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SOME PROPERTIES of $C_p(X)$

Vesko VALOV

Abstract: There is found out a necessary and sufficient condition on a space X for $C_p(X)$ to have the following property (τ): any family of G_τ-subsets of $C_p(X)$ contains a dense subfamily of cardinality $\leq \tau$.

Key words: $C_p(X)$, property (τ), property $(\tau)^*$.

Classification: 54C35, 54C30

Introduction. B. Efimov [5] proved that every dyadic space X has the following property (referred below as "the property (τ)"): any family $\mathcal{F}$ of G_τ-subsets of X contains a subfamily $\mathcal{H}$ of cardinality $|\mathcal{H}| \leq \tau$ such that the set $\cup\{F:F\in\mathcal{H}\}$ is dense in $\cup\{F:F\in\mathcal{F}\}$ (such a subfamily will be called a sense subfamily). Here τ is an infinite cardinal .This result was generalized by E. Ščepin [8] to the class of all continuous images of k-metrizable compacta. In [10] we consider an extension of the class of all k-metrizable compact spaces, namely the class of all completely regular spaces with lattices of d-open [9] mappings. Every element X of the latter class has the following property: any family $\mathcal{F}$ of subsets of X with $\chi(F,X)\leq\tau$ for every $F\in\mathcal{F}$ contains a dense subfamily $\mathcal{H}$ of cardinality $|\mathcal{H}|\leq\tau$. The following question arises in connection with the last fact: Does every space with a lattice of d-open mappings have the property (τ) ?

In the present paper we answer this question in negative by finding a necessary and sufficient condition on a space X for $C_p(X)$ to have the property (τ) (let us note that $C_p(X)$ always has a lattice of d-open mappings [11]). We also give a negative answer to the following problem of A. Arhangel'skii ([1],problem 7.26): Is it true that there exists a continuous injection from X into an Eberlein compactum provided $C_p(X)$ contains a dense

σ-compact space?

Notations. We shall consider only completely regular spaces and continuous mappings.

Given spaces X and Y we denote by $C_p(X.Y)$ the set of all mappings from X to Y, endowed with the topology of pointwise convergence. If $A\subset X$ then p_A will denote the mapping from $C_p(X,Y)$ to $C_p(A,Y)$ defined by $p_A(f)=f|A$ for every $f\in C_p(X,Y)$. $C_p(X)$ stands for $C_p(X,R)$, where R is the real line with the usual topology. Let f be a mapping from X to Z. By f^* is denoted the dual mapping from $C_p(Z,Y)$ to $C_p(X,Y)$.

A space X is said to be τ-Lindelöf if every open covering of X contains a subcovering of cardinality $\le\tau$. We write $hl(X)\le\le\tau$ if X is hereditarily τ-Lindelöf. By $d(X)$ is denoted the density of X and by $hd(X)$ the hereditary density of X.

If $\mathcal{F}$ is a family of subsets of X then $\cup\mathcal{F}$ denotes the set $\cup\{F:F\in\mathcal{F}\}$.

The property (τ) and related properties of $C_p(X)$. We need the following result which is proved by P. Zenor [13].

Lemma 1. For a space X and an infinite cardinal τ the following are equivalent:

(i) $hl(X^{\omega_0})\le\tau$ ($hd(X^{\omega_0})\le\tau$, respectively);

(ii) $hd(C_p(X,Y)\le\tau$ ($hl(C_p(X,Y)\le\tau$, respectively) for every space Y with $w(Y)\le\tau$.

Remark 1. Let a set $M\subset C_p(X)$ separate the points from the closed subsets of X i.e. for every closed P in X and every $x\notin P$ there exists $f\in M$ with $f(x)=0$ and $f(P)=1$. Suppose $hd(M)\le\tau$. Using the Veličko arguments ([12], the proof of Theorem 1) one can show that $hl(X^n)\le\tau$ for each $n\in N$.

Lemma 2. Let $\mathcal{K}=\cup\{\mathcal{K}_n:n\in N\}$, where $\mathcal{K}_n$ are families of subsets of $C_p(X,R^\tau)$. Suppose $\{A(n):n\in N\}$ is an increasing sequene of closed subsets of X such that $p^{-1}_{A(n+1)}(p_{A(n+1)}(F))=F$ for every $F\in\mathcal{K}_n$ and every $n\in N$. Then $f\in\overline{\cup\mathcal{K}}$ provided $p_{A(n)}(f)\in\overline{p_{A(n)}(\cup\mathcal{K}_n)}$ for each $n\in N$.

