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ON ONE CONSTRUCTION OF ALL QUASIFIELDS OF ORDER 9

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Abstract: An alternative construction of all quasifields of order 9 is given based on one concept of Dempwolff and Reifart ([1], p. 138).

Key words: Dispersing matrices over GF(3), quasigroups, quasifields.

AMS subject classification (1980): Primary: 51A40, 51E15

An approach to the construction of all quasifields of order 9 is given by M. Hall in [2], reconstructed by H. Lüneburg in [3], § 8. As a main tool, there is used in [3] a convenient representation of a spread (describing a translation plane P) with help of some admissible subset of GL(X) where X is a vector space over the kernel of P. In the sequel we give an alternative construction of all quasifields of order 9 stimulated by some aspects of the article [1] by Dempwolff and Reifart. We shall find both binary operations of the quasifields directly (some matrices over GF(3) are needed in our procedure, too).

As well-known, every quasifield of order p^n can be obtained as follows: We take an n-dimensional vector space V over GF(p) and a set $\mathfrak{M}$, $p^n - 1$ matrices from GL(n,p) such that AB^{-1} is fixed - point - free on V for $A, B \in \mathfrak{M}$, $A \neq B$. Then there is a bijection $\mu: \mathfrak{M} \to V \setminus \{0\}$ such that V together with binary operations + (vector addition on V) and $\cdot$ ($v \cdot w = \mu^{-1}(v)w$ for all v,w ∈ V and v ≠ 0, $\underline{0}\cdot w = \underline{0}$) is a quasifield (cf. [1], p. 138).

0. Denote elements of GF(3) by 0, 1, 2 and investigate the 2-dimensional vector space $V_2(3)$ of ordered couples

$$(0,0) =: \underline{0},\ (1,0) =: \underline{1},\ (2,0) =: \underline{2},$$
$$(0,1) =: \underline{3},\ (1,1) =: \underline{4},\ (2,1) =: \underline{5},$$
$$(0,2) =: \underline{6},\ (1,2) =: \underline{7},\ (2,2) =: \underline{8} \text{ over GF}(3).$$

The set of these vectors $\underline{0}, \underline{1}, \ldots, \underline{8}$ will be designated by S and the chosen ordering of vectors will be called natural.
For the component - wise addition of vectors in $V_2(3)$ we can write the corresponding Cayley table (Table 1, without heading).

+									
	0	1	2	3	4	5	6	7	8
	1	2	0	4	5	3	7	8	6
	2	0	1	5	3	4	8	6	7
	3	4	5	6	7	8	0	1	2
	4	5	3	7	8	6	1	2	0
	5	3	4	8	6	7	2	0	1
	6	7	8	0	1	2	3	4	5
	7	8	6	1	2	0	4	5	3
	8	6	7	2	0	1	5	3	4

Tab. 1.

So $(S, +, \underline{0})$ is an elementary abelian 3-group of order 9.

Further consider the set of all non-singular 2×2 matrices over GF(3): $\mathcal{M} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \text{GF}(3),\ \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0 \right\}$.
Obviously $\# \mathcal{M} = 48$ as there are eight possibilities for the first (non - zero) row of our matrix and six possibilities for the second row: $(\gamma, \delta) \neq \lambda \cdot (\alpha, \beta)$, $\lambda \neq 1, 2$.
The set $\mathcal{M}$ with respect to matrix multiplication forms a group of order 48 sometimes denoted by GL(2,3) or by $GL_2(3)$.

If x is a non - zero vector from $V_2(3)$ and $M \in \mathcal{M}$, then $M.x^T = y^T$ (T denotes transposing) and y is also a non - zero vector from $V_2(3)$.

Now let $M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ be a matrix and $(1,0)$, $(0,1)$ vectors of the canonic basis. Then $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}$ and $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\beta\\ \delta\end{pmatrix}$. For $(\alpha, \gamma) = i$ and $(\beta, \delta) = j$ we shall introduce a convenient denotation $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} =: M_{i,j}$. So $M_{1,3} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is a unit matrix.

