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WEAK UNIFORM ROTUNDITY IN ORLICZ SPACES

A. KAMIŃSKA and W. KURC

Abstract: In the paper there are presented criteria for weak uniform rotundity in the sense of Smulian [13] and in the sense of Cudia [2] of Orlicz spaces in the case of atomless measure. They enable us to prove that a number of geometric properties (WUR, WLUR, WUR and WLUR in the sense of Cudia, URWC, URED, LUR, HR, E, MLUR, R) coincide for Orlicz spaces whenever reflexivity is assumed. Some results concerning these properties for general Banach spaces are also proved.

Key words: Banach spaces, Orlicz spaces, rotundity, relfexivity.

Classification: 46B20, 46B25

Introduction. In many applications of Orlicz spaces, L_φ, some geometric conditions are usually imposed. They are, for example, rotundity (R), uniform rotundity (UR), Radon-Riesz property with the rotundity (HR), weak uniform rotundity in the sense of Smulian (WUR) and in the sense of Cudia (CWUR). It is of great importance to have expressed them in terms of the Young's function φ. Regarding the uniform rotundity (UR) and the rotundity we refer to [7] and [16] for results and further bibliography. For the local uniform rotundity the basic step was done in [8] in the case of atomless measure (cf. Theorem 0.1 below). Taking into account the well known implications concerning the above mentioned properties as well as the weak local uniform rotundity (WLUR), midpoint local uniform rotundity (MLUR), uniform rotundity in every direction (URED), we get all these properties, except UR, WUR and CWUR, coincide for Orlicz spaces in the case of atomless measure. Moreover, they also coincide with the weak local uniform rotundity in the sense of Cudia and the SE property because LUR implies CWLUR and the SE property lies between CWLUR and R. Let us explain that the SE property means that the unit sphere consists of only strongly exposed points (see [11]).

In this paper it is further proved that the stronger properties CWUR and WUR coincide for Orlicz spaces under consideration and that both of them imply the reflexivity of this space. As a consequence it follows that for Orlicz spaces, for atomless measure, all above mentioned properties, except UR, coincide when the reflexivity is assumed. We can add the E-spaces to these equivalences as well, since they are nothing but CWUR spaces.

We shall give definitions and notations that will be used in the paper. For an arbitrary Banach space X (real) with a norm $\|\cdot\|$ let B_X denote the unit ball and let S_X denote its surface. A Banach space X is said to be WUR if for every $0 \neq x^*$, $x^* \in X^*$, and $\varepsilon > 0$ there exists $\delta(x^*, \varepsilon) > 0$ such that if $x, y \in S_X$ and $x^*(x-y) \geq \varepsilon$ then $\|\frac{x+y}{2}\| \leq 1 - \delta(x^*, \varepsilon)$ (cf. [14],[12]). We say that X is CWUR if for every $x^* \in S_{X^*}$, $\varepsilon > 0$ there exists $\delta(x^*, \varepsilon) > 0$ such that if $x, y \in S_X$ and $\|x-y\| \geq \varepsilon$ then $|x^*(\frac{x+y}{2})| \leq$ $\leq 1 - \delta(x^*, \varepsilon)$ ([2]). In the above definitions, S_X can be equivalently replaced by B_X. X is said to have the Radon-Riesz property (H) if for each $x \in X$ and $(x_n) \subset X$ such that $\|x_n\| \to \|x\|$ and $x_n \to x$ weakly, we have $x_n \to x$ in the norm. If moreover X is rotund then it is called an HR space or a space with the HR-property. Following [4], a Banach space X is called an E-space (we will also say that X has the E-property) if for each $x^* \in X$ and each sequence (x_n) in S_X, $\|x_n - x_m\| \to 0$ for $n, m \to +\infty$, whenever $x^*(x_n) \to \|x^*\|$. For definitions of other geometric properties we refer to [13],[14],[2],[15],[3].

Let (T, Σ, μ) be a positive measure space with atomless measure μ. The Orlicz space L_φ is the subspace of all measurable functions $x: T \to R$ (R - real line) such that $I_\varphi(\lambda x) =$ $= \int_T \varphi(\lambda |x(t)|) d\mu < +\infty$ for some $\lambda > 0$ depending on x. Here $\varphi: R_+ \to [0, +\infty]$ is a Young's function, i.e. $\varphi(0) = 0$, φ is convex, left-continuous and not identical to infinity away from zero. The space L_φ equipped with the Luxemburg norm $\|x\|_\varphi = \inf\{\lambda > 0:$ $: I_\varphi(x/\lambda) \leq 1\}$ is a Banach space. If $A \in \Sigma$, then $L_\varphi(A)$ denotes the set $\{x \chi_A : x \in L_\varphi\}$. Let $\psi(v) = \sup_{u \geq 0}(u \cdot v - \varphi(u))$ for $v \geq 0$. This so-called complementary function to φ is again a Young's function and its complementary function is φ. If φ and ψ take only finite values we have the following relations