Proof. Put $A=\bigcup\{A(n):n\in N\}$. Let $U=\{f\in C_p(X,R^\tau):f(x_i)\in V_i, i=1,\dots,k\}$ be a neighbourhood of f in $C_p(X,R^\tau)$. We can suppose that A meets $\{x_i:i=1,\dots,k\}$. Then $U=U_1\cap U_2$, where $U_1=\{f\in C_p(X,R^\tau): :f(x_i)\in V_i$ for each $x_i\in A\}$ and $U_2=\{f\in C_p(X,R^\tau):f(x_i)\in V_i$ for each $x_i\notin A\}$. Choose m such that $A\cap\{x_i:i=1,\dots,k\}\subset A(m)$. Since $p_{A(m)}(U)$ is open in $p_{A(m)}(C_p(X,R^\tau))$ we have $p_{A(m)}(U)\cap \cap p_{A(m)}(\bigcup\mathcal{K}_m)\neq\emptyset$. Hence $p_{A(m)}(F_0)\cap p_{A(m)}(U)\neq\emptyset$ for some $F_0\in\mathcal{K}_m$. Then $U_1\cap F_0\neq\emptyset$. Let $g_1\in U_1\cap F_0$. There exists $g_2\in U_2$ such that $g_2|A(m+1)=g_1|A(m+1)$. Thus, $g_2\in F_0\cap U$. Therefore $f\in\overline{\bigcup\mathcal{K}}$.

Let τ be an infinite cardinal. A space X has the property $(\tau)^*$ if $hl(A^n)\leq\tau$ for every closed A in X with $d(A)\leq\tau$ and every $n\in N$.

Theorem 1. For a space X and an infinite cardinal τ the following are equivalent:

(i) X has the property $(\tau)^*$;

(ii) $C_p(X,R^\lambda)$ has the property (τ) for every $\lambda\leq\tau$;

(iii) $C_p(X,R)$ has the property (τ).

Proof. (i) $\longrightarrow$ (ii). Let $\mathcal{F}$ be a family of G_τ-subsets of $C_p(X,R^\lambda)$, where $\lambda\leq\tau$. We can suppose that for every $F\in\mathcal{F}$ there exists a set $A(F)\subset X$ of cardinality $\leq\tau$ with $p^{-1}_{A(F)}(p_{A(f)}(F))=F$.

For every $n\in N$ we construct a set $B(n)\subset X$ and a subfamily $\mathcal{K}_n\subset\mathcal{F}$ such that:

1) $B(n)\subset B(n+1)$, $|B(n)|\leq\tau$ and $|\mathcal{K}_n|\leq\tau$;

2) $p^{-1}_{A(n+1)}(p_{A(n+1)}(F))=F$ for every $F\in\mathcal{K}_n$, where $A(k)=\overline{B(k)}$ for $k\in N$;

3) $p_{A(n)}(\bigcup\mathcal{K}_n)$ is dense in $p_{A(n)}(\bigcup\mathcal{F})$.

Assume we have already defined $B(n)$ and $\mathcal{K}_n$ satisfying 1) - - 3). Put $B(n+1)=B(n)\cup(\bigcup\{A(F):F\in\mathcal{K}_n\})$. Obviously $|B(n+1)|\leq\tau$ and $F=p^{-1}_{A(n+1)}(p_{A(n+1)}(F))$ for every $F\in\mathcal{K}_n$. By Lemma 1 we have $hd(C_p(A(n+1),R^\lambda))\leq\tau$. So, there exists a subfamily $\mathcal{K}_{n+1}$ of $\mathcal{F}$ such that $|\mathcal{K}_{n+1}|\leq\tau$ and $p_{A(n+1)}(\bigcup\mathcal{K}_{n+1})$ is dense in $p_{A(n+1)}(\bigcup\mathcal{F})$. Denote $\mathcal{K}=\bigcup\{\mathcal{K}_n:n\in N\}$, $B=\bigcup\{B(n):n\in N\}$ and $A=\bigcup\{A(n):n\in N\}$. Let $f\in p^{-1}_A(p_A(\overline{\bigcup\mathcal{F}}))$. Since