We say the matrix $M \in \mathcal{M}$ is dispersing if $M x^T \neq x^T$ $\forall x \in V_2(3) \setminus \{(0,0)\}$. If a matrix $M \in \mathcal{M}$ is not dispersing and not unit it preserves two non - zero vectors. The unit matrix $M_{1,3}$ fixes the whole $V_2(3)$. As usual we shall call the set of all eigenvalues of a matrix its spectrum. It may be shown that there are 27 dispersing matrices. Their spectrum is either void (for 18 matrices) or consists just of $2 \in GF(3)$. If a matrix is not dispersing then its spectrum is either $\{1\}$ or $\{1,2\}$, where $1,2 \in GF(3)$. Under the group order of a matrix we shall understand its order as of an element in $GL_2(3)$. The set $\mathcal{M}$ can be decomposed onto the following subsets

$\mathcal{M}_I$ - the set of all matrices with the group order 8, $\#\mathcal{M}_I = 12$;
$\mathcal{M}_{II}$ - the set of all matrices with the group order 6, $\#\mathcal{M}_{II} = 8$;
$\mathcal{M}_{III}$ - the set of all matrices with the group order 4, $\#\mathcal{M}_{III} = 6$;
$\mathcal{M}_{IV}$ - the set of all matrices with the group order 3, $\#\mathcal{M}_{IV} = 8$;
$\mathcal{M}_V$ - the set of all matrices with the group order 2, $\#\mathcal{M}_V = 13$;
$\mathcal{M}_{VI}$ - the set consisting just of the unit matrix.

$\mathcal{M}_I = \{M_{3,7}, M_{4,1}; M_{3,4}, M_{5,1}; M_{4,5}, M_{5,8}; M_{6,5}, M_{8,2}; M_{6,8}, M_{7,2}; M_{7,4}, M_{8,7}\}$ contains only dispersing matrices all with void spectrum (neighbouring matrices are mutually inverse and similarly in what follows).

$\mathcal{M}_{II} = \{M_{2,7}, M_{2,8}; M_{3,5}, M_{7,1}; M_{4,2}, M_{6,4}; M_{5,6}, M_{8,6}\}$ contains only dispersing matrices with spectrum $\{2\}$.

$\mathcal{M}_{III} = \{M_{3,2}, M_{6,1}; M_{4,7}, M_{8,5}; M_{5,4}, M_{7,8}\}$. Here the matrices are dispersing and all have void spectrum.

$\mathcal{M}_{IV} = \{M_{1,4}, M_{1,5}; M_{3,8}, M_{8,1}; M_{4,3}, M_{7,3}; M_{5,2}, M_{6,7}\}$. All these matrices have spectrum $\{1\}$.

$\mathcal{M}_V = \{M_{1,6}; M_{1,7}; M_{1,8}; M_{2,3}; M_{2,4}; M_{2,5}; M_{3,1}; M_{4,6}; M_{5,3}; M_{6,2}; M_{7,6}; M_{8,3}\} \cup \{M_{2,6}\}$.

All the matrices are involutory. The matrices of the first subset have spectrum $\{1,2\}$ whereas $M_{2,6}$ is a dispersing matrix with spectrum $\{2\}$. Now we shall construct such sets $\mathcal{M}_{\mathfrak{X}}$ consisting of eight matrices of $GL_2(3)$ that $\mathcal{M}_{\mathfrak{X}}$ operates minimally transitively on the set of all non - zero vectors of $V_2(3)$. This means that to every couple x, y of non - zero vectors of $V_2(3)$ there is just one matrix $M_{i,j} \in \mathcal{M}_{\mathfrak{X}}$ such that $M_{i,j}\, x^T = y^T$. The last equation can be re-written in the form $\mathcal{U}(x,y) = i$ where $\mathcal{U}$ is a quasigroup operation on the set $S \setminus \{0\}$. By our convention concerning the matrix denotation we have $M_{i,j}\,(1,0)^T = i$ so that $\mathcal{U}(1,y) = y$ for all $y \in S \setminus \{0\}$. Thus the quasigroup $(S \setminus \{0\}, \mathcal{U})$ has a left unit. For $\mathcal{U}(x,y) = c$ we write $\mathcal{U}^{-1}(x,c) = x \cdot c = y$. If $M_{1,3} \in \mathcal{M}_{\mathfrak{X}}$, then $M_{1,3}\, x^T = x^T$, $\forall x \in V_2(3)$ so that $\mathcal{U}(x,x) = 1$ and $\mathcal{U}^{-1}(x,1) = x \cdot 1 = x$. Thus the operation $(\cdot)$, inverse from right to $\mathcal{U}$, possesses a both - side unit and consequently is a loop operation. Thus $(S \setminus \{0\}, \cdot, 1)$ is a loop.