$$(0.1) \qquad (u-v)p(v) \le \varphi(u) \le u\,p(u)$$

for all $u,v \in R_+$, where $u \ge v$, and

$$(0.2) \qquad \begin{aligned} u\,p(u) &= \varphi(u) + \psi(p(u)), \\ v\,q(v) &= \varphi(q(v)) + \psi(v) \end{aligned}$$

for all $u,v \in R_+$, where p and q are left derivatives of φ and ψ, respectively (we put $p(0)=q(0)=0$). In the above expressions we can replace the left derivatives p and q by the right derivatives $\bar{p}$ and $\bar{q}$. Let us note that $q(v)=\inf\{s \in R_+ : p(s) \ge v\}$. Recall that a Young's function φ is said to satisfy a Δ_2-condition for all arguments (for large arguments) if φ takes only finite values and there is a constant $K>0$ (resp. there are constants K, $u_0>0$) such that $\varphi(2u) \le K\varphi(u)$ for all $u \in R_+$ (for all $u \ge u_0$). We will say simply in the following that φ satisfies a Δ_2-condition if (i) φ fulfils the Δ_2-condition for large arguments and $\mu(T)=+\infty$ or (ii) φ fulfils the Δ_2-condition for all arguments and $\mu(T)<+\infty$. Let us also note that if ψ_1 is a conjugate function to a Young's function φ_1, then $L_\varphi \subset L_{\varphi_1}$ implies $L_{\psi_1} \subset$ $\subset L_\psi$.

Let us introduce the following definition. We will say that the modular I_φ is weakly uniformly rotund (WUR) if for every $x^* \in L^*_\varphi$ $x^* \neq 0$, and $\varepsilon > 0$ there exists $\delta(x^*,\varepsilon) > 0$ such that if $I_\varphi(x)=I_\varphi(y)=1$ and $x^*(x-y) \ge \varepsilon$ then $I_\varphi\left(\frac{x+y}{2}\right) \le 1 - \delta(x^*,\varepsilon)$.

The following known result will be needed.

0.1. Theorem ([8],[9]). The following conditions are equivalent:

(i) L_φ is LUR.

(ii) L_φ is URED.

(iii) L_φ is R.

(iv) The function φ is strictly convex and satisfies the Δ_2-condition.

0.2. Theorem. The Orlicz space L_φ is reflexive if and only if both φ and its complementary function ψ satisfy the Δ_2-condition.

0.3. Lemma. If $\varphi : R_+ \longrightarrow R_+$ is strictly convex on an interval

$[0,a]$, then for each $\varepsilon > 0$, $d_1, d_2 \in (0,a]$ $(d_1 < d_2)$ there is $p \in (0,1)$ such that

$$\varphi\left(\left|\frac{u+v}{2}\right|\right) \leq (1-p)\, \frac{\varphi(|u|)+\varphi(|v|)}{2}$$

whenever $|u-v| \geq \varepsilon(|u| \vee |v|)$ and $|u| \vee |v| \in [d_1, d_2]$.

1. Weak uniform rotundity in the sense of Cudia (CWUR). The following two theorems are known (cf. [2] pp.296 and 310) in a somewhat implicit form. We complete them with direct proofs.

1.1. Theorem. A Banach space is CWUR if and only if it is an E-space.

Proof. Let a Banach space X be CWUR and let us assume for a moment that X is not an E-space. Then $x^*(x_n) \longrightarrow 1$ and $\|x_{n_i} - x_{n_j}\| \geq a$ for some $x^* \in S_{X^*}$, a sequence $(x_n) \subset S_X$, $a > 0$ and a subsequence (n_i). From the definition of CWUR we get a contradiction. Conversely, let now X be an E-space which is not CWUR. Then there exist $x^* \in S_X$, $\varepsilon > 0$ and sequences (x_n), (y_n) in S_X such that $x^*\left(\frac{x_n+y_n}{2}\right) \longrightarrow 1$ and $\|x_n - y_n\| \geq \varepsilon$. Since $x^*\left(\frac{x_n+y_n}{2}\right) \leq \max\{x^*(x_n), x^*(y_n)\} \leq 1$, then by passing to a subsequence we get $x^*(x_{n_i}) \longrightarrow 1$ and $x^*(y_{n_i}) \longrightarrow 1$. Hence both these sequences tend to some x and y, respectively, in S_X since X is complete. On the other hand $\left|x^*\left(\frac{x_n+y_n}{2}\right)\right| \leq \left\|\frac{x_n+y_n}{2}\right\| \leq 1$. Therefore $\frac{x+y}{2} \in S_X$. We prove that x=y by defining a sequence (z_n) as follows. For n odd we put $z_n = (1 - \frac{1}{2n})x + \frac{1}{2n}y$ and for n even we put $z_n = (1-\frac{1}{2n})y + \frac{1}{2n}x$. We have $\|z_n\| \nearrow 1$ and $x^*(z_n) = 1$. Since X is an E-space we conclude that $\|z_n - z_m\| \longrightarrow 0$ when $n,m \longrightarrow +\infty$. On the other hand, $\|z_n - z_{n+1}\| = (1-\frac{1}{n})\|x-y\|$ - a contradiction if one assumes that $x \neq y$. Thus x=y. Next, we have $\|x_{n_i} - y_{n_i}\| \leq \|x_{n_i} - x\| + \|x - y_{n_i}\|$ and $\|x_n - y_n\| \geq \varepsilon$. This contradiction ends the proof.

