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SPECTRAL ANALYSIS OF VARIATIONAL INEQUALITIES

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Abstract: We investigate solvability of variational inequality

(1) $u \in K: \langle \lambda u - Au - g(u,\lambda) - f, v-u \rangle \geq 0 \quad \forall v \in K,$

where K is a closed convex cone in a Hilbert space; A, g are completely continuous mappings, A linear, and λ is a real parameter. As a consequence we get some assertions on the existence of bifurcation points and eigenvalues for corresponding problems. These assertions are very close to the results of M. Kučera [1, 2].

Key words: Variational inequality, bifurcation point, eigenvalue.

Classification: 49H05, 73H10

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1. Introduction. In this paper we study solvability of variational inequalities of the following type:

(1) $u \in K: \langle \lambda u - Au - g(u,\lambda) - f, v-u \rangle \geq 0 \quad \forall v \in K,$

where K is a closed convex cone in a real separable Hilbert space H with the scalar product $\langle \cdot , \cdot \rangle$, λ is a real parameter, $A: H \longrightarrow H$ is a completely continuous linear mapping, $g: H \times R \longrightarrow H$ is a completely continuous (nonlinear) map and $f \in H$ is a right-hand side. As a corollary of our considerations we get some assertions on the existence of higher eigenvalues and bifurcation points for corresponding problems.

We remind that $\lambda_0 \in \mathbb{R}$ is a bifurcation point of the variational inequality

(2) $u \in K: \langle \lambda u - Au - g(u,\lambda), v-u \rangle \geq 0 \quad \forall v \in K$,

if there exists a sequence (u_n, λ_n) of solutions of (2) such that $0 \neq u_n \longrightarrow 0$, $\lambda_n \longrightarrow \lambda_0$.

An element $\lambda_0 \in \mathbb{R}$ is an eigenvalue of the operator A on the cone K, if the problem

(3) $u \in K: \langle \lambda_0 u - Au, v-u \rangle \geq 0 \quad \forall v \in K$

has a non-trivial solution $u_0 \neq 0$. The vector u_0 is called eigenvector corresponding to λ_0.

We shall denote by $\sigma_K(A)$ the set of all eigenvalues of the inequality (3) (i.e. the set of all eigenvalues of the operator A on the cone K) and we put $\sigma_K^+(A) = \sigma_K(A) \cap \mathbb{R}^+$, where $\mathbb{R}^+ = \{t \in \mathbb{R}; t > 0\}$.

There are known (to the author) two methods concerning higher eigenvalues or bifurcation points for variational inequalities - the method of E. Miersemann (see e.g. [3, 4, 5]) which consists in a generalization of Krasnoselskij sup-min principle and can be used only for symmetric operator A, and the method of M. Kučera which is based on Dancer's global bifurcation theorem (see e.g. [1, 2]). In our paper, the problem (1) is reformulated (for $\lambda > 0$) to the operator equation $Tu = 0$, where the operator $T: H \longrightarrow H$ depends on λ, A, g, f and K, and solvability of this equation is investigated using the Leray-Schauder degree. As a corollary we get some results on bifurcation points which are very close to the results of M. Kučera.

Main results are formulated in Section 2; in Section 3 we show that for special cones we obtain more information . Finally, let us mention that our method can be used also in another situation (see [7]).

2. <u>General theory</u>. In the whole section we assume that H is a real separable Hilbert space, $K \subset H$ a closed convex cone with its vertex at the origin, $A: H \longrightarrow H$ a completely continuous linear operator, $g: H \times \mathbb{R} \longrightarrow H$ a completely continuous operator and $\lambda \in \mathbb{R}$.

First we remind some properties of the set $\sigma_K(A)$: The set $\sigma_K(A)$ is bounded by $\pm \|A\|$. It can be easily shown that the set $\sigma_K^+(A)$ is closed in $\mathbb{R}^+$, nevertheless the set $\sigma_K^-(A)$ need not be closed in $\mathbb{R}^-$ (see Example 1). Each positive bifurcation point of (2) belongs to $\sigma_K(A)$, if $\frac{g(u,\lambda)}{\|u\|} \longrightarrow 0$ for $u \longrightarrow 0$ (locally uniformly in λ). The set $\sigma_K(A)$ may contain an interval (see Example 3). If the operator A is symmetric and positive, the set $\sigma_K(A)$ is non-empty, it may contain a non-zero accumulation point (see [6]) and it may also consist of only one point, even for dim H = $+\infty$ (see [6]).

In what follows we shall deal only with $\lambda > 0$; this restriction is substantial in our method. The problem (1) can be rewritten as

$$u \in K: \quad \langle \frac{1}{\lambda}(Au+g(u,\lambda)+f) - u, v-u \rangle \leqq 0 \qquad \forall v \in K.$$

Using characterization of the projection P_K on the set K we get that our problem is equivalent to the problem

$$(4) \qquad Tu = 0,$$

where $Tu = T(\lambda,f,g,A,K)u = u-P_K(\frac{1}{\lambda}(Au+g(u,\lambda)+f))$.

