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COVARIANT APPROACH TO NATURAL TRANSFORMATIONS OF WEIL FUNCTORS

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Abstract: We deduce that the natural transformations of a Weil functor T^B into a Weil functor T^A are bijectively related with the B-admissible A-velocities. The covariant character of such an approach is suitable for some concrete problems in differential geometry.

Key words: Weil algebra, near point, Weil functor, generalized velocity.

Classification: 58A05, 58A20

The geometrical significance of Weil algebras and the related functors [9], has been recently underlined by the theorem that all product-preserving functors from the category of connected smooth manifolds into itself are determined by a finite number of Weil algebras [1],[3]. In both these papers it is also proved that the natural transformations of Weil functors are bijectively related with the homomorphisms of the corresponding Weil algebras. However, such a description is of contravariant character, while in the greater part of differential geometric problems the covariant approach is to be used. That is why we present an independent proof of the latter result, in which we replace the near A-points of A. Weil [9], by an equivalent concept of an A-velocity generalizing the classical k^r-velocities by C. Ehresmann [2]. Our idea of a B-admissible A-velocity originated in the admissible separated jets used in the special case of the natural transformations of the iterated k^r-velocities functors [8]. We deduce that all natural transformations of the Weil functors can be characterized by means of certain reparametrizations, a special case of which we discussed for the second tangent functor in [5]. The basic purpose of our example is to show that such reparametrizations are suitable for some concrete problems in differential geometry. - Our consideration is in the category

C^{∞} and all manifolds are assumed to be paracompact.

1. In the algebra $R[x_1,\dots,x_k]$ of all real polynomials in k variables, consider the ideal $\langle x_1,\dots,x_k\rangle$ generated by $x_1,\dots,x_k$ and its (r+1)-st power $\langle x_1,\dots,x_k\rangle^{r+1}$. By [9], a Weil algebra can be defined as a factor algebra $A=R[x_1,\dots,x_k]/\mathcal{A}$, where $\mathcal{A}$ is any ideal satisfying $\langle x_1,\dots,x_k\rangle \supset \mathcal{A} \supset \langle x_1,\dots,x_k\rangle^{r+1}$. We shall need the following modification of this approach. Let E(k) be the algebra of all germs of smooth functions on R^k at zero, $\underline{m}(k)$ be the ideal of all germs vanishing at zero and $\underline{m}(k)^{r+1}$ be its (r+1)-st power. Any ideal $\mathcal{A}$ in E(k) satisfying $\underline{m}(k)\supset\mathcal{A}\supset\underline{m}(k)^{r+1}$ will be called a Weil ideal and the corresponding Weil algebra is defined by $A=E(k)/\mathcal{A}$. Since $E(k)/\underline{m}(k)^{r+1}=R[x_1,\dots,x_k]/\langle x_1,\dots$ $\dots,x_k\rangle^{r+1}$, our second definition of a Weil algebra is equivalent to the first one.

Let M be a manifold and A be a Weil algebra. According to [9], a near A-point on M means an algebra homomorphism $X:C^{\infty}M \to A$, where $C^{\infty}M$ is the algebra of all smooth functions on M, see also [6]. All near A-points on M form a fibred manifold $M^A \to M$ and every smooth map $f:M\to N$ is extended into $f^A:M^A\to N^A$ by composition with the induced homomorphism $f^*:C^{\infty}N\to C^{\infty}M$, i.e. $f^A(X)=$ $=X\circ f^*$. This defines the Weil functor corresponding to the Weil algebra A. For $\mathcal{A}=\underline{m}(k)^{r+1}$, such a functor coincides with the k^r-velocities functor by C. Ehresmann, which assigns to every manifold M the space $T^r_kM=J^r_0(R^k,M)$ of all r-jets of R^k into M with source zero and the extension $T^r_kf:T^r_kM\to T^r_kN$ of any map $f:M\to N$ is defined by the composition of jets, [2]. The covariant approach to an arbitrary Weil functor is based on the following definition, the basic idea of which is due to A. Morimoto. Let E(M,x) be the set of all germs of smooth functions on a manifold M at a point x.