1.2. Theorem. If a Banach space is weakly uniformly rotund in the sense of Cudia (CWUR) then X is reflexive.

Proof. We shall prove that each functional $x^* \in X^*$ achieves its norm on S_X. Recall that by James theorem this is equivalent to the reflexivity of X. For any $x^*\in S_{X^*}$ let $(x_n)\subset S_X$ be such that $1=\lim_{m\to+\infty} x^*(x_n)$. First, suppose (x_n) is not a Cauchy sequence. Then $\|x_{n_i}-x_{m_i}\|\geq a$ for some subsequences (n_i), (m_i) and some $a>0$. Since X is CWUR there exists $\delta>0$ such that $\left|x^*\left(\frac{x_{n_i}+x_{m_i}}{2}\right)\right|\leq 1-\delta$ for all i. The left-hand side tends to 1, so we get a contradiction. Thus (x_n) must be a Cauchy sequence, so $x_n\to x$ for some $x\in S_X$. It follows that $1=x^*(x)$, as desired.

1.3. Theorem. If X is CWUR (equivalently, X is an E-space), then it is a HR-space. If X is reflexive, the converse is also true.

Proof. That CWUR implies HR is an easy consequence of the definitions. Let now X be an HR-space and assume for the contrary that there exist $x^*\in S_{X^*}$, $\varepsilon>0$ and sequences (x_n), (y_n) in S_X such that $\|x_n-y_n\|\geq\varepsilon$ and $1-\frac{1}{n}\leq\left|x^*\left(\frac{x_n+y_n}{2}\right)\right|\leq 1$. Since X is reflexive, B_X is weakly sequentially compact. So, there is a subsequence (n_k) and $x,y\in B_X$ such that $x_{n_k}\to x$ and $y_{n_k}\to y$ weakly. Therefore $1=\left|x^*\left(\frac{x+y}{2}\right)\right|\leq\left\|\frac{x+y}{2}\right\|\leq\frac{\|x\|+\|y\|}{2}\leq 1$, and hence $x=y$ since X is rotund. On the other hand, the Radon-Riesz property H implies that $\|x_{n_k}-y_{n_k}\|\to 0$ while $\|x_n-y_n\|\geq\varepsilon$. This contradiction finishes the proof.

1.4. Theorem. The Orlicz space L_φ is weakly uniformly rotund in the sense of Cudia if and only if it is rotund and reflexive.

Proof. If L_φ is CWUR then itis rotund. It is a simple consequence of the definitions. From Theorem 1.2 it follows that L_φ is reflexive. On the other hand if L_φ is rotund then it is LUR, by Theorem 0.1. So it has the property HR ([13]). If L_φ is also reflexive, then it is CWUR, by Theorem 1.3.

2. Weak uniform rotundity (WUR).

For the proof of the main theorem in this section, a number of auxiliary facts is needed.

2.1. Lemma. If an arbitrary Banach space X contains an isomorphic copy of l_1 then X is not WUR.

Proof. Let $X_0 \subset X$ be isomorphic to l_1. The space X_0 with the induced norm cannot be uniformly rotund because it is not reflexive. So there exist sequences (x_n), (y_n) in X_0 and $\varepsilon > 0$ such that

$$(2.1) \qquad \|x_n\| = \|y_n\| = 1, \quad \|x_n - y_n\| \geq \varepsilon \;, \left\|\frac{x_n+y_n}{2}\right\| \longrightarrow 1.$$

In the space l_1, weak and strong convergences are equivalent. So X_0 has this property. Since $\|x_n - y_n\| \geq \varepsilon$, there exists $x^* \in X_0^*$ such that $x^*(x_n - y_n) \not\longrightarrow 0$. We extend x^* to the whole of X and find an increasing sequence (n_k) of natural numbers and a constant $\varepsilon_1 > 0$ such that $x^*(x_{n_k} - y_{n_k}) \geq \varepsilon_1$, taking the functional x^* with the opposite sign if necessary. Hence and by 2.1 it is seen that X is not weakly uniformly rotund.