Note that this rewriting can be made also for a general closed convex set K. If K is a cone with its vertex at 0, then

$$Tu = u - \frac{1}{\lambda} P_K(Au+g(u,\lambda)+f).$$

We want to use Leray-Schauder degree in (4), so that we need some apriori estimates for solutions of the equation (4). Before we prove such estimates, let us introduce the following

<u>Definition</u>. Let $K,\tilde{K} \subset H$. We shall write $\Delta(K,\tilde{K}) \leqq \varepsilon$, if the following two conditions are fulfilled:

(5) $(\forall x \in K) \qquad \operatorname{dist}(x,\tilde{K}) \leq \varepsilon \max(1, \|x\|)$

(6) $(\forall \tilde{x} \in \tilde{K}) \qquad \operatorname{dist}(\tilde{x},K) \leq \varepsilon \max(1, \|\tilde{x}\|)$.

Lemma 1. Let $K \subset H$ be a closed convex cone with its vertex at 0, let $\tilde{K} \subset H$ be a closed convex set, $\Delta(K,\tilde{K}) \leq \varepsilon$. Then

$$\|P_K u - P_{\tilde{K}} \tilde{u}\| \leq (\varepsilon + 2\sqrt{\varepsilon + \varepsilon^2}) \cdot \max(1, \|u\| + \varepsilon)$$

for any $u \in H$.

(See [11].)

Lemma 2 (Apriori estimates). Let $I \subset \mathbb{R}^+ - \sigma_K(A)$ be a compact interval, $\frac{g(u,\lambda)}{\|u\|} \to 0$ for $\|u\| \to \infty$ uniformly for $\lambda \in I$. Then for every $M > 0$ there exist $\varepsilon, R > 0$ such that for each $\lambda \in I$, $s,t \in \langle 0,1\rangle$, $f \in H$, $\|f\| < M$, and arbitrary closed convex set $\tilde{K} \subset H$ with $\Delta(K,\tilde{K}) \leq \varepsilon$ the following estimate is true:

$$[(1-s)T(\lambda,f,tf,A,K) + sT(\lambda,f,tg,A,\tilde{K})]\, u = 0 \Longrightarrow \|u\| < R.$$

Proof. By a contradiction: suppose there exist $u_n \in H, \|u_n\| \to \infty$, $\lambda_n \in I$, $s_n, t_n \in \langle 0,1\rangle$, $\|f_n\| < M$, closed convex sets $\tilde{K}_n$ with $\Delta(K,\tilde{K}_n) \leq \frac{1}{n}$ such that

$$[(1-s_n)T(\lambda_n,f_n,t_n g,A,K) + s_n T(\lambda_n,f_n,t_n g,A,\tilde{K}_n)]\, u_n = 0.$$

Using Lemma 1 we get

(7) $$u_n = \frac{1}{\lambda_n} P_K(Au_n + t_n g(u_n,\lambda_n) + f_n) + r_n,$$

where $r_n = o(\|u_n\|)$ $(n \to \infty)$.

We may suppose $w_n = \frac{u_n}{\|u_n\|} \to w$, $\lambda_n \to \lambda \in I$.

Dividing (7) by $\|u_n\|$ we get

(8) $$w_n = \frac{1}{\lambda_n} P_K\left(Aw_n + \frac{t_n g(u_n,\lambda_n)}{\|u_n\|} + \frac{f_n}{\|u_n\|}\right) + \frac{r_n}{\|u_n\|}.$$

The right-hand side in (8) converges strongly to $\frac{1}{\lambda} P_K Aw$, thus $w_n \to w$, $w = \frac{1}{\lambda} P_K Aw$ (i.e. $w \in K$, $\langle \lambda w - Aw, v-w\rangle \geq 0 \quad \forall v \in K$). Since $\|w_n\| = 1$, we have $w \neq 0$, **thus** $\lambda \in \sigma_K(A)$, which gives us a contra-

Corollary. Put $B_R = \{u \in H; \|u\| < R\}$. If $\lambda \in \mathbb{R}^+ - \sigma_K(A)$, $\frac{g(u,\lambda)}{\|u\|} \longrightarrow 0$ ($\|u\| \longrightarrow \infty$), $f \in H$ and $\Delta(K,\tilde{K}) \leqq \varepsilon$, where ε is sufficiently small, then the Leray-Schauder degree $\deg(T(\lambda,f,g,A,\tilde{K}),0,B_R)$ is well defined for R sufficiently large and this degree does not depend on $\lambda, f, g, \tilde{K}$ in the following way: Let λ_1, λ_2 belong to the same component of $\mathbb{R}^+ - \sigma_K(A)$, $f \in H$, $\frac{g(u,\lambda_1)}{\|u\|} \longrightarrow 0$ (for $\|u\| \longrightarrow \infty$) and $\Delta(K,\tilde{K}) \leqq \varepsilon$, where ε is sufficiently small. Then (for sufficiently large R) we have

$$\deg(T(\lambda_1,f,g,A,\tilde{K}),0,B_R) = \deg(T(\lambda_2,0,0,A,K),0,B_R).$$

Proof. The assertion is a consequence of homotopy-invariance property of Leray-Schauder degree.

Remark 1. If $\lambda \in \mathbb{R}^+ - \sigma_K(A)$, then $d(\lambda)=\deg(T(\lambda,0,0,A,K),0,B_R)$ is well defined for any $R>0$ and does not depend on R.

Remark 2. In the sequel we shall deal only with the cone K, nevertheless, using Corollary of Lemma 2, many of our results can be proved for convex sets which are "close" to the cone K (e.g. if $d(\lambda) \neq 0$, then the problem (1) will have a solution also when we shift or turn the cone K a little bit).
We shall write briefly $T(\lambda,f,g)$ instead of $T(\lambda,f,g,A,K)$.

