langle Q_i,\wp_i,\mu_i\rangle$, $i=1,2$, be W-spaces. Let $P=P_1\times P_2=\langle Q,\wp,\mu\rangle$. Let $A\in\mathrm{dom}\,\bar{\mu}$. For $x\in Q_1$ let $f(x)$ be equal to the $\bar{\mu}_2$-measure of $\{y\in Q_2:(x,y)\in A\}$ if this set is $\bar{\mu}_2$-measurable, and to zero if not. Then f is $\bar{\mu}_1$-measurable, $w(A.P)=$ $=w(f.P_1)$ and $d(f.P_1)\leqq d(A.P)$.

<u>Proof</u>. The first two assertions follow at once from well-known theorems. Put $B=\{x\in Q_1:f(x)>0\}$, $A'=A\cap(B\times Q_2)$. Clearly, $\bar{\mu}(A\setminus A')=0$, hence $A'.P=A.P$. Put $a=d(A.P)$. Let U consist of all $((x_1,x_2),(y_1,y_2))\in A'\times A'$ such that $\wp_1(x_1,y_1)>a$. Clearly, $[\mu\times\mu](U)=0$. Let T consist of all $(x_1,y_1)\in B\times B$ such that $\wp(x_1,y_1)>a$. For any $(x_1,y_1)\in T$, the set of all $(x_2,y_2)\in Q_2\times Q_2$ such that $((x_1,x_2),(y_1,y_2))\in U$ is equal to $\{(x_2,y_2)\in Q_2\times Q_2:$ $:((x_1,x_2),(y_1,y_2))\in A'\times A'\}=\{z\in Q_2:(x_1,z)\in A'\}\times\{z\in Q_2:(y_1,z)\in$ $\in A'\}$, and therefore its $[\mu_2\times\mu_2]$-measure is positive. Together with $[\mu\times\mu](U)=0$, this implies, by well-known theorems, $[\mu_1\times\mu_1](T)=0$, which proves $d(B.P_1)\leqq a$, hence $d(f_1.P)\leqq a$.

4.4. <u>Fact</u>. Let $P_i=\langle Q_i,\wp_i,\mu_i\rangle$, $i=1,2$, be W-spaces. Let

$P=P_1\times P_2$, $P=\langle Q,\varrho,\mu\rangle$. Then, for any $\sigma>0$, $E(\sigma * P)\geqq wP_2\cdot$ $\cdot E(\sigma * P_1)$.

Proof. By 2.3, it suffices to show that $\eta^*(\sigma * P)\geqq wP_2\cdot$ $\cdot\,\eta(\sigma * P_1)$. We can assume that $wP_2>0$ and $\eta^*(\sigma * P)<\infty$. Choose a number $b>\eta^*(\sigma * P)$ and choose a pure ω-partition $(A_k\cdot P:k\in K)$ of P such that $H(\bar\mu A_k:k\in K)<b$, $d(A_k\cdot P)\leqq\sigma$. By 4.3, there are $\bar\mu_1$-measurable functions f_k, $k\in K$, such that $d(f_k\cdot P_1)\leqq d(A_k\cdot P)\leqq\sigma$ and $w(f_k\cdot P_1)=w(A_k\cdot P)$. Clearly, $((f_k/wP_1)\cdot P_1:k\in K)$ is a partition of P_1. Since $d(f_k\cdot P_1)\leqq\sigma$ for all k, we get $\eta(\sigma * P_1)\leqq$ $\leqq H(w(f_k\cdot P_1)/wP_2:k\in K)=H(w(A_k\cdot P)/wP_2:k\in K)<b/wP_2$. This proves $\eta^*(\sigma * P)\geqq WP_2\cdot\eta(\sigma * P_1)$.

4.5. Proposition. Let P_1 and P_2 be non-null W-spaces. Let $P=P_1\times P_2$. Then $\max(ud(P_1),ud(P_2))\leqq ud(P)\leqq ud(P_1)+ud(P_2)$, $\max(ld(P_1),ld(P_2))\leqq ld(P)\leqq ud(P_1)+ld(P_2)$. If P_1 and P_2 are dimension exact, then $\max(Rd(P_1),Rd(P_2))\leqq ld(P)\leqq ud(P)\leqq Rd(P_1)+Rd(P_2)$.

This is an immediate consequence of 4.2 and 4.4.

4.6. Definition. Let P be a W-space or a metric space. If there exists a function $f:R^*_+\longrightarrow N$ such that $(\log f(\varepsilon))/|\log\varepsilon|\longrightarrow 0$ for $\varepsilon\longrightarrow 0$ and, for all sufficiently small $\varepsilon>0$, there is an $(\varepsilon,f(\varepsilon))$-partition of P (respectively, an $(\varepsilon,f(\varepsilon))$-covering of P consisting of Borel sets), then we will say that P satisfies SGC ("slow growth condition").

Remark. There are countable topologically discrete metric spaces which do not satisfy SGC. On the other hand, there exist infinite-dimensional compact metric spaces satisfying SGC.