2.2. Lemma. The space $L_1(T)$ contains an isometric copy of l_1. We omit the simple proof. Let us note only that we can produce a sequence of pairwise disjoint sets of finite and positive measure since the measure is atomless.

The proof of the following lemma is similar to the proof of Lemma 1 in [7] and is therefore also omitted.

2.3. Lemma. The space L_φ is WUR if and only if the modular I_φ is WUR and φ satisfies the Δ_2-condition.

2.4. Lemma. If the complementary function to φ satisfies the Δ_2-condition, then $\frac{\varphi(u)}{u} \longrightarrow +\infty$ if $u \longrightarrow +\infty$.

Proof. Since the function $\frac{\varphi(u)}{u}$ is nondecreasing for $u > 0$, we can replace in the thesis "lim" by "sup". Let $\sup_u \frac{\varphi(u)}{u} \leq \leq a < +\infty$. Then $\psi(v) = \sup_{u \geq 0} (u\,v - \varphi(u)) = +\infty$ - contradiction with the Δ_2-condition assumed for ψ

2.5. Lemma. If $\mu(T) < +\infty$ and $\frac{\varphi(u)}{u} \longrightarrow +\infty$ when $u \longrightarrow +\infty$, for $\varphi: R_+ \longrightarrow R_+$, then for each $\varepsilon > 0$ there exist constants $a,b > 0$ such that if $\int_T |x(t)-y(t)|\,d\mu \geq \varepsilon$ and $I_\varphi(x) = I_\varphi(y) = 1$, then $\int_A |x(t)-y(t)|\,d\mu \geq \frac{\varepsilon}{4}$, where $A = \{t \in T : a \leq |x(t)| \vee |y(t)| \leq b\}$.

Proof. We have $\varphi(u) \geq n \cdot u$ for all $u \geq u_n$, $n \in N$, and some sequence (u_n). Let m be a natural number such that $\varepsilon - \frac{6}{m} \geq \frac{\varepsilon}{2}$. Let us put $A_1 = \{t \in T : |x(t)| \leq u_m\}$, $A_2 = \{t \in T : |y(t)| \leq u_m\}$, $A_3 = \{t \in T : \frac{\varepsilon}{8\,\mu(T)} \leq |x(t)| \vee |y(t)|\}$. Then $\int_{T\setminus A_1} |x(t)|\,d\mu \leq \leq \frac{1}{m} \int_{T\setminus A_1} \varphi(|x(t)|)\,d\mu \leq \frac{1}{m}$ and $\int_{T\setminus A_2} |y(t)|\,d\mu \leq \frac{1}{m}$. Moreover, we have

$$\int_{T\setminus A_1}|y(t)|d\mu = \int_{(T\setminus A_1)\cap(T\setminus A_2)}|y(t)|d\mu + \int_{A_2\setminus A_1}|y(t)|d\mu \le$$

$$\int_{T\setminus A_2}|y(t)|d\mu + \int_{T\setminus A_1}|x(t)|d\mu \le \frac{2}{m},$$

by the definition of the sets A_1 and A_2. Similarly, $\int_{T\setminus A_2}|x(t)|d\mu \le \frac{2}{m}$. Hence

$$\int_{T\setminus(A_1\cap A_2)}|x(t)-y(t)|d\mu \le \int_{T\setminus A_1}|x(t)|d\mu + \int_{T\setminus A_1}|y(t)|d\mu +$$

$$+\int_{T\setminus A_2}|x(t)|d\mu + \int_{T\setminus A_2}|y(t)|d\mu \le \frac{6}{m}.$$

Since we have $\int_T |x(t)-y(t)|d\mu \ge \varepsilon$ by assumption, so

$$\int_{A_1\cap A_2}|x(t)-y(t)|d\mu \ge \varepsilon - \frac{6}{m} \ge \frac{\varepsilon}{2}.$$

If $t\notin A_3$ then $|x(t)|\le \frac{\varepsilon}{8\mu(T)}$ and $|y(t)|\le\frac{\varepsilon}{8\mu(T)}$. Then we get

$$\int_{(A_1\cap A_2)\setminus A_3}|x(t)-y(t)|d\mu \le \frac{\varepsilon}{8\mu(T)}\mu(T\setminus A_3)+\frac{\varepsilon}{8\mu(T)}\mu(T\setminus A_3)\le\frac{\varepsilon}{4}.$$

By putting $A=A_1\cap A_2\cap A_3$ and $a=\frac{\varepsilon}{8\mu(T)}$, $b=u_m$, we get the desired inequality $\int_A |x(t)-y(t)|d\mu \ge \frac{\varepsilon}{2}-\frac{\varepsilon}{4}=\frac{\varepsilon}{4}$.