Lemma 3 (On bifurcations). Let $\lambda^1, \lambda^2 \in \mathbb{R}^+ - \sigma_K(A)$, $\lambda^1 < \lambda^2$, $\frac{g(u,\lambda^i)}{\|u\|} \longrightarrow 0$ (for $u \longrightarrow 0$, $i=1,2$), $g(0,\lambda)=0$ for $\lambda \in \langle \lambda^1, \lambda^2 \rangle$, $d(\lambda^1) \neq d(\lambda^2)$. Then there exists a bifurcation point $\lambda_0 \in \langle \lambda^1, \lambda^2 \rangle$ of the variational inequality (2).

Proof. First we prove (by a contradiction) that the equation $T(\lambda^i,0,tg)u = 0$ does not have solution for $0 \neq u \in B_\varepsilon$ (ε sufficiently small), $t \in \langle 0,1\rangle$ and $i=1,2$.

Suppose e.g. there exist $0 \neq u_n \to 0$ and $t_n \in \langle 0,1\rangle$ such that $T(\lambda^1,0,t_n g)u_n = 0$, i.e. $u_n = \frac{1}{\lambda^1} P_K(Au_n+t_n g(u_n,\lambda^1))$. Dividing this equation by $\|u_n\|$ and passing to the limit (we may suppose $\frac{u_n}{\|u_n\|} \to w$) we get $\frac{u_n}{\|u_n\|} \to w = \frac{1}{\lambda^1} P_K Aw$, which gives us a contradiction, since $\lambda^1 \notin \sigma_K(A)$.

Now suppose that there is no bifurcation point $\lambda_0 \in \langle \lambda^1,\lambda^2\rangle$. Then the equation $T(\lambda,0,g) = 0$ is not solvable for $\lambda \in \langle \lambda^1,\lambda^2\rangle$ in $B_\varepsilon - \{0\}$ for sufficiently small ε and using the homotopy-invariance property of Leray-Schauder degree we get

$$d(\lambda^1) = \deg(T(\lambda^1,0,0),0,B_\varepsilon) = \deg(T(\lambda^1,0,g),0,B_\varepsilon) =$$
$$= \deg(T(\lambda^2,0,g),0,B_\varepsilon) = \deg(T(\lambda^2,0,0),0,B_\varepsilon) = d(\lambda^2),$$

a contradiction.

Theorem 1. Let $\lambda > \max(\sigma_K(A) \cup \{0\})$. Then $d(\lambda) = 1$.

Proof. Choose $\Lambda > \|A\|$. By Corollary of Lemma 2 we get $d(\lambda)=d(\Lambda)$. Using the homotopy-invariance property of Leray-Schauder degree for the homotopy

$$H(t,u) = u - \frac{t}{\Lambda} P_K Au$$

we get

$$d(\Lambda) = \deg(T(\Lambda,0,0),0,B_R) = \deg(I - \frac{1}{\Lambda} P_K A,0,B_R) = \deg(I,0,B_R)= 1$$

(we have $H(t,u) \neq 0$ for $u \in \partial B_R$, since $\|\frac{t}{\Lambda} P_K Au\| < \|u\|$ for $u \neq 0$).

Lemma 4. Let K be not a subspace of H (i.e. the linear hull span $K \neq K$) and let $\lambda < \inf_{\|u\|=1} \langle Au,u\rangle$. Then the variational inequality

(9) $u \in K$: $\langle \lambda u-Au-f,v-u\rangle \geqq 0 \quad \forall v \in K$

does not have solution for suitable f.

Proof. First we shall prove that there exists $0 \neq u_0 \in K$ such that $\langle u,u_0\rangle \geqq 0$ for any $u \in K$.

Choose $v_0 \in \text{span } K - K$. Using Hahn-Banach theorem for the convex sets K and $\{v_0\}$ in $\overline{\text{span } K}$, we obtain an element $u_1 \in \overline{\text{span } K}$, $u_1 \neq 0$, such that $\langle u, u_1 \rangle \geqq 0$ for each $u \in K$. Using the characterization of the projection P_K we get that it is sufficient to put $u_0 = P_K u_1$.

Now we shall prove that the inequality (9) does not have solution for $f = u_0$. Suppose there exists $u \in K$ such that

$$(10) \quad \langle \lambda u - Au - u_0, v-u \rangle \geqq 0 \quad \forall v \in K.$$

Putting $v=0$ and $v=2u$ we get $\langle \lambda u - Au - u_0, u \rangle = 0$, so that

$$\lambda \|u\|^2 - \langle Au, u \rangle = \langle u_0, u \rangle \geqq 0.$$

Since $\lambda < \inf_{\|u\|=1} \langle Au, u \rangle$, we have $u=0$.

Putting $v=u_0$ in (10), we get now $-\langle u_0, u_0 \rangle \geqq 0$, which gives us a contradiction.

<u>Corollary</u>. Let $\dim H < \infty$, span $K \neq K$, $g(0,\lambda) \equiv 0$, $\frac{g(u,\lambda)}{\|u\|} \longrightarrow 0$ (for $u \longrightarrow 0$). Then there exists a bifurcation point of (2). Particularly, $\sigma_K(A) \neq \emptyset$.