4.7. Fact. If a W-space or a metric space satisfies SGC, then so does each of its subspaces. The metric space R^n, n=1,2, ..., satisfies SGC.

4.8. Proposition. Let $P=\langle Q,\varrho,\mu\rangle$ be a weakly Borel metric W-space and let $\langle Q,\varrho\rangle$ be separable. If $\langle Q,\varrho\rangle$ satisfies SGC, then so does P.

Proof. Let $f:R^*_+\longrightarrow N$ be a function possessing (with respect to $\langle Q,\varrho\rangle$) the properties described in 4.6. For each $\varepsilon\in R^*_+$, let $(X_{\varepsilon k}:k\in K_\varepsilon)$ be an $(\varepsilon,f(\varepsilon))$-covering of $\langle Q,\varrho\rangle$ consisting of Borel sets. Clearly, all K_ε are countable, hence we can assume $K_\varepsilon=N$. For $n\in N$, put $Y_{\varepsilon,n}=X_{\varepsilon n}\setminus\bigcup(X_{\varepsilon k}:k<n)$. It is easy to see that $(Y_{\varepsilon k}:k\in N)$ is an $(\varepsilon,f(\varepsilon))$-partition of P.

4.9. Proposition. Let P_1 and P_2 be W-spaces (respectively, metric spaces). If both P_1 and P_2 satisfy SGC, then so does $P=P_1 \times P_2$.

Proof. Let P_i be W-spaces (the other case is analogous). Let $f_i:R_+^* \rightarrow N$ possess, with respect to P_i, the properties described in 4.6. It is easy to see that $f=f_1 f_2$ possesses these properties with respect to P, since if $(X_{ik}:k \in K_i)$ is an $(\varepsilon, f_i(\varepsilon))$-partition of P_i, i=1,2, then $(X_{1k} \times X_{2j}:(k,j) \in K_1 \times K_2)$ is an $(\varepsilon, f(\varepsilon))$-partition of P.

4.10. Theorem. Let P_1 and P_2 be W-spaces satisfying SGC. If both P_1 and P_2 are dimension-exact, then so is $P=P_1 \times P_2$, and $Rd(P_1 \times P_2)=Rd(P_1)+Rd(P_2)$.

Proof. Let $P_i = \langle Q_i, \rho_i, \mu_i \rangle$, $P = \langle Q, \rho, \mu \rangle$. For i=1,2, let f_i possess, with respect to P_i, the properties described in 4.6. For $n \in N$, put $\varepsilon_n = 2^{-n}$, $p_n^{(i)} = f_i(\varepsilon_n)$, $p_n = p_n^{(1)} p_n^{(2)}$. For i=1,2, $n \in N$, let $(X_{nk}^{(i)}:k \in K_n^{(i)})$ be an $(\varepsilon_n, p_n^{(i)})$-partition of P_i. By 2.6, $\lim H(\bar{\mu}_i X_{nk}^{(i)}:k \in K_n^{(i)}) = Rw(P_i)$. Put $K_n = K_n^{(1)} \times K_n^{(2)}$; for $(k,j) \in K_n$, put $Y_{nkj} = X_{nk}^{(1)} \times X_{nj}^{(2)}$. Clearly, $(Y_{nkj}:(k,j) \in K_n)$ is an (ε_n, p_n)-partition of P. Since $\bar{\mu} Y_{nkj} = \bar{\mu}_1 X_{nk}^{(1)} \cdot \bar{\mu}_2 X_{nj}^{(2)}$, we get, by 1.13B, $\lim H(\bar{\mu} Y_{nkj}:(k,j) \in K_n) = wP_2 \cdot Rw(P_1) + wP_1 \cdot Rw(P_2)$, hence, by 2.6, $Rw(P) = wP_2 \cdot Rw(P_1) + wP_1 \cdot Rw(P_2)$, which proves the theorem.

References

[1] J. BALATONI, A. RÉNYI: On the notion of entropy (Hungarian), Publ.Math.Inst.Hungarian Acad.Sci. 1(1956), 9-40. - English translation: Selected papers of Alfred Rényi, vol.I,pp.558-584,Akadémiai Kiadó,Budapest, 1976.

[2] M. KATĚTOV: Extended Shannon entropies I, Czechosl.Math.J. 33(108)(1983), 564-601.

[3] M. KATĚTOV: On extended Shannon entropies and the epsilon entropy, Comment.Math.Univ.Carolinae 27(1986), 519-534.

[4] A. RÉNYI: On the dimension and entropy of probability distributions, Acta Math.Acad.Sci.Hung. 10(1959), 193-215.

[5] A. RÉNYI: Dimension, entropy and information, Trans.2nd Prague Conf. Information Theory, pp.545-556.Prague ,1960.

[6] A. RÉNYI, J. BALATONI: Über den Begriff der Entropie, Arbeiten zur Informationstheorie, pp.117-134.

Deutscher Verlag der Wissenschaften, Berlin, 1957.

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