2.6. Lemma. Let φ satisfy a Δ_2-condition and let $B\in\Sigma$, $\varepsilon>0$ and $p\in(0,1)$ be such that

$$I_\varphi((x-y)\chi_B)\ge\varepsilon \text{ and } I_\varphi\left(\frac{x+y}{2}\right)\le 1-\frac{p}{2}\,(I_\varphi(x\chi_B)+I_\varphi(y\chi_B))$$

where x, y are arbitrary measurable functions with $I_\varphi(x)=I_\varphi(y)=1$. Then there exists a constant $q\in(0,1)$ (more precisely $q=\frac{p}{2k}$) such that $I_\varphi\left(\frac{x+y}{2}\right)\le 1-q$.

Proof. By the Δ_2-condition there exist constants $c,k>0$ such that $\varphi(2c)\mu(T)<\frac{\varepsilon}{2}$ (we put $0\cdot\infty=0$ if $c=0$ and $\mu(T)=+\infty$) and $\varphi(2u)\le k\varphi(u)+\varphi(2c)$ for each $u\in R_+$. Then

$$\varepsilon\le I_\varphi((x-y)\chi_B)\le\frac{k}{2}(I_\varphi(x\chi_B)+I_\varphi(y\chi_B))+\varphi(2c)\mu(T)\le$$

$$\le\frac{k}{2}(I_\varphi(x\chi_B)+I_\varphi(y\chi_B))+\frac{\varepsilon}{2}.$$

Hence and by the assumption of our lemma we get $I_\varphi\left(\frac{x+y}{2}\right)\le 1-\frac{p}{2k}$.

2.7. Lemma. If ψ takes only finite values and does not fulfil a Δ_2-condition, then there exist sequences $(u_n)\subset(0,+\infty)$, $(b_n)\subset(0,1)$ such that $b_n\downarrow 0$ and

$$(2.2)\qquad \psi((1+b_n)u_n) > 2^n\,\psi(u_n)$$

for each $n\in N$ and

$$(2.3)\qquad \lim_{m\to+\infty}\frac{\varphi(la_n)}{\varphi(a_n)} = 1$$

for each $l\in(0,1)$, where $a_n=q((1+b_n)u_n)$. If ψ does not satisfy a Δ_2-condition for large arguments, then $u_n\uparrow+\infty$.

Proof. For arbitrary sequences $(b_m),(c_m)\subset(0,+\infty)$ such that $b_m\downarrow 0$, $c_m\downarrow 0$ and $c_m<b_m$ there exists a sequence $(u_m)\subset(0,+\infty)$ such that

$\psi((1+b_m)u_m)>2^m\psi((1+c_m)u_m)$ because ψ does not fulfil the Δ_2-condition. It is evident that we may choose (u_m) in such a way that $u_m\uparrow+\infty$ whenever ψ does not satisfy a Δ_2-condition for large arguments. By applying inequalities (0.1) we obtain $q((1+b_m)u_m)(1+b_m)u_m> 2^m\,q(u_m)c_m\,u_m$. Now, putting $c_m=\frac{1}{m}$, $b_m=\frac{2}{m}$ we have

$$(2.4)\qquad q((1+b_m)u_m) > d_m\,q(u_m)$$

where $d_m=\frac{2^m}{m+2}\longrightarrow+\infty$ as $m\longrightarrow+\infty$. Let us note that even if $q(u_m)=0$, we always have $q((1+b_m)u_m)>0$. Suppose, there exist $l,r\in(0,1)$ such that $lq((1+b_m)u_m)\le q(ru_m)$. However, in virtue of (2.4) we have

$$ld_m<\frac{lq((1+b_m)u_m)}{q(u_m)}\le\frac{q(ru_m)}{q(u_m)}\le 1$$

if $q(u_m)>0$ or $q((1+b_m)u_m)=0$ if $q(u_m)=0$. So we get a contradiction, because $d_m\longrightarrow+\infty$ and $q((1+b_m)u_m)>0$. Then, for sequences $(r_n),(l_n)\subset(0,1)$ such that $r_n\uparrow 1$, $l_n\downarrow 0$ there is a subsequence (m_n) of natural numbers such that

$$l_n q((1+b_{m_n})u_{m_n}) > q(r_n u_{m_n}).$$

By putting $a_n=q((1+b_{m_n})u_{m_n})$ and applying the inequalities $p(q(u))\le u$, $\bar p(q(u))\ge u$ we have

$$(2.5)\qquad \frac{\bar p(l_n a_n)}{p(a_n)}\ge\frac{\bar p(q(r_n u_{m_r}))}{(1+b_{m_n})u_{m_n}}\ge\frac{r_n}{1+b_{m_n}}\longrightarrow 1,$$

as $n\longrightarrow+\infty$. Let $l\in(0,1)$ be arbitrary. We have

$$\frac{\varphi(la_n)}{\varphi(a_n)} \geq \frac{(1-l_n)a_n\bar{p}(l_n a_n)}{a_n p(a_n)} \longrightarrow 1,$$

by (2.5) and inequalities (0.1). Putting $(u_n)=(u_{m_n})$, we proved the lemma.