Proof. We may suppose $\inf_{\|u\|=1} \langle Au, u \rangle > 0$ (instead of the mapping A we may consider the mapping $A+tI$, where $t>0$ is sufficiently large). Choose $\lambda^1 \in (0, \inf_{\|u\|=1} \langle Au, u \rangle)$, $\lambda^2 > \|A\|$. By Lemma 4 we have $d(\lambda^1)=0$, by Theorem 1 $d(\lambda^2)=1$. Now it is sufficient to use Lemma 3 and notice that for $\dim H < \infty$ each bifurcation point belongs to the set $\sigma_K(A)$.

Note that the condition $\frac{g(u,\lambda)}{\|u\|} \longrightarrow 0$ (for $u \longrightarrow 0$) is sufficient to be supposed for $\lambda = \lambda^1, \lambda^2$.

<u>Lemma 5</u>. Let $0 \neq u_0 \in K$, $A^* u_0 = \lambda_0 u_0$, $\lambda_0 > 0$ (where A^* is the adjoint of A). Then the variational inequality

$$(11) \quad u \in K: \langle \lambda_0 u - Au - u_0, v-u \rangle \geqq 0 \quad \forall v \in K$$

does not have solution.

Proof (by a contradiction) Putting $v=u+u_0$ in (11), we get

$0 \leqq \langle \lambda_0 u - Au - u_0, u_0 \rangle = \langle u, \lambda_0 u_0 - A^* u_0 \rangle = \|u_0\|^2 = -\|u_0\|^2$,
a contradiction.

<u>Corollary</u>. Let $0 \neq u_0 \in K$, $A^* u_0 = \lambda_0 u_0$, $\lambda_0 \in \mathbb{R}^+ - \sigma_K(A)$, $\frac{g(u,\lambda)}{\|u\|} \to 0$ for $u \to 0$. Then there exists a bifurcation point λ of (2) with $\lambda > \lambda_0$.

Proof. It is sufficient to use Lemma 5, Theorem 1 and Lemma 3 as in Corollary of Lemma 4.

<u>Exercise 1</u>. Let $K \subset \{u \in H; \langle u, u_K \rangle \geqq \varepsilon \|u\|\}$, where $\varepsilon > 0$, $0 \neq u_K \in H$, and let $\langle Au, u \rangle > 0$ for $u \neq 0$. Prove that $\sigma_K(A) \neq \emptyset$.
Hint: Put $C = \{u \in K; \langle u, u_K \rangle = 1\}$ and

$$Su = \frac{P_K Au}{\langle P_K Au, u_K \rangle} \quad \text{for } u \in C.$$

Then use Schauder fixed point theorem.

Main results of this section are the following two theorems and their corollaries.

<u>Theorem 2</u>. Let $\lambda_k > 0$ be a simple eigenvalue of the operator A, let the corresponding eigenvector $u_k \in K^0$, let $K \neq H$. The eigenspace $Ker(\lambda_k I - A^*)$ is generated by a vector v_k and we assume $v_k \in K^0$, $\langle v_k, u_k \rangle > 0$ (for A symmetric we put $v_k = u_k$). Then the following assertions hold:

(a) The eigenvalue λ_k is an isolated point of $\sigma_K(A)$.

(b) Put $\lambda_k^+ = \inf\{\lambda \in \sigma_K(A); \lambda > \lambda_k\}$. If $\lambda \in (\lambda_k, \lambda_k^+)$, then $d(\lambda) = (-1)^{\beta_k}$, where $\beta_k = \sum_{\lambda > \lambda_k} \dim(\bigcup_{p=1}^{\infty} Ker(\lambda I - A)^p)$.

(c) Put $\lambda_k^- = \sup(\{\lambda \in \sigma_K(A); \lambda < \lambda_k\} \cup \{0\})$.

If $\lambda \in (\lambda_k^-, \lambda_k)$, then $d(\lambda) = 0$.

For $\lambda < \lambda_k$ sufficiently close to λ_k, the inequality

(12) $u \in K: \langle \lambda u - Au - v_k, v - u \rangle \geqq 0 \quad \forall v \in K$
does not have solution.

Proof. (a) Suppose there exist $\lambda^n \in \sigma_K^+(A) - \{\lambda_k\}$, $\lambda^n \longrightarrow \lambda_k$. Then there exist $u^n \in K$, $\|u^n\| = 1$, such that

$$\langle \lambda^n u^n - Au^n, v-u^n \rangle \geq 0 \quad \forall v \in K,$$

or equivalently

$$(13) \quad u^n = \frac{1}{\lambda^n} P_K A u^n.$$

Since λ_k is an isolated point of $\sigma(A)$ (the spectrum of the operator A), we have $\lambda^n u^n \neq Au^n$ for $n \geq n_0$; thus $u^n \in \partial K$ for $n \geq n_0$. We may suppose $u^n \longrightarrow w$. Passing to the limit in (13) we get

$$w = \frac{1}{\lambda_k} P_K A w, \quad u^n \longrightarrow w \in \partial K.$$

Thus

$$(14) \quad 0 \neq w \in \partial K, \quad \langle \lambda_k w - Aw, v-w \rangle \geq 0 \quad \forall v \in K.$$

Choose $z \in H$. Then $v_k + tz \in K$ for sufficiently small $t > 0$ and putting $v = w + v_k + tz$ in (14) we get

$$0 \leq t \langle \lambda_k w - Aw, z \rangle + \langle w, \lambda_k v_k - A^* v_k \rangle = t \langle \lambda_k w - Aw, z \rangle,$$

thus $\lambda_k w = Aw$, which gives us a contradiction, since $u_k \in K^0$ and λ_k is a simple eigenvalue of A.