1. Let us take the matrix $M_{4,5}$ and form the group $\langle M_{4,5} \rangle$ generated by this matrix. We get the set of eight matrices $\mathcal{M}_1 = \{M_{1,3};\ M_{2,6};\ M_{3,2};\ M_{4,5};\ M_{5,8};\ M_{6,1};\ M_{7,4};\ M_{8,7}\}$. Each of them maps $V_2(3)$ onto $V_2(3)$ and in Table 2 we have the graph of all of these mappings and subsequently also the multiplication table for $\mathcal{U}$.

y								
(2,2)	8	4	5	2	3	7	6	1
(1,2)	7	5	8	6	2	4	1	3
(0,2)	6	3	2	4	7	1	5	8
(2,1)	5	7	4	3	1	8	2	6
(1,1)	4	8	7	1	6	5	3	2
(0,1)	3	6	1	8	5	2	7	4
(2,0)	2	1	3	7	8	6	4	5
(1,0)	1	2	6	5	4	3	8	7
0	(1,0)	(2,0)	(0,1)	(1,1)	(2,1)	(0,2)	(1,2)	(2,2)
								x

Tab. 2

·							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	6	2	5	8	1	4	7
4	8	5	6	1	7	2	3
5	7	8	1	3	4	6	2
6	3	1	7	4	2	8	5
7	5	4	2	6	8	3	1
8	4	7	3	2	5	1	6

Tab. 3

From $\mathcal{U}$ we get easily over to the operation $\mathcal{U}^{-1} = \cdot$ described in Table 3. $(S \setminus \{0\}, \cdot, 1)$ is a cyclic group of order 8. If we enlarge the operation $\cdot$ onto the whole S by setting $0 \cdot x = x \cdot 0 = 0$, then $(S, +, \cdot)$ is a Galois field GF(9) deduced from GF(3) with help of the irreducible polynomial $\xi^2 + 1$, if we denote linear polynomials ξ, $\xi + 1$, $\xi + 2$, 2ξ, $2\xi + 1$, $2\xi + 2$, respectively, by symbols 3, 4, 5, 6, 7, 8, respectively.

2. An analogous situation occurs if we generate the cyclic group of order 8 by the matrix $M_{3,7}$. We obtain the set $\mathcal{M}_2 = \{M_{1,3}; M_{2,6}; M_{3,7}; M_{4,1}; M_{5,4}; M_{6,5}; M_{7,8}; M_{8,2}\}$. The graph of the eight mappings is described in Table 4 and the corresponding multiplication $\cdot$ in Table 5.

8	4	7	2	6	5	3	1
7	5	3	8	2	6	1	4
6	3	2	5	4	1	8	7
5	7	6	4	1	3	2	8
4	8	5	1	3	7	6	2
3	6	1	7	8	2	4	5
2	1	8	6	5	4	7	3
1	2	4	3	7	8	5	6

0

Tab. 4

·							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	6	7	1	4	5	8	2
4	8	1	5	6	2	3	7
5	7	4	6	2	8	1	3
6	3	5	2	8	7	4	1
7	5	8	3	1	4	2	6
8	4	2	7	3	1	6	5

Tab. 5

$(S, +, \cdot)$ is a Galois field GF(9) deduced from GF(3) using the irreducible polynomial $\xi^2 + \xi + 2$ by denotations of § 1.

3. Also in the third case we have a situation similar to both preceding ones. The cyclic group of order 8 can be generated by the matrix $M_{3,4}$. We get the set $\mathfrak{M}_3 = \{M_{1,3}; M_{2,6}; M_{3,4}; M_{4,7}; M_{5,1}; M_{6,8}; M_{7,2}; M_{8,5}\}$. The corresponding tables of mappings and of the multiplication $\cdot$ are presented im Tables 6 - 7.

	8	4	6	2	5	3	7	1
	7	5	4	3	2	8	1	6
	6	3	2	7	8	1	4	5
	5	7	8	6	1	4	2	3
	4	8	3	1	7	6	5	2
	3	6	1	5	4	2	8	7
	2	1	7	4	6	5	3	8
	1	2	5	8	3	7	6	4
0								

Tab. 6

$\cdot$							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	6	4	7	1	8	2	5
4	8	7	2	3	5	6	1
5	7	1	3	8	2	4	6
6	3	8	5	2	4	1	7
7	5	2	6	4	1	8	3
8	4	5	1	6	7	3	2

Tab. 7

$(S, +, \cdot)$ is a Galois field GF(9) derived from GF(3) using the irreducible quadratic polynomial $\xi^2 + 2\xi + 2$ by denotations from § 1. The fields from §§ 1 - 3 are mutually isomorphic. If we write the isomorphism as product of cycles, we have $\sigma = (354)(678)$, $\sigma^2 = (345)(687)$ as it can be easily verified. If we want to work with GF(9) it is convenient to use concrete tables for both field operations. We shall prefer Tables 1 and 3.