$p_{A(n)}(f) \in \overline{p_{A(n)}(\cup\mathcal{F})} = \overline{p_{A(n)}(\cup\mathcal{K}_n)}$, it follows from Lemma 2 that $f \in \overline{\cup\mathcal{K}}$, i.e. $p_A^{-1}(p_A(\overline{\cup\mathcal{F}})) = \overline{\cup\mathcal{K}} = \overline{\cup\mathcal{F}}$. The implication is proved. Let us note that $p_B^{-1}(\overline{\cup\mathcal{F}}) = \overline{\cup\mathcal{F}}$ because B is dense in A; hence $\overline{\cup\mathcal{F}}$ is a union of G_τ-subsets of $C_p(X,R^\lambda)$.

(ii) $\longrightarrow$ (iii). This implication is obvious.

(iii) $\longrightarrow$ (i). Suppose A is a closed subset of X with $d(A) \le \tau$. Then there is an injection from $p_A(C_p(X))$ into a space of weight $\le \tau$, so that every point of $p_A(C_p(X))$ is a G_τ-set in $p_A(C_p(X))$. Hence $hd(p_A(C_p(X)) \le \tau$. It follows from Remark 1 that $hl(A^n) \le \tau$ for each $n \in N$.

Corollary 1. Let X be a monolithic space in the sense of Arhangel'skii [2]. Then $C_p(X)$ has the property (τ) for every infinite cardinal τ.

Corollary 2. For a space X and an infinite cardinal τ the following conditions are equivalent:

(i) $C_p(X)$ has the property (τ);

(ii) $C_p(X^\lambda)$ has the property (τ) for every $\lambda \le \tau$.

Proof. In view of Theorem 1 it is enough to prove that X^λ has the property (τ) provided X does. Let A be a closed subset of X^λ with $d(A) \le \tau$. Put $B = \overline{\cup\{q_s(A) : s < \lambda\}}$, where $q_s : X^\lambda \longrightarrow X$ is the projection onto the s-th factor. Obviously, $d(B) \le \tau$, so $hl(B^n) \le \tau$ for every $n \in N$. Hence, $hl(B^\lambda) \le \tau$ (see [13], Theorem 3). Since $A \subset B^\lambda$ we have $hl(A^n) \le \tau$ for each $n \in N$.

Lemma 3. Suppose $X = \prod\{X_s : s \in S\}$, where $|S| \le \tau$. Then X has the property (τ) provided $\prod\{X_s : s \in B\}$ does for every finite set $B \subset S$.

The proof of Lemma 3 is similar to the proof of Theorem 3* from [13].

Theorem 2. Let X^n have the property (τ) for every $n \in N$. Then $C_p(X,Z)$ has the property $(\tau)^*$ for every space Z with $w(Z) \le \tau$.

Proof. Suppose A is closed in $C_p(X,Z)$ and $d(A) \le \tau$. Put $p = \Delta\{f : f \in A\}$ and $Y = p(X)$. Since $d(A) \le \tau$, there exists an injection from Y to a space of weight not greater than τ. So, every point of Y^{ω_0} is a G_τ-set in Y^{ω_0}. By Lemma 3, X^{ω_0} has the property (τ). Thus, $hd(Y^{\omega_0}) \le \tau$ because Y^{ω_0} is a continuous image of X^{ω_0}. It follows from Lemma 1 that $hl(C_p(X,Z^{\omega_0}) \le \tau$. Next, for every $f \in A$ there exists $h_f \in C_p(Y,Z)$ such that $f=h_f \circ p$. Denote $B=\{h_f : f \in A\}$. Clearly, B^{ω_0} can be considered as a subspace of $C_p(Y,Z^{\omega_0})$; hence $hl(B^{\omega_0}) \le \tau$. But $p^*(B)=A$, so $hl(A^{\omega_0}) \le \tau$.

Corollary 3. A space X has the property $(\tau)^*$ if and only if $C_p(C_p(X))$ has the property $(\tau)^*$.

Proof. Since $C_p(C_p(X))$ contains X, all we have to prove is the "only if" part. Suppose X has the property $(\tau)^*$. By Theorem 1, $C_p(X,R^{\omega_0})$ has the property (τ), so the space $C_p(X)^{\omega_0}$ does, too. The Theorem 2 completes the proof.