(b) Let $\lambda > \lambda_k$, $\lambda \notin \sigma_K(A) \cup \sigma(A)$. Then u_k is a regular solution of the equation $Tu \equiv T(\lambda, (\lambda - \lambda_k)u_k, 0)u = 0$, i.e. the mapping T is of the class C^1 in the neighbourhood of u_k and the Fréchet derivative $T'(u_k) = I - \frac{1}{\lambda} A$ is an isomorphism. Thus for sufficiently large $R > 0$ and sufficiently small $\varepsilon > 0$ we get (using Leray-Schauder index of isolated solution)

$$d(\lambda) = \deg(T, 0, B_R - \overline{B_\varepsilon(u_k)}) + \deg(T, 0, B_\varepsilon(u_k)) =$$

$$= \deg(T, 0, B_R - \overline{B_\varepsilon(u_k)}) + (-1)^{\beta_k}.$$

Since $d(\lambda)$ is constant on (λ_k, λ_k^+), it is sufficient to prove that $\deg(T, 0, B_R - \overline{B_\varepsilon(u_k)}) = 0$ for λ sufficiently close to λ_k ($\lambda > \lambda_k$). We shall prove (by a contradiction) that for λ suf-

ficiently close to λ_k ($\lambda > \lambda_k$), the equation Tu=0 does not have solution different from u_k.
Suppose that for $\lambda^n \searrow \lambda_k$ ($\lambda^n \neq \lambda_k$) there exist $u^n \neq u_k$ such that

$$(15) \quad T(\lambda^n, (\lambda^n - \lambda_k)u_k, 0)u^n = 0,$$

i.e.

$$(16) \quad u_k + u^n \in K, \ \langle \lambda^n u^n - Au^n - (\lambda^n - \lambda_k)u_k, v - u^n \rangle \geqq 0 \quad \forall v \in K.$$

Since $(\lambda^n I - A)$ is an isomorphism for $n \geqq n_0$ and $u = u_k$ is the solution of the equation $(\lambda^n I - A)u = (\lambda^n - \lambda_k)u_k$, the vector u^n cannot solve this equation and thus $u^n \in \partial K$ (each solution $u \in K^0$ of the inequality (9) is also a solution of the corresponding equation $\lambda u - Au = f$).
Putting $v = u^n + v_k$ in (16) we get

$$0 \leqq \langle \lambda^n u^n - Au^n, v_k \rangle - (\lambda^n - \lambda_k) \langle u_k, v_k \rangle =$$
$$= \langle u^n, \lambda^n v_k - A^* v_k \rangle - (\lambda^n - \lambda_k) \langle u_k, v_k \rangle =$$
$$= (\lambda^n - \lambda_k)(\langle u^n, v_k \rangle - \langle u_k, v_k \rangle).$$

Hence

$$(17) \quad \langle u^n, v_k \rangle \geqq \langle u_k, v_k \rangle > 0.$$

Dividing (15) by $\|u^n\|$ we get

$$(18) \quad \frac{u^n}{\|u^n\|} = \frac{1}{\lambda^n} P_K\left(A \frac{u^n}{\|u^n\|} + \frac{\lambda^n - \lambda_k}{\|u^n\|} u_k\right).$$

We may suppose $\frac{u^n}{\|u^n\|} \longrightarrow w$, from (17) it follows $\frac{\lambda^n - \lambda_k}{\|u^n\|} \longrightarrow 0$.
Passing to the limit in (18) we get

$$w = \frac{1}{\lambda_k} P_K Aw, \ 0 \neq w \in \partial K,$$

which gives us a contradiction as in the proof of (a).

(c) It is sufficient to prove that for $\lambda < \lambda_k$, close to λ_k, the inequality (12) does not have solution.
Suppose the contrary. Then there exist $\lambda^n \nearrow \lambda_k$ ($\lambda^n \neq \lambda_k$) and

u^n such that

(19) $$u^n = \frac{1}{\lambda^n} P_K(Au^n+v_k),$$

or, equivalently,

(20) $$u^n \in K, \ \langle \lambda^n u^n - Au^n - v_k, v-u^n \rangle \geq 0 \quad \forall v \in K.$$

Putting $v=u^n+v_k$ in (20) we get

$$0 \leq \langle \lambda^n u^n - Au^n - v_k, v_k \rangle = \langle u^n, \lambda^n v_k - A^* v_k \rangle - \langle v_k, v_k \rangle =$$

$$= (\lambda^n - \lambda_k) \langle u^n, v_k \rangle - \langle v_k, v_k \rangle .$$

Thus

(21) $$\langle u^n, v_k \rangle = - \frac{1}{\lambda_k - \lambda^n} \|v_k\|2 \longrightarrow -\infty .$$

Hence $\|u^n\| \longrightarrow \infty$ and we may suppose $\frac{u^n}{\|u^n\|} \rightharpoonup w$. Passing to the limit in (19) we get $w = \frac{1}{\lambda_k} P_K Aw$, $\|w\| = 1$; using (21) we get $\langle w, v_k \rangle \leqq 0$.