<u>Definition 1</u>. Let $A=E(k)/\mathcal{A}$ be a Weil algebra. Two maps $g,h:R^k\to M$, $g(0)=h(0)=x$, are said to be A-equivalent, if

(1) $$\varphi\circ g-\varphi\circ h\in\mathcal{A}$$

for every germ $\varphi\in E(M,x)$. Such an equivalence class will be denoted by j^Ag and called an A-velocity on M. The point g(0) will be said to be the target of j^Ag.

Denote by T^AM the set of all A-velocities on M. It is easy to see that $T^A R=A$. The target map is a projection $T^A M \to M$. Further, for every $f:M \to N$ we define $T^A f: T^A M \to T^A N$ by $T^A f(j^A g)=$ $=j^A(f \circ g)$. Obviously, T^A is a functor. Every A-velocity $j^A g \in$ $\in T^A M$ determines a near A-point $X \in M^A$ by

$$X(\varphi)=j^A(\varphi \circ g) \quad \text{for all } \varphi \in C^\infty M. \tag{2}$$

Morimoto proved that (2) is an identification $M^A \approx T^A M$, which is a natural equivalence of functors $-^A$ and T^A, [7].

2. Let $\mathcal{B} \subset E(p)$ be another Weil ideal. Denote by 0_A or 0_B the zero element of $A=E(k)/\mathcal{A}$ or $B=E(p)/\mathcal{B}$, respectively. Given a map $f:(R^k,0) \to (R^p,0)$, it holds $f^*(\mathcal{B}) \subset \mathcal{A}$ if and only if

$$j^A(\varphi \circ f)=0_A \quad \text{for all } \varphi \in \mathcal{B}. \tag{3}$$

Indeed, $j^A(\varphi \circ f)=0_A$ means $\varphi \circ f \in \mathcal{A}$. Since $j^A(\varphi \circ f)=$ $=T^A\varphi(j^A f)$, condition (3) depends on $j^A f$ only.

<u>Definition 2</u>. If (3) holds, then $j^A f \in T^A_0 R^p$ is called a B-admissible A-velocity.

A practical procedure for finding the B-admissible A-velocities is based on the following lemma.

<u>Lemma 1</u>. Let φ_α, $\alpha=1,\dots,q$ be a system of generators of the ideal $\mathcal{B}$. If

$$j^A(\varphi_\alpha \circ f)=0 \quad \text{for all } \alpha=1,\dots,q, \tag{4}$$

then $j^A f$ is a B-admissible A-velocity.

<u>Proof</u>. Every $\varphi \in B$ is of the form $\sum_{\alpha=1}^{q} \varphi_\alpha h_\alpha$, $h_\alpha \in E(p)$. Since $f^*(\varphi_\alpha) \in \mathcal{A}$ by assumption and $\mathcal{A}$ is an ideal, we have

$f^*(\varphi)=\sum_{\alpha=1}^{q} f^*(\varphi_\alpha)f^*(h_\alpha) \in \mathcal{A}$, QED.

<u>Lemma 2</u>. If $j^A f \in T^A_0 R^p$ is B-admissible, then $j^A(g \circ f)$ depends only on $j^B g$ for every map $g:R^p \to M$.

<u>Proof</u>. By Definition 1, $j^B g=j^B h$ means $\varphi \circ g-\varphi \circ h \in \mathcal{B}$ for all $\varphi \in E(M,x)$, $x=g(0)$. Then $\varphi \circ g \circ f-\varphi \circ h \circ f \in f^*(\mathcal{B}) \subset \mathcal{A}$, QED.

<u>Proposition 1</u>. Let $X=j^A f$ be a B-admissible A-velocity. Then the maps

(5) $$i^X_M : T^B M \longrightarrow T^A M, \quad j^B g \longmapsto j^A(g \circ f)$$

determine a natural transformation $i^X : T^B \longrightarrow T^A$.