Combining Corollary 3 and Theorem 1 we get: $C_p(X)$ has the property (τ) if and only if $C_p(C_p(X))$ has the property $(\tau)^*$. Therefore, Theorem 2 is inversable in the case $X=C_p(Y)$ for some Y. The following example shows that this is not the case in general.

Example 1. Let X be a nonmetrizable Eberlein compactum. Then $C_p(X)$ has the property $(\omega_0)^*$ being a monolithic space (see [1]). Suppose X has the property (ω_0). Then X is separable, because X contains a dense metrizable space (ss [6]). But every separable Eberlein compactum is metrizable [3] - a contradiction.

Let τ be an infinite cardinal. A space X is said to have the property $[\tau]$ if for every family $\mathcal{F}$ of G_τ-subsets of X and every $x \in \overline{\cup\mathcal{F}}$ there is a subfamily $\mathcal{H}$ of $\mathcal{F}$ such that $x \in \overline{\cup\mathcal{H}}$ and $|\mathcal{H}| \le \tau$. X has the property $[\tau]^*$ if $l(A^n) \le \tau$ for every $n \in N$, where A is a closed subset of X with $d(A) \le \tau$.

Theorem 3. For a space X and an infinite cardinal τ the following conditions are equivalent:

(i) X has the property $[\tau]^*$;

(ii) $C_p(X,R^\lambda)$ has the property $[\tau]$ for every $\lambda \le \tau$;

(iii) $C_p(X)$ has the property $[\tau]$.

Proof. (i) $\longrightarrow$ (ii). Let $\mathcal{F}$ be a family of G_τ-subsets of

$C_p(X,R^{\lambda})$ and $f \in \overline{\cup \mathcal{F}}$. Using the same notations as in the proof of Theorem 1, for every $n \in N$ we construct a set $B(n) \subset X$ and a family $\mathcal{K}_n \subset \mathcal{F}$ such that:

1) $|B(n)| \le \tau$, $|\mathcal{K}_n| \le \tau$ and $B(n+1)=B(n) \cup (\cup\{A(F): F \in \mathcal{K}_n\})$;

2) $p^{-1}_{A(n+1)} p_{A(n+1)}(F))=F$ for every $F \in \mathcal{K}_n$;

3) $p_{A(n)}(f) \in \overline{p_{A(n)}(\cup \mathcal{K}_n)}$.

The following result (actually established by Arhangel´skii [2]) will be used below: If $l(X^n) \le \tau$ for every $n \in N$, then $t(C_p(X,R^{\lambda})) \le \tau$ where $\lambda \le \tau$.

Suppose we have defined $B(n)$ and $\mathcal{K}_n$. Put $B(n+1)=B(n) \cup (\cup\{A(F): F \in \mathcal{K}_n\})$ and $A(n+1)=\overline{B(n+1)}$. Then $l(A(n+1)^k \le \tau$ for every $k \in N$ because $d(A(n+1)) \le \tau$. By the result mentioned above (Arhangel´skii [2]), $t(C_p(A(n+1),R^{\lambda})) \le \tau$. Since $p_{A(n+1)}(f) \in \overline{p_{A(n+1)}(\cup \mathcal{F})}$, there exists a subfamily $\mathcal{K}_{n+1}$ of $\mathcal{F}$ such that $p_{A(n+1)}(f) \in \overline{p_{A(n+1)}(\cup \mathcal{K}_{n+1})}$ and $|\mathcal{K}_{n+1}| \le \tau$. Thus, $B(n)$ and $\mathcal{K}_n$ are constructed for every $n \in N$. Let $\mathcal{K} = \cup\{\mathcal{K}_n : n \in N\}$. Obviously $|\mathcal{K}| \le \tau$. It follows from Lemma 2 that $f \in \overline{\cup \mathcal{K}}$.

(ii) $\longrightarrow$ (iii). This implication is obvious.

(iii) $\longrightarrow$ (i). Suppose A is closed in X with $d(A) \le \tau$. Since every point of $p_A(C_p(X))$ is a G_{τ}-set in $p_A(C_p(X))$ and the mapping p_A is open (see [1]), we have $t(p_A(C_p(X)) \le \tau$. Observe that $p_A(C_p(X))$ separates the points from the closed sets in A. Now, the following result (actually established by E. Pytkeev [7]) completes the proof: Let $t(M) \le \tau$ and M separate the points from the closed subsets in Y, where $M \subset C_p(Y)$. Then $l(Y^n) \le \tau$ for every $n \in N$.