Since u_k is the only (normalized) solution of the equation $\lambda_k u= =Au$ lying in K and $\langle u_k, v_k \rangle > 0$, we have $w \in \partial K$. This gives us a contradiction as in the proof of (a).

In the following theorem we shall use notation from Theorem 2. The proof of Theorem 3 is very similar to the proof of Theorem 2, so that we shall just sketch it.

<u>Theorem 3</u>. Let $K \neq H$, let $\lambda_k > 0$ be a simple eigenvalue of the operators A, A^*, let the corresponding eigenvectors $u_k, v_k \in K^0$ and $\langle u_k, v_k \rangle < 0$. Then the following assertions hold:

(a) The eigenvalue λ_k is an isolated point of $\sigma_K(A)$.

(b) If $\lambda \in (\lambda_k, \lambda^+_k)$, then $d(\lambda)=0$.

For $\lambda > \lambda_k$ sufficiently close to λ_k the inequality (12) does not have solution.

(c) If $\lambda \in (\lambda^-_k, \lambda_k)$, then $d(\lambda) = (-1)^{\gamma_k}$, where $\gamma_k = \sum_{\lambda \geq \lambda_k} \dim(\bigcup_{p=1}^{\infty} \operatorname{Ker}(\lambda I-A)^p)$.

Sketch of the proof.

(a) The proof is the same as in Theorem 2.

(b) Suppose there exist $\lambda^n \searrow \lambda_k$ ($\lambda^n \neq \lambda_k$) and $u^n \in K$ such that

(22) $u^n = \frac{1}{\lambda^n} P_K(Au^n+v_k)$.

Putting $v=u^n+v_k$ in the variational inequality corresponding to (22) we get $(\lambda^n-\lambda_k)\langle u^n,v_k\rangle \geq \|v_k\|^2$, hence $\|u^n\| \longrightarrow \infty$ and $\langle w,v_k\rangle \geq 0$ (where we suppose $\frac{u^n}{\|u^n\|} \longrightarrow w$).

Passing to the limit in (22) we get $\|w\| = 1$, $w = \frac{1}{\lambda_k} P_K Aw$, which gives us a contradiction as in the proof of Theorem 2(c).

(c) For $\lambda < \lambda_k$ (close to λ_k) we have

$d(\lambda) = \deg(T(\lambda,(\lambda-\lambda_k)u_k,0),0,B_R-\overline{B_\varepsilon(u_k)}) + (-1)^{\gamma_k}$.

Suppose there exist $\lambda^n \nearrow \lambda_k$ ($\lambda^n \neq \lambda_k$) and $u^n \in \partial K$ such that

(23) $u^n = \frac{1}{\lambda^n} P_K(Au^n+(\lambda^n-\lambda_k)u_k)$.

Putting $v=u^n+v_k$ in the corresponding variational inequality we get $\langle u^n,v_k\rangle \leq \langle u_k,v_k\rangle < 0$. Passing to the limit in (23) we obtain $w = \frac{1}{\lambda_k} P_K Aw$, where $0 \neq w \in \partial K$ ($w = \lim \frac{u^n}{\|u^n\|}$), which gives us a contradiction.

<u>Corollary</u>. Let λ_i, λ_j be simple positive eigenvalues of the operators A, A^* ($\lambda_i < \lambda_j$), let the corresponding eigenvectors $u_i,v_i,u_j,v_j \in K^0$, $\langle u_i,v_i\rangle \cdot \langle u_j,v_j\rangle > 0$. Let $g(0,\lambda) \equiv 0$, $\frac{g(u,\lambda)}{\|u\|} \longrightarrow 0$ (for $u \longrightarrow 0$, $\lambda \in (\lambda_i, \lambda_j)$). Then there exists a bifurcation point $\lambda \in (\lambda_i, \lambda_j)$ for the variational equality (2).

Proof. Using **Theorems 2, 3, we get** $d(\lambda^1) \neq d(\lambda^2)$ for suitable $\lambda_i < \lambda^1 < \lambda^2 < \lambda_j$. Now it is sufficient to use Lemma 3.

Remark 3. Some of the assertions of Theorems 2, 3 can be proved (in the same way) also under weaker assumptions, e.g. the following assertion is true:

Proposition 1. Let $\lambda_k > 0$ be an eigenvalue of the operator A, let $v_k \in \mathrm{Ker}(\lambda_k I - A^*) \cap K^0$. Suppose $\langle v_k, u\rangle > 0$ for any $u \in \mathrm{Ker}(\lambda_k I - A) \cap K$, $u \neq 0$. Then $\lambda_k^- < \lambda_k$, $d(\lambda)=0$ for $\lambda \in (\lambda_k^-, \lambda_k)$ and for $\lambda < \lambda_k$ close to λ_k, the inequality (12) does not have solution.

Open problem 1. Let $\lambda \in \mathbb{R}^+ - \sigma_K(A)$, $d(\lambda)=0$. Find some general assumptions under which there necessarily exists $f \in H$ such that the inequality (9) is not solvable. Very special assumptions of this type are given in Exercise 2.
The connection between the Leray-Schauder degree and the number of solutions of a similar problem is studied e.g. in [8,9,10].