Proof. By Lemma 2, i^X_M is well defined. Given a map $h: M \to N$, we have $i^X_N(T^B h(j^B g)) = i^X_N(j^B(h \circ g)) = j^A(h \circ g \circ f) = T^A h(i^X_M(j^B g))$, so that i^X is a natural transformation, QED.

3. From now on we restrict ourselves to the natural transformations $i: T^B \longrightarrow T^A$ with the property that every $i_M: T^B M \longrightarrow T^A M$ is a base-preserving morphism of fibred **manifolds**. We are going to deduce that Proposition 1 determines all such natural **transformations of** T^B into T^A.

There is a distinguished element $1_p = j^B(\mathrm{id}_{R^p})$ in $T^B_0 R^p$. Obviously, it holds $j^B f = j^B(f \circ \mathrm{id}_{R^p}) = T^B f(1_p)$. Since Weil functors are product-preserving, every singleton pt is transformed into a singleton, i.e. $t^B(pt) = pt$.

Lemma 3. If $\tilde{0}: pt \longrightarrow R$ is the map transforming a singleton into $0 \in R$, then $0_B = T^B \tilde{0}(pt)$.

Proof. Let $\hat{0}: R^p \longrightarrow R$ denote the constant map of R^p into $0 \in R$. This can be factorized by $\hat{0} = \tilde{0} \circ ct$, where $ct: R^p \longrightarrow pt$ is the unique map. Then $0_B = j^B(\hat{0}) = T^B \hat{0}(1_p) = T^B \tilde{0}(T^B(ct)(1_p)) = T^B \tilde{0}(pt)$, QED.

Lemma 4. Every natural transformation $i: T^B \longrightarrow T^A$ satisfies $i_R(0_B) = 0_A$.

Proof. The naturality condition on $\tilde{0}: pt \longrightarrow R$ gives a commutative diagram

$$\begin{array}{ccc} pt & \xrightarrow{T^B \tilde{0}} & T^B R \\ \| & & \downarrow i_R \\ pt & \xrightarrow{T^A \tilde{0}} & T^A R \end{array}$$

Hence Lemma 4 follows from Lemma 3, QED.

Lemma 5. For every natural transformation $i: T^B \longrightarrow T^A$, $i_{R^p}(1_p) \in T^A_0 R^p$ is a B-admissible A-velocity.

Proof. Consider $\varphi : R^p \longrightarrow R$ such that its germ at zero belongs to $\mathfrak{B}$. The naturality condition on φ gives a commutative diagram

$$\begin{array}{ccc} T^B R^p & \xrightarrow{\;T^B\varphi\;} & T^B R \\ {\scriptstyle i_{R^p}}\downarrow & & \downarrow{\scriptstyle i_R} \\ T^A R^p & \xrightarrow{\;T^A\varphi\;} & T^A R \end{array}$$

Clockwise we obtain $i_R(T^B\varphi(1_p)) = i_R(0_B) = 0_A$ by Lemma 4. If $i_{R^p}(1_p) = j^A f$, then counterclockwise we find $0_A = T^A\varphi(j^A f)$, which is equivalent to (3), QED.

Proposition 2. There is a bijection between the natural transformations $i : T^B \longrightarrow T^A$ and the B-admissible A-velocities given by

(6) $$i_{R^p}(1_p) = j^A f.$$

Proof. We have to prove that every natural transformation $i : T^B \longrightarrow T^A$ is determined by the B-admissible A-velocity (6). For every $j^B g \in T^B M$, the naturality condition on $g : R^p \longrightarrow M$ gives a commutative diagram

$$\begin{array}{ccc} T^B R^p & \xrightarrow{\;T^B g\;} & T^B M \\ {\scriptstyle i_{R^p}}\downarrow & & \downarrow{\scriptstyle i_M} \\ T^A R^p & \xrightarrow{\;T^A g\;} & T^A M \end{array}$$

For $1_p \in T^B_0 R^p$, we obtain $i_M(j^B g) = j^A(g \circ f)$, QED.