Remark that the natural ordering of the set $\mathfrak{M}_k$ $(k = 1,2,3)$ by first indices of matrices of $\mathfrak{M}_k$ the second indices become the ordering given by the third row of the multiplication table. This is a consequence of our convention and of operating the matrix on the vector $(0,1) = \underline{3}$.

4. If we take the matrices $M_{1,3}$, $M_{2,6}$ together with all the matrices of $\mathfrak{M}_{II}$, we can see that they form a group with respect to matrix multiplication that is isomorphic to the

quaternion group. The corresponding mappings and operations are described in Tables 8 - 9.

8	4	7	2	3	5	6	1
7	5	4	6	2	8	1	3
6	3	2	5	8	1	4	7
5	7	8	3	1	4	2	6
4	8	5	1	6	7	3	2
3	6	1	7	4	2	8	5
2	1	3	4	5	6	7	8
1	2	6	8	7	3	5	4

$\underline{0}$

Tab. 8

$\cdot$							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	6	2	7	4	1	8	5
4	8	5	2	6	7	3	1
5	7	8	3	2	4	1	6
6	3	1	5	8	2	4	7
7	5	4	6	1	8	2	3
8	4	7	1	3	5	6	2

Tab. 9

$(S, +, \cdot)$ is a nearfield of order 9 satisfying only the right distributivity law

$$(x + y)\, z = x\, z + y\, z \qquad \forall x, y, z \in S .$$

5. If we choose pairs of matrices of group order 8 from $\mathcal{M}_1$, $\mathcal{M}_2$, resp. $\mathcal{M}_3$ we obtain two sets $\mathcal{M}_5 = \{M_{1,3}; M_{2,6}; M_{3,7}; M_{4,5}; M_{5,1}; M_{6,8}; M_{7,4}; M_{8,2}\}$, $\mathcal{M}_6 = \{M_{1,3}; M_{2,6}; M_{3,4}; M_{4,1}; M_{5,8}; M_{6,5}; M_{7,2}; M_{8,7}\}$. These sets form neither groups nor quasigroups with respect to matrix multiplication because the product of two dispersing matrices need not be a dispersing matrix, e. g. $M_{3,7} \cdot M_{5,1} = M_{4,3}$ and this matrix preserves the vectors $(0,1)$, $(0,2)$. We verify easily that $M_{2,6} \cdot \mathcal{M}_5 = \mathcal{M}_6$. Both sets of matrices operate on the set of all non - zero vectors of $V_2(3)$ strictly transitively and the corresponding tables are: Tables 10 - 11 for $\mathcal{M}_5$ and Tables 12 - 13 for $\mathcal{M}_6$. In both cases $(S, +, \cdot)$ is a right quasifield satisfying only the right distributivity law $(x + y)\, z = x\, z + y\, z$ $\forall x, y, z \in S$. In the well-known Appendix of the article [1] we find the denotations for multiplication tables $R \cdot$ (our Table 9), $T \cdot$ (our Table 11) and $S \cdot$ (our Table 13). The corresponding quasifields are the "first" examples of Hall quasifields. They can be obtained from $GF(3)$ so that the addition + is defined as

in Table 1 whereas the multiplication $\circ$ is defined as follows:

$(a,b)\circ(c,d) = (ac,bc)$ for $d = 0$,

$(a,b)\circ(c,d) = (ac - bd^{-1}f(c), ad - bc + br)$ for $d \neq 0$.

$f(\xi)$ is one of three irreducible quadratic polynomials $\xi^2 - r\xi - s$. In GF(3) it holds in addition $d^{-1} = d$ for $d \neq 0$.