2.8. Theorem. The Orlicz space L_φ is weakly uniformly rotund if and only if it is rotund and reflexive.

Proof. Let L_φ be rotund and reflexive. From [16] (cf. also Theorem 0.1) it follows that φ is strictly convex and satisfies the Δ_2-condition. From Theorem 0.2 we have moreover that the conjugate function ψ satisfies the Δ_2-condition. Hence the dual space L_φ^* is isometrically isomorphic to L_ψ with the Orlicz norm and therefore it is isomorphic to L_ψ. More precisely, for each $x^* \in L_\varphi^*$ there exists $z \in L_\psi$ such that $x^*(x)=\int_T x(t)\, z(t)d\mu$ for all $x \in L_\varphi$ and the dual norm is equivalent to $\|\cdot\|_\psi$ ([10],[12]). Now, let $x,y \in L_\varphi$ be such that $I_\varphi(x)=I_\varphi(y)=1$ and $x^*(x-y) \geq \varepsilon$ for some $\varepsilon \in (0,1)$ and $x^* \in L_\varphi^*$. So, we have $\int_T (x(t)-y(t))z(t)d\mu \geq \varepsilon$ for some $z \in L_\psi$. We know that the set $\mathcal{B}$ of all bounded functions with supports being of finite measure is dense in L_ψ, because ψ satisfies the Δ_2-condition ([10],[12]). Hence, and in virtue of $\|x-y\|_\varphi \leq 2$ one can choose $z_0 \in \mathcal{B}$ such that $|\int_T (x(t)-y(t))z_0(t)d\mu| \geq \frac{\varepsilon}{2}$ if $x^*(x-y) \geq \varepsilon$. Let $z_0(t)=z_0(t)\chi_{T_0}$ and $|z_0(t)| \leq M$ where T_0 is some set of finite measure and $M>1$. Therefore we have $\int_{T_0} |x(t)-y(t)|d\mu \geq \frac{\varepsilon}{2M}$. Applying Lemma 2.5 with T_0 and $\frac{\varepsilon}{M}$ in place of T and ε, there exist constants a, b such that

$$\int_A |x(t)-y(t)|d\mu \geq \frac{\varepsilon}{8M} \tag{2.6}$$

where $A=\{t \in T_0 : a \leq |x(t)| \vee |y(t)| \leq b\}$.

For $\alpha = \|\chi_{T_0}\|_\psi\, 8M/\varepsilon$ we find constants K, $c \geq 0$ such that

$$\varphi(\alpha u) \leq K\varphi(u) + \varphi(\alpha c) \tag{2.7}$$

and $\varphi(\alpha c)\mu(T) < \frac{1}{2}$, because φ fulfils the Δ_2-condition (recall that $0\cdot+\infty = 0$ if $c=0$ and $\mu(T)=+\infty$).

Now let $B=\{t \in A : |x(t)-y(t)| \geq (|x(t)| \vee |y(t)|)\cdot \frac{\varepsilon}{4MK}$. Since $a \leq |x(t)| \vee |y(t)| \leq b$ for $t \in A$ and φ is strictly convex on R, so

$$\varphi\left(\left|\frac{x(t)+y(t)}{2}\right|\right) \le (1-p)\ \frac{\varphi(|x(t)|)+\varphi(|y(t)|)}{2}$$

for each $t \in B$, for some $p=p(\varepsilon,a,b)=p(\varepsilon,z) \in (0,1)$ by Lemma 0.3. Hence and by the assumption $I_\varphi(x)=I_\varphi(y)=1$ we have

$$I_\varphi\left(\frac{x+y}{2}\right) \le 1 - \frac{p}{2}(I_\varphi(x\chi_B)+I_\varphi(y\chi_B)). \tag{2.8}$$

If $t \in A\setminus B$ then $\varphi(|x(t)-y(t)|) \le (\varphi(|x(t)|)+\varphi(|y(t)|))\cdot\frac{\varepsilon}{4MK}$. So

$$I_\varphi((x-y)\chi_{A\setminus B}) \le \frac{\varepsilon}{2MK}. \tag{2.9}$$

Now, apply the Hölder's inequality to (2.6). We obtain $\|(x-y)\chi_A\|_\varphi \cdot \|\chi_{T_0}\|_\psi \ge \int_A |x(t)-y(t)|d\mu \ge \frac{\varepsilon}{8M}$. Hence, immediately, $I_\varphi(\alpha(x-y)\chi_A)=I_\varphi((x-y)\chi_A \cdot \|\chi_{T_0}\|_\psi \cdot \frac{8M}{\varepsilon}) \ge 1$. But, by (2.7), $1 \le I_\varphi(\alpha(x-y)\chi_A) \le K\ I_\varphi((x-y)\chi_A)+\frac{1}{2}$ which implies