4. To find all natural transformations of T^B into T^A, we first have to determine all B-admissible A-velocities by means of (4). Then the natural transformations are given by (5), which represents a kind of reparametrization of the B-velocities. As a very simple illustration of this procedure, we determine all natural transformations between the functor T^2_1 of the classical 1^2-velocities and the iterated tangent functor TT. Since the classical tangent functor is the Weil functor of the algebra of dual numbers $D = R[t]/\langle t^2 \rangle$, the iterated tangent functor TT is the Weil functor of the tensor product $D \otimes D = R[t,\tau]/\langle t^2, \tau^2 \rangle$ according

to a general theorem [9]. On the other hand, T^2_1 corresponds to the Weil algebra $E=R[u]/\langle u^3\rangle$.

As an auxiliary result, we first determine all natural transformations from T into T^2_1. Every E-velocity on R at zero can be written as $t=hu+ku^2$. Since the Weil ideal of D is generated by t^2, such an E-velocity is D-admissible if and only if $(hu+ku^2)^2 \in \langle u^3\rangle$. This implies $h=0$. Having some local coordinates x^i on a manifold M, the induced coordinates ξ^i on TM or a^i, b^i on T^2_1M are given by $x^i+\xi^i t$ or $x^i+a^iu+b^iu^2$, respectively. According to (5), the coordinate expression of all natural transformations $T \to T^2_1$ is

$$a^i=0, \quad b^i=k\xi^i, \quad k\in R. \tag{7}$$

Geometrically, this represents the constant multiples of the well-known injection of TM into the kernel of the jet projection $T^2_1M \to TM$.

Now we discuss the natural transformations $TT \to T^2_1$. Every E-velocity on R^2 at zero can be written as $t=h_1u+k_1u^2$, $\tau=h_2u+k_2u^2$. Since the Weil ideal of $D\otimes D$ is generated by t^2 and τ^2, the admissibility condition implies $h_1=h_2=0$ similarly as above. Denote by c^i, d^i, e^i the induced local coordinates $x^i+c^it+d^i\tau+e^it\tau$ on TTM. Using (5), we deduce that all natural transformations $TT \to T^2_1$ are

$$a^i=0, \quad b^i=k_1c^i+k_2d^i, \quad k_1,k_2\in R. \tag{8}$$

There are two well-known natural projections of TTM into TM. The geometrical meaning of (8) is that we take any linear combination with constant coefficients of both projections and apply the kernel injection into T^2_1M.

Finally we determine all natural transformations from T^2_1 into TT. Every $D\otimes D$-velocity on R at zero can be written as $u=k_1t+k_2\tau+k_3t\tau$. The E-admissibility condition requires $(k_1t+k_2\tau+k_3t\tau)^3\in\langle t^2,\tau^2\rangle$. This is always satisfied, so that any such velocity is E-admissible. This leads to the following

3-parameter system of natural transformations $T_1^2 \to TT$

(9) $$c^i = k_1 a^i, \quad d^i = k_2 a^i, \quad e^i = k_3 a^i + 2k_1 k_2 b^i$$

For $k_1 = k_2 = 1$, $k_3 = 0$, we obtain the classical injection $v_M : T_1^2 M \to \to TTM$ transforming any 2-jet of a curve $\gamma : R \to M$ at zero into the tangent vector of the induced curve $T\gamma$ on TM. There are two natural vector bundle structures on TTM over TM. For $k_3 = 0$, (9) represents the composition $v_M(k_1, k_2)$ of v_M with constant multiplication by k_1 with respect to the first structure and by k_2 with respect to the second structure. Further, we can compose the jet projection $T_1^2 M \to TM$, the kernel injection $TM \to T_1^2 M$ and $v_M : T_1^2 M \to TTM$. Clearly, (9) is the sum of a constant multiple of the latter map with $v_M(k_1, k_2)$.

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