Open problem 2. Let λ^1, λ^2 belong to the same component of $\mathbb{R}^+ - \sigma_K(A)$, let there exist $f^1 \in H$ such that the inequality (9) does not have solution for $\lambda = \lambda^1$, $f=f^1$.
Does there necessarily exist a right-hand side f^2 such that the inequality (9) does not have solution for $\lambda = \lambda^2$, $f=f^2$?
A partial answer to this question is given in the following

Lemma 6. The set

$X = \{\lambda \in \mathbb{R}^+ - \sigma_K(A);\ (9)\text{ is solvable for any } f \in H\}$

is closed in $\mathbb{R}^+ - \sigma_K(A)$.

Proof. Let $\lambda^n \to \lambda$ in $\mathbb{R}^+ - \sigma_K(A)$, let $\lambda^n \in X$, $f \in H$. We shall find a solution of (9). Since $\lambda^n \in X$, there exist $u^n \in H$ such that

(24) $u^n = \frac{1}{\lambda^n} P_K(Au^n+f)$.

Suppose $\|u^n\| \to \infty$. Then passing to the limit in (24) divided by $\|u^n\|$ we get $w = \frac{1}{\lambda} P_K Aw$, where $w = \lim \frac{u^n}{\|u^n\|}$, which gives us a contradiction with $\lambda \notin \sigma_K(A)$. Thus we may suppose $u^n \rightharpoonup u_0$ and passing to the limit in (24) we get $u_0 = \frac{1}{\lambda} P_K(Au_0+f)$, hence u_0 is the solution of (9).

Remark 4. If $\lambda > \max(\sigma_K(A) \cup \{0\})$, then $d(\lambda)=1$ (according to Theorem 1) and thus the inequality (9) is solvable for any $f \in H$. One can easily prove that for $\lambda > \max_{\|u\| \leq 1} \langle Au,u \rangle$ the solution is unique (the operator $\lambda I-A$ is strictly monotone). Nevertheless, for $\lambda < \max_{\|u\| \leq 1} \langle Au,u \rangle$ we may lose the uniqueness: Suppose e.g. A is symmetric and positive, let λ_1 be the first eigenvalue of the operator A, let its multiplicity be odd and $\mathrm{Ker}(\lambda_1 I-A) \cap K = \{0\}$. Choose $\lambda \in (0, \lambda_1)$ such that $\lambda > \max \sigma_K(A) = \max_{\substack{u \in K \\ \|u\| \leq 1}} \langle Au,u \rangle$ and $\lambda > \max(\sigma(A) - \{\lambda_1\})$. Choose $u_0 \in K^0$ and put $f = (\lambda I-A)u_0$. Then

$$1 = d(\lambda) = \deg(T(\lambda,f,0),0,B_R) =$$
$$= \deg(T(\lambda,f,0),0,B_\varepsilon(u_0)) + \deg(T(\lambda,f,0),0,B_R-\overline{B_\varepsilon(u_0)}) =$$
$$= -1 + \deg(T(\lambda,f,0),0,B_R-\overline{B_\varepsilon(u_0)}),$$

thus there exists a solution of (9) in $B_R-\overline{B_\varepsilon(u_0)}$, i.e. the inequality (9) has at least two solutions.

Remark 5. The results of E. Miersemann on higher eigenvalues and bifurcation points are (in the symmetric case) stronger than Corollary of Theorem 2. As a corollary of his results (see [5]) we obtain the following

Proposition 2. Let A be symmetric, let $\lambda_k > \lambda_{k+1} > 0$ be two consecutive eigenvalues of A, let $\mathrm{Ker}(\lambda_{k+1} I-A) \cap K^0 \neq \emptyset$,

$Ker(\lambda_k I-A) \not\subset K$. Then there exists $\lambda \in \sigma_K(A) \cap (\lambda_{k+1}, \lambda_k)$. If the assumption $Ker(\lambda_k I-A) \not\subset K$ fails, we can use the following

Lemma 7. Let A be symmetric, let $\lambda_{k-p} > \lambda_{k-p+1} \geqq \dots$ $\dots \geqq \lambda_k > \lambda_{k+1} > 0$ be consecutive eigenvalues of A, let $Ker(\lambda_{k+1} I-A) \cap K^0 \neq \emptyset$, $V \equiv \bigoplus_{i=k-p+1}^{k} Ker(\lambda_i I-A) \subset K$, $Ker(\lambda_{k-p} I-A) \not\subset K$.
Then there exists an eigenvalue $\lambda \in \sigma_K(A) \cap (\lambda_{k+1}, \lambda_{k-p})$ with an eigenvector $w \in V^{\perp}$.

Proof. Put $\tilde{H}=V^{\perp}$, $\tilde{K}= \tilde{H} \cap K$, $\tilde{A}= A/_{\tilde{H}}$. Then we can use Proposition 2 for $\tilde{H},\tilde{K},\tilde{A}$ to obtain an eigenvalue $\lambda \in \sigma_{\tilde{K}}(\tilde{A})$ with an eigenvector $w \in \tilde{K}$. Denote $P: H \longrightarrow \tilde{H}$ the orthogonal projection of H onto $\tilde{H}$. Choose $v \in K$. Then $Pv \in \tilde{K}$, hence $\langle \lambda w-Aw, v-w \rangle =$
$= \langle \lambda w-\tilde{A}w, v-w \rangle = \langle \lambda w-\tilde{A}w, Pv-w \rangle \geqq 0$.

Note that analogous results to Proposition 2 and Lemma 7 hold also for the existence of bifurcation points of the corresponding non linear problems.