<u>Corollary 4</u>. For a space X and an infinite cardinal τ the following are equivalent:

(i) $C_p(X)$ has the property $[\tau]$;

(ii) $C_p(X^n)$ has the property $[\tau]$ for every $n \in N$.

Proof. In view of Theorem 3, it suffices to show that X^n has the property $[\tau]^*$ for every $n \in N$ provided X does. The last proposition can be proved by the same arguments as Corollary 2 was.

<u>Theorem 4</u>. Let τ be an infinite cardinal and $C_p(C_p(X))$ have the property $[\tau]^*$. Then $C_p(X)$ has the property $[\tau]$.

Proof. Let $f \in \overline{\bigcup \mathcal{F}}$, where $\mathcal{F}$ is a family of G_τ-subsets of $C_p(X)$. Using the same notations as in the proof of Theorem 1, we define sets $B(n) \subset X$ and subfamilies $\mathcal{K}_n$ of $\mathcal{F}$ with the following properties:

1) $B(n) \subset B(n+1)$, $|B(n)| \leq \tau$ and $|\mathcal{K}_n| \leq \tau$;

2) $p^{-1}_{A(n+1)}(p_{A(n+1)}(F))=F$ for every $F \in \mathcal{K}_n$;

3) $p_{A(n)}(f) \in \overline{p_{A(n)}(\bigcup \mathcal{K}_n)}$.

Suppose we have already constructed $B(i)$ and $\mathcal{K}_i$ for $i \leq n$. Put $B(n+1)=B(n) \cup (\bigcup\{A(F):F \in \mathcal{K}_n\})$ and $A(n+1)=\overline{B(n+1)}$. Since $d(A(n+1)) \leq \tau$, there exists a one-to-one mapping from $P(n+1)= =p_{A(n+1)}(C_p(X))$ onto a space of weight $\leq \tau$, so $d(C_p(P(n+1))) \leq \tau$. Hence, $l(C_p(P(n+1))) \leq \tau$ because $C_p(P(n+1))$ is closed in $C_p(C_p(X))$. By a result of Asanov [4], we have $t(P(n+1)) \leq \tau$. Therefore, there is a subfamily $\mathcal{K}_{n+1}$ of $\mathcal{F}$ such that $p_{A(n+1)}(f) \in \overline{p_{A(n+1)}(\bigcup \mathcal{K}_{n+1})}$ and $|\mathcal{K}_{n+1}| \leq \tau$. Hence $B(n)$ and $\mathcal{K}_n$ are defined for every $n \in N$. Denote $\mathcal{K} = \bigcup\{\mathcal{K}_n : n \in N$. Obviously $|\mathcal{K}| \leq \tau$. It follows from Lemma 2 that $f \in \overline{\bigcup \mathcal{K}}$.

<u>Corollary 5</u>. A space X has the property $[\tau]^*$ provided $C_p(C_p(X))$ does.

<u>Example 2</u>. There exists a space X with the property $[\omega_0]$ but $C_p(X)$ does not have the property $[\omega_0]^*$.

Let X be a discrete space of cardinality c. Then $C_p(X)=R^c$. The space R^c does not have the property $[\omega_0]^*$ because it is separable and non-normal.

Finally, we give a negative answer to the question of Arhangel'skii ([1], problem 7.26), mentioned above. Let X be a non-metrizable Eberlein compactum. Using the arguments of Arhangel'skii ([1], the proof of Proposition 6.9) one can show that $C_p(C_p(X))$ contains a σ-compact subspace (this follows also from a more general result of V. Tkačuk). Suppose that there exists an injection j from $C_p(X)$ into an Eberlein compactum Y. Then the closure of $j(C_p(X))$ in Y is also an Eberlein compactum satisfying the countable chain condition. Hence $w(j(C_p(X))) \leq \omega_0$ (see [3]). Next, it is well known that $C_p(X)$ contains a σ-compact dense subset M. Therefore $d(M) \leq \omega_0$ because M is a countable union of compact metrizable spaces. Consequently X is metrizable - a contradiction.

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