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ESTIMATOR OF VARIANCE OF WIENER PROCESS BASED ON ITS INTEGRAL

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Abstract: A consistent unbiased estimator of the variance of a Wiener process is suggested. The estimator is based on observations of the path of its integral. Some properties of this estimator are studied.

Key words: Wiener process, consistent unbiased estimator, Kalman filter.

Classification: 60J65, 60J60

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1. Introduction. Let $W(t)$, $t \geq 0$ be the standard Wiener process. Introduce the process $I(t) = \sigma \int_0^t W(s)ds$, $t \geq 0$, where $\sigma > 0$ is a parameter. The problem is to construct an estimator of the unknown parameter σ^2 based on observations $I(t_1), \dots, I(t_n)$, $0 = t_0 < t_1 < \dots < t_n$.

2. Fundamental relations. It is easily seen that $I(t)$, $t \geq 0$ is a Gaussian process with zero expectation. An elementary calculation gives the covariance function

$$(1) \quad R(t,t+h) = \mathrm{cov}(I(t), I(t+h)) = \frac{t^3}{3} + \frac{t^2 h}{2}, \quad t \geq 0,\ h \geq 0.$$

Denote $D_i = \dfrac{I(t_{i+1}) - I(t_i)}{t_{i+1} - t_i}$, $i = 0, 1, \dots, n-1$,

$$\Delta D_i = D_{i+1} - D_i, \quad i = 0, 1, \dots, n-2.$$

Next lemma yields a useful decomposition of the random variables D_i.

Lemma. The random variables D_i, $i=0,1,\dots,n-1$, can be expressed in the following way, $D_i = \sigma(W_i + Y_i)$, where Y_i, $i=0,1,\dots,n-1$, are mutually independent random variables having normal distribution $N(0,(t_{i+1}-t_i)/3)$ and $W_i=W(t_i)$, $i=0,1,\dots,n-1$. Moreover, $\mathrm{cov}(W_i,Y_j)=0$ for $j \geq i$ and $\mathrm{cov}(W_i,Y_j)=(t_{j+1}-t_j)/2$ for $j<i$.

Proof. The following equality is obvious.

$$\frac{1}{\sigma} D_i=W(t_i)+(t_{i+1}-t_i)^{-1} \int_{t_i}^{t_{i+1}} (W(s)-W(t_i))ds.$$

Set $Y_i=(t_{i+1}-t_i)^{-1} \int_{t_i}^{t_{i+1}}(W(s)-W(t_i))ds$. Now, computing the variances and covariances $\mathrm{cov}(W_i,Y_j)$ using (1) and independence of increments of the Wiener process, we accomplish the proof.

Using the lemma we get some properties of differences ΔD_i.

Property 1. The variables ΔD_i, $i=0,1,\dots,n-2$ are normally distributed with zero expectation and variances $E(\Delta D_i)^2 = \sigma^2(t_{i+2}-t_i)/3$.

Property 2. $\mathrm{Cov}(\Delta D_i, \Delta D_{i+1})= \sigma^2(t_{i+2}-t_{i+1})/6$, $i=0,1,\dots,n-3$. ΔD_i, ΔD_j are for $|i-j| \geq 2$ mutually independent.

3. Construction of the estimator. We shall employ the theory of the Kalman filter. It can be easily verified that the sequence Y_i^0, $i=0,1,\dots,n-2$, defined by the following model, has the same distribution as $\Delta D_i/\sigma$, $i=0,1,\dots,n-2$.

(2) Let $X_{i+1}=aX_i+U_{i+1}$

$$Y_i^0=cX_i+U_i^0, \quad i=0,1,\dots,n-2,$$

where $a = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix}$, $c = (1,1)$, $X_i = \begin{pmatrix} X_i^1 \\ X_i^2 \end{pmatrix}$,

assume that

$U_i= \begin{pmatrix} 0 \\ u_i \end{pmatrix}$ such that $U_i \sim N(0,d_i)$, where $d_i=\begin{pmatrix} 0, & 0 \\ 0, & (t_{i+2}-t_{i+1})/6 \end{pmatrix}$,

$U_i^0 \sim N(0,d_i^0)$, where $d_i^0=(t_{i+2}-t_i)/6$,

$X_{-1}^j \sim N(0,t_1/6)$, $j=1,2$,

u_i, U_i^o, $i=0,1,\ldots,n-2$ and X_{-1}^j, $j=1,2$ are mutually independent.
We shall construct the Kalman filter for model (2). (See e.g. Åström [1].)
Denote $r_i=E(X_i-\hat{X}_i)(X_i-\hat{X}_i)'$ and $r_i^-=E(X_i-\hat{X}_i^-)(X_i-\hat{X}_i^-)'$. Kalman equation is

(3) $\quad \hat{X}_i=\hat{X}_i^-+k_i(Y_i^o-c\,\hat{X}_i^-)$, where $\hat{X}_{i+1}^-=a\,\hat{X}_i$,

$$\hat{X}_{-1}=\begin{pmatrix}0\\0\end{pmatrix},\quad r_{-1}=\begin{pmatrix}t_1/6, & 0\\ 0, & t_1/6\end{pmatrix},\quad k_i=r_i^-\,c'(c\,r_i^-\,c'+d_i^o)^{-1}.$$

For the matrices r_i and r_i^- we have

(4) $\quad r_{i+1}^-=a\,r_i\,s'+d_{i+1}$ and $r_{i+1}=r_{i+1}^- - k_{i+1}\,c\,r_{i+1}^-$.