$$I_\varphi((x-y)\chi_A) \ge \frac{1}{2K}. \tag{2.10}$$

By combining inequalities (2.9) and (2.10) we obtain $I_\varphi((x-y)\chi_B) \ge \frac{1}{2K} - \frac{\varepsilon}{2MK} = \frac{1}{2K}(1-\frac{\varepsilon}{M}) > 0$. Hence and in virtue of (2.8) and Lemma 2.6 we get

$$I_\varphi\left(\frac{x+y}{2}\right) \le 1-q$$

where the constant $q=\frac{p}{4K^2}(1-\frac{\varepsilon}{M}) \in (0,1)$ depends only on x^*, ε and φ. Thus L_φ is WUR by Lemma 2.3.

In order to prove the necessity of the reflexivity and rotundity in the theorem, let us first note that in virtue of Theorems 0.1 and 0.2 it is enough to show that ψ satisfies the Δ_2-condition. Assume that ψ takes only finite values and does not fulfil the Δ_2-condition (the case when ψ takes infinity will be considered further). There exist sequences (u_n), (b_n) with the same properties as in Lemma 2.7. Let A_n be pairwise disjoint sets such that $\mu(A_n)=\frac{1}{\psi((1+b_n)u_n)}$ where $n \in N$. Such choice of A_n is always possible since μ is atomless and $u_n \uparrow +\infty$ if $\mu(T) < +\infty$. Putting

$$z(t)=\sum_{n=1}^{\infty} u_n \chi_{A_n}(t),$$

we have $I_\psi(z)=\sum_{n=1}^{\infty}\frac{\psi(u_n)}{\psi((1+b_n)u_n)} < \sum_{n=1}^{\infty}\frac{1}{2^n} = 1$, by (2.2) in Lemma

2.7. It means that $z\in L_{\psi}$. Let $\bar{x}_n(t)=a_n\chi_{A_n}(t)$, where $a_n=q((1+b_n)u_n)$. We have

$$\int_T \bar{x}_n(t)z(t)d\mu = a_n u_n\,\mu(A_n) = \frac{q((1+b_n)u_n)(1+b_n)u_n}{(1+b_n)\psi((1+b_n)u_n} =$$

$$= \frac{\varphi(a_n)+\psi((1+b_n)u_n)}{(1+b_n)\psi((1+b_n)u_n)} = \frac{1}{1+b_n}(I_{\varphi}(\bar{x}_n)+1)$$

by the equality in (0.2). On the other hand, $\int_T \bar{x}_n(t)z(t)d\mu \leq$
$\leq I_{\varphi}(\bar{x}_n)+\dfrac{\psi(u_n)}{\psi((1+b_n)u_n)} < I_{\varphi}(\bar{x}_n)+\dfrac{1}{2^n}$, by the Young's inequality and (2.2). From the two inequalities above we get $I_{\varphi}(\bar{x}_n)+\dfrac{1}{2^n} >$
$> \dfrac{I_{\varphi}(\bar{x}_n)+1}{1+b_n}$, $I_{\varphi}(\bar{x}_n)\geq \dfrac{1}{b_n}(1-\dfrac{1}{2^{n-1}}\geq \dfrac{1}{2}$ for n large (we can always suppose that $b_n\leq 1$. So, we find $B_n\subset A_n$ such that

(2.11) $$\frac{1}{2}\leq \varphi(a_n)\mu(B_n)\leq 1.$$

Let $B_{1,n}$, $B_{2,n}$ be sets such that $B_n=B_{1,n}\cup B_{2,n}$ and $\mu(B_{1,n})=$ $=\mu(B_{2,n})$. Let $l\in(0,1)$ be arbitrary but fixed. Since $\varphi(a_n)\mu(B_{1,n})+$ $+\varphi(la_n)\mu(B_{2,n})\leq 1$ one can find a set C_n and a number c_n such that $\mu(C_n)<+\infty$, $C_n\cap B_n=\emptyset$ and

(2.12) $$\varphi(a_n)\mu(B_{1,n})+\varphi(la_n)\mu(B_{2,n})+\varphi(c_n)\mu(C_n)=1.$$