8	4	6	2	5	7	3	1
7	5	3	8	2	4	1	6
6	3	2	4	7	1	8	5
5	7	4	6	1	3	2	8
4	8	7	1	3	6	5	2
3	6	1	5	8	2	7	4
2	1	8	7	6	5	4	3
1	2	5	3	4	8	6	7

$\underline{0}$

Tab. 10

$\circ$							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	6	7	5	1	8	4	2
4	8	1	6	3	5	2	7
5	7	4	1	8	2	6	3
6	3	5	7	2	4	8	1
7	5	8	2	4	1	3	6
8	4	2	3	6	7	1	5

Tab. 11

8	4	5	2	6	3	7	1
7	5	8	3	2	6	1	4
6	3	2	7	4	1	5	8
5	7	6	4	1	8	2	3
4	8	3	1	7	5	6	2
3	6	1	8	5	2	4	7
2	1	7	6	8	4	3	5
1	2	4	5	3	7	8	6

$\underline{0}$

Tab. 12

$\circ$							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	6	4	1	8	5	2	7
4	8	7	5	1	2	6	3
5	7	1	6	3	8	4	2
6	3	8	2	4	7	1	5
7	5	2	3	6	4	8	1
8	4	5	7	2	1	3	6

Tab. 13

6. Further it is possible to find sets containing besides the unit matrix still seven matrices. The first and second of them are of group order 8, the third has group order 4. These three matrices are taken always from one of the sets $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$. The remaining four matrices are from $\mathfrak{M}_{II}$; they are dispersing and with group order 6. So we obtain the following six sets:

$$\mathfrak{M}_7 = \{M_{1,3};\ M_{2,7};\ M_{3,5};\ M_{4,2};\ M_{5,8};\ M_{6,1};\ M_{7,4};\ M_{8,6}\},$$
$$\mathfrak{M}_8 = \{M_{1,3};\ M_{2,7};\ M_{3,5};\ M_{4,1};\ M_{5,6};\ M_{6,4};\ M_{7,8};\ M_{8,2}\},$$
$$\mathfrak{M}_9 = \{M_{1,3};\ M_{2,7};\ M_{3,4};\ M_{4,2};\ M_{5,6};\ M_{6,8};\ M_{7,1};\ M_{8,5}\},$$
$$\mathfrak{M}_{10} = \{M_{1,3};\ M_{2,8};\ M_{3,2};\ M_{4,5};\ M_{5,6};\ M_{6,4};\ M_{7,1};\ M_{8,7}\},$$
$$\mathfrak{M}_{11} = \{M_{1,3};\ M_{2,8};\ M_{3,7};\ M_{4,2};\ M_{5,4};\ M_{6,5};\ M_{7,1};\ M_{8,6}\},$$
$$\mathfrak{M}_{12} = \{M_{1,3};\ M_{2,8};\ M_{3,5};\ M_{4,7};\ M_{5,1};\ M_{6,4};\ M_{7,2};\ M_{8,6}\}.$$

The corresponding tables are:

a/ for $\mathfrak{M}_7$:

8	4	5	3	2	7	6	1
7	5	2	6	4	3	1	8
6	3	8	2	7	1	5	4
5	7	3	8	1	2	4	6
4	8	7	1	6	5	2	3
3	6	1	4	5	8	7	2
2	1	4	7	3	6	8	5
1	2	6	5	8	4	3	7

<u>0</u>

Tab. 14

•							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	7	5	2	8	1	4	6
4	6	8	3	1	7	2	5
5	8	2	7	3	4	6	1
6	5	7	1	4	2	8	3
7	4	1	5	6	8	3	2
8	3	4	6	2	5	1	7

Tab. 15

b/ for $\mathfrak{M}_8$:

8	4	7	3	2	6	5	1
7	5	2	8	6	3	1	4
6	3	5	2	4	1	8	7
5	7	3	4	1	2	6	8
4	8	6	1	5	7	2	3
3	6	1	7	8	5	4	2
2	1	8	5	3	4	7	6
1	2	4	6	7	8	3	5

<u>0</u>

Tab. 16

•							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	7	5	1	6	4	8	2
4	6	8	5	2	1	3	7
5	8	2	6	4	7	1	3
6	5	7	2	3	8	4	1
7	4	1	3	8	5	2	6
8	3	4	7	1	2	6	5

Tab. 17

c/ for $\mathfrak{M}_9$:

8	4	6	7	2	3	5	1
7	5	2	3	4	8	1	6
6	3	5	2	8	1	7	4
5	7	8	6	1	2	4	3
4	8	3	1	5	6	2	7
3	6	1	4	7	5	8	2
2	1	4	5	6	7	3	8
1	2	7	8	3	4	6	5