It follows from the theory of the Kalman filter that $(Y_i^o-c\,\hat{X}_i^-)$, $i=0,1,\ldots,n-2$, are mutually independent random variables having normal distribution $N(0,c\,r_i^-\,c'+d_i^o)$. Therefore the sum $\sum_{i=0}^{n-2}(Y_i^o-c\,\hat{X}_i^-)^2/(c\,r_i^-c'+d_i^o)$ has χ^2-distribution with (n-1) degrees of freedom and $\frac{1}{n-1}\sum_{i=0}^{n-2}(Y_i^o-c\,\hat{X}_i^-)^2/(c\,r_i^-\,c'+d_i^o)\xrightarrow{n\to\infty}1$ a.s.
Now we can construct the estimator of σ^2.

<u>Theorem 1</u>. Let ΔD_i be as before. Replace Y_i^o by ΔD_i in Kalman equation (3). Introduce the following variable

$$S_n^2=\frac{1}{n-1}\sum_{i=0}^{n-2}(\Delta D_i-c\,\hat{X}_i^-)^2/(c\,r_i^-\,c'+d_i^o)$$

Then S_n^2 is a consistent unbiased estimator of σ^2.
Moreover, $\frac{n-1}{\sigma^2}S_n^2$ has χ^2-distribution with (n-1) degrees of freedom.

Proof of the theorem follows from the preceding reasoning.

<u>Remark</u>. Kalman equation yields the orthogonalization of the sequence D_i, $i=0,1,\ldots,n-2$.

We shall derive the form of S_n^2 suitable for computation. It follows from (4) that

(5) $\quad c\,r_{i+1}^-c'=(r_i)_{22}+(t_{i+3}-t_{i+2})/6$, where $(r_i)_{22}$ denotes the element on the position (2,2) in the matrix r_i.
Using (4) to compute $(r_i)_{22}$ we obtain the recurrent formula for

variances $v_i=c\, r_i^- c'+d_i^0$. Namely,

(6) $v_{i+1}=(t_{i+3}-t_{i+1})/3-(36\, v_i)^{-1}(t_{i+2}-t_{i+1})^2$ with the initial condition $v_0=t_2/3$.

Because $\hat{X}_{i+1}^-=a\, \hat{X}_i^-+a\, k_i(Y_i^0-c\, \hat{X}_i^-)$ and $a^2=\begin{pmatrix}0, & 0\\ 0 & 0\end{pmatrix}$, we have a $\hat{X}_i^-=\begin{pmatrix}0\\0\end{pmatrix}$ for each $i=0,1,\dots,n-2$. Consequently, $c\, \hat{X}_{i+1}^-=$ $=c\, a\, k_i(Y_i^0-c\, \hat{X}_i^-)$. After a short computation we get

(7) $c\, \hat{X}_{i+1}^-=(t_{i+2}-t_{i+1})(Y_i^0-c\, \hat{X}_i^-)/(6\, v_i)$.

Theorem 2. Let v_i, $i=0,1,\dots,n-2$, be as in (6). Let $Z_{i+1}=$ $=\Delta D_{i+1}-(t_{i+2}-t_{i+1})Z_i/(6\, v_i)$, $Z_0=\Delta D_0$. Then the estimator S_n^2 has the form

$$S_n^2=\frac{1}{n-1}\sum_{i=0}^{n-2} Z_i^2/v_i.$$

Proof. The theorem is an immediate consequence of (6),(7) and of Theorem 1.

Remark. (Special case.) Let $t_{i+1}-t_i=K$, $i=0,1,\dots,n-1$, where K is a positive constant. Then the estimator S_n^2 can be computed as follows.

$S_n^2=\frac{1}{K(n-1)}\sum_{i=0}^{n-2} Z_i'^2/v_i'$, where $v_{i+1}'=2/3-1/(36\, v_i')$,

$v_0'=2/3$, $Z_{i+1}'=\Delta D_{i+1}-Z_i'/(6\, v_i')$ and $Z_0'=\Delta D_0$.

Note that $v_1'=0.625$, $v_i'=0.622$, $i=2,3,\dots$.

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