Let us put

$$x_n=a_n\chi_{B_{1,n}}-la_n\chi_{B_{2,n}}+c_n\chi_{C_n}$$

$$y_n=la_n\chi_{B_{1,n}}-a_n\chi_{B_{2,n}}+c_n\chi_{C_n}$$

Then $I_{\varphi}(x_n)=I_{\varphi}(y_n)=1$, by (2.12). Moreover

$$\int_T(x_n(t)-y_n(t))z(t)d\mu = \int_{B_n}(1-l)a_n u_n d\mu =$$

$$=(1-l)\frac{q((1+b_n)u_n)(1+b_n)u_n)}{1+b_n}\mu(B_n)\geq \frac{1-l}{1+b_n}\varphi(a_n)\mu(B_n)\geq \frac{1-l}{4},$$

for all $n\in N$, by (2.11) and the fact that $b_n\in(0,1)$. Since $\frac{1+l}{2}\in(0,1)$,

$$\varphi\left(\frac{1+l}{2}a_n\right)\geq r_n\frac{1+l}{2}\varphi(a_n)$$

for some sequence $(r_n)\subset(0,1)$ satisfying $r_n\longrightarrow 1$, by (2.4) in Lemma 2.7. Hence

$$I_\varphi\left(\frac{x_n+y_n}{2}\right)\geq r_n\,\frac{1+1}{2}\varphi(a_n)\,\mu(B_{1,n})+r_n\,\frac{1+1}{2}\varphi(a_n)\,\mu(B_{2,n})+$$

$$\varphi(c_n)\,\mu(C_n)\geq r_n\,\mu(B_{1,n})\,\frac{\varphi(a_n)+\varphi(1a_n)}{2}+$$

$$+r_n\,\mu(B_{2,n})\,\frac{\varphi(a_n)+\varphi(1a_n)}{2}+\varphi(c_n)\,\mu(C_n)=r_n+\varphi(c_n)\,\mu(C_n)(1-r_n)\longrightarrow 1$$

as $n\to+\infty$. Hence and in virtue of Lemma 2.3 we have shown that L_φ is not WUR.

Now let $\psi(v)=+\infty$ for $v>v_0$ and $\psi(v)<+\infty$ for $v<v_0$, where v_0 is some positive number (the assumption that φ is not identically equal to zero implies that $v_0>0$). Then $L_\psi(A)\subset L_\infty(A)$ for $A\in\Sigma$. Hence $L_1(A)\subset L_\varphi(A)$. But $L_\varphi(A)\subset L_1(A)$ for each $A\in\Sigma$ of finite measure. Thus $L_1(A)$ is isomorphic to $L_\varphi(A)\subset L_\varphi$, where A is some set of finite measure. Hence and by Lemma 2.2 the space L_φ contains an isomorphic copy of l_1. So, in virtue of Lemma 2.1, L_φ cannot be WUR, which ends the proof of the theorem.

Remark. Let us note that a much simpler (but indirect) proof of the necessity can be given using the fact that a Banach space X whose dual X^* contains an isomorphic copy of c_0, contains an isomorphic copy of l_1 ([1]). The fact that if L_φ is WUR then the Young's function φ satisfies the Δ_2-condition follows from Theorem 0.1, since each WUR-space is rotund. If ψ does not satisfy the Δ_2-condition, then $L_\psi(A)$ is equivalent to $(L_\varphi(A))^*$ (see [16]) and hence contains an isomorphic copy of c_0. Consequently $L_\varphi(A)$ contains an isomorphic copy of l_1, where $A\in\Sigma$ is any set of finite and positive measure. Since L_φ contains an isomorphic copy of l_1, it cannot be WUR by Theorem 2.1 - a contradiction.

Let us isolate the following properties: (*) φ satisfies the Δ_2-condition and is strictly convex, (**) φ and its conjugate function satisfy the Δ_2-condition and φ is strictly convex.

2.9. Theorem. The following properties concerning the Orlicz spaces L_φ are equivalent: (**), WUR, CWUR, E-property.

Proof. It suffices to apply Theorem 2.8, Theorem 1.3 and Theorem 0.2.

For the sake of completeness we formulate in the sequel a theorem which is an immediate consequence of Theorem 0.1 and some well known implications (cf. [13] and the remarks in the introduction).

2.10. Theorem. The following properties concerning the Orlicz spaces L_φ are equivalent: $(*)$, R, MLUR, HR, LUR, URED, CWLUR, WLUR.

As a corollary we get

2.11. Theorem. If the Orlicz space L_φ is reflexive then all properties from the above theorems and the property URWC coincide.

Remark. Let us note that from the proofs given above it follows that we can also deal with the measure space (T,Σ,μ) which is not purely atomic, by making evident modifications in these proofs if necessary. So, Theorem 2.9, Theorem 2.10 and Theorem 2.11 become unchanged for such measures.

Let us also mention that after preparing this paper we have been learned on the paper [5] of N. Herrndorf concerning LUR Orlicz spaces for vector-valued functions with results similar to Theorem 0.1. Recall that this theorem together with Theorem 1.2 and Theorem 1.3 play the crucial role in this paper in the proofs of Theorems 2.9-2.11.

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