0

Tab. 18

•							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	7	4	2	6	8	1	5
4	6	7	3	2	5	8	1
5	8	1	7	4	2	3	6
6	5	8	1	3	4	2	7
7	4	2	5	8	1	6	3
8	3	5	6	1	7	4	2

Tab. 19

d/ for $\mathfrak{M}_{10}$:

8	4	2	7	3	6	5	1
7	5	8	2	6	4	1	3
6	3	5	4	2	1	7	8
5	7	4	3	1	8	6	2
4	8	6	1	5	2	3	7
3	6	1	8	7	5	2	4
2	1	3	5	8	7	4	6
1	2	7	6	4	3	8	5

0

Tab. 20

•							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	8	2	5	6	4	1	7
4	7	5	6	2	1	8	3
5	6	8	1	4	7	3	2
6	4	1	7	3	8	2	5
7	3	4	2	8	5	6	1
8	5	7	3	1	2	4	6

Tab. 21

e/ for $\mathfrak{M}_{11}$:

8	4	2	7	6	5	3	1
7	5	3	2	4	6	1	8
6	3	8	5	2	1	7	4
5	7	6	8	1	3	4	2
4	8	5	1	3	2	6	7
3	6	1	4	7	8	2	5
2	1	4	6	5	7	8	3
1	2	7	3	8	4	5	6

0

Tab. 22

•							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	8	7	2	4	5	1	6
4	7	1	3	6	2	8	5
5	6	4	7	2	8	3	1
6	4	5	1	8	7	2	3
7	3	8	5	1	4	6	2
8	5	2	6	3	1	4	7

Tab. 23

f/ for $\mathfrak{M}_{12}$:

8	4	2	3	5	6	7	1
7	5	4	2	6	3	1	8
6	3	8	7	2	1	4	5
5	7	3	8	1	4	6	2
4	8	6	1	7	2	5	3
3	6	1	5	4	8	2	7
2	1	7	4	3	5	8	6
1	2	5	6	8	7	3	4

$\underline{0}$

Tab. 24

•							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	8	5	7	1	4	2	6
4	7	8	2	3	1	6	5
5	6	2	3	8	7	4	1
6	4	7	5	2	8	1	3
7	3	1	6	4	5	8	2
8	5	4	1	6	2	3	7

Tab. 25

We have deduced multiplication tables for six quasifields. These quasifields are mutually isomorphic. If we take as a starting quasifield e. g. that of table 17 (i. e. the Hall's denotation U.) then the corresponding isomorphisms are given by the following permutations of the set $\{3, 4, 5, 6, 7, 8\} = GF(9) \setminus GF(3)$: $\tau_2 = (354)(678)$, $\tau_3 = (345)(687)$, $\tau_4 = (36)(47)(58)$, $\tau_5 = (37)(48)(56)$ and $\tau_6 = (38)(46)(57)$. All the isomorphisms of these quasifields with kernels different from GF(3) form a group isomorphic with the symmetric group S_3.

7. It is possible to find also such sets which contain also non - dispersing matrices, for instance $\mathfrak{M}_{13} = \{M_{1,4}; M_{2,7}; M_{3,5}; M_{4,6}; M_{5,2}; M_{6,1}; M_{7,8}; M_{8,3}\}$. The corresponding tables are Tables 26 - 27.

Final remarks: The operation from Table 27 is only a quasigroup one and not a loop one. As it is easily seen $(S, +, \cdot)$ is a right quasifield. Table 27 is isotopic with Table 21: $(\varphi, \psi, \chi) = ((345)(687), id, id)$. We can verify that the natural ordering of the set $\mathfrak{M}_{\varkappa}$ by the first indices of the matrices $M_{i,j} \in \mathfrak{M}_{\varkappa}$ induces a "dispersing" order of second indices given by some of 3^{rd} to 8^{th} rows of multiplication tables "2k + 1"; k = 1, 2, ..., 12.

	8	4	7	3	2	1	6	5
	7	5	2	6	8	3	4	1
	6	3	4	2	5	8	1	7
	5	7	3	1	4	2	8	6
	4	8	1	5	6	7	2	3
	3	6	8	7	1	4	5	2
	2	1	5	8	3	6	7	4
	1	2	6	4	7	5	3	8
o								

Tab. 26

•							
1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
4	7	5	6	2	1	8	3
5	6	8	1	4	7	3	2
3	8	2	5	6	4	1	7
8	5	7	3	1	2	4	6
6	4	1	7	3	8	2	5
7	3	4	2	8	5	6	1

Tab. 27

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