3. Special cones. We shall assume all general assumptions from Section 2 and, moreover, we shall suppose $K = \{u \in H; \langle u,w_i \rangle \geqq \geqq 0, i=1,\dots,n\}$, where $w_i \neq 0$ $(i=1,\dots,n)$.

Lemma 8. Let $K = \{u \in H; \langle u,w_1 \rangle \geqq 0\}$, $w_1 \neq 0$, let $\lambda \notin \sigma(A)$. Put $F(\lambda) = \langle R(\lambda,A)w_1,w_1 \rangle$, where $R(\lambda,A) = (\lambda I-A)^{-1}$. Then

(i) the inequality (9) is (uniquely) solvable for any $f \in H$ iff $F(\lambda) > 0$;

(ii) $\lambda \in \sigma_K(A)$ iff $F(\lambda) = 0$.

Proof. Denote $R(\lambda,A)w_1 = u_1$. Obviously, an element $u \in K$ is the solution of (9) iff $\lambda u-Au-f = tw_1$, or, equivalently, $u = R(\lambda,A)f+ tu_1$, where ($u \in K^0$ and $t=0$) or ($u \in \partial K$ and $t \geqq 0$).

Suppose $F(\lambda) > 0$, i.e. $u_1 \in K^0$. Choose $f \in H$. If $R(\lambda,A)f \in K$, it is sufficient (and necessary) to put $u=R(\lambda,A)f$; if $R(\lambda,A)f \notin K$, we put $u=R(\lambda,A)f + tu_1$, where $t = -\dfrac{\langle R(\lambda,A)f,w_1\rangle}{\langle u_1,w_1\rangle}$.

Suppose $F(\lambda)=0$. Then $u_1 \in \partial K$, $\lambda u_1 - Au_1 = w_1$, i.e. u_1 is an eigenvector corresponding to $\lambda \in \sigma_K(A)$.

Obviously $\lambda \in \sigma_K(A) - \sigma(A)$ implies $F(\lambda)=0$.

If $F(\lambda) < 0$, then for $R(\lambda,A)f \in K^0$ we have two solutions $(u^1=R(\lambda,A)f,\ u^2=R(\lambda,A)f+tu_1$, where $t = -\dfrac{\langle R(\lambda,A)f,w_1\rangle}{\langle u_1,w_1\rangle} > 0)$, for $R(\lambda,A)f \in \partial K$ we obtain the unique solution $u=R(\lambda,A)f$ and for $R(\lambda,A)f \notin K$, the inequality (9) is not solvable.

Lemma 9. Let the assumptions of Lemma 8 be fulfilled. Then the function $F(\lambda)$ is real-analytic. If, moreover, A is symmetric, then $F(\lambda)$ is strictly decreasing on each component of the set $\mathbb{R} - \sigma(A)$.

Proof. The analyticity of $F(\lambda)$ is obvious.

Let A be symmetric. Using the resolvent identity we get

$$F'(\lambda) = -\langle R^2(\lambda,A)w_1,w_1\rangle = -\|R(\lambda,A)w_1\|^2 < 0.$$

Lemma 10. Let the assumptions of Lemma 8 be fulfilled, let A be symmetric, $0 \neq \lambda_k \in \sigma(A)$, $\mathrm{Ker}(\lambda_k I-A) \subset \partial K$. Then the function $F(\lambda)$ has a removable singularity in $\lambda = \lambda_k$.

Proof. Denote P the orthogonal projection of H onto $\tilde{H} = (\mathrm{Ker}(\lambda_k I-A))^{\perp}$, put $\tilde{A} = A/_{\tilde{H}}$. Then $w_1 \in \tilde{H}$, $A(\tilde{H}) \subset \tilde{H}$, thus $R(\lambda,A)w_1 = R(\lambda,\tilde{A})w_1$ and $F(\lambda)=\tilde{F}(\lambda)$ for $\lambda \notin \sigma(A)$, where $\tilde{F}(\lambda) = \langle R(\lambda,\tilde{A})w_1,w_1\rangle$ is real-analytic on $\mathbb{R} - \sigma(\tilde{A})$.

Theorem 4. Let K be a halfspace, $K = \{u \in H;\ \langle u,w_1\rangle \geqq 0\}$, let A be symmetric.

(i) Let $\lambda_{k-p} > \lambda_{k-p+1} \geq \dots \geq \lambda_k > \lambda_{k+1} > 0$ be consecutive eigenvalues of the operator A ($0 \leq p < k$), let $\mathrm{Ker}(\lambda_i I-A) \subset K$ for $i=k-p+1,\dots,k$ and $\mathrm{Ker}(\lambda_i I-A) \cap K^0 \neq \emptyset$ for $i=k-p,k+1$. Then there exists the unique $\lambda_0 \in (\lambda_{k+1}, \lambda_{k-p}) \cap \sigma_K(A)$ for which there exists an eigenvector u_0 (of the variational inequality (3)) such that u_0 is not solution of the equation $\lambda_0 u-Au=0$. Moreover, we can choose $u_0 \perp \bigoplus_{i=k-p+1}^{k} \mathrm{Ker}(\lambda_i I-A)$. For $\lambda \in (\lambda_{k+1}, \lambda_0)-\sigma(A)$ the inequality (9) has the unique solution for any $f \in H$; for $\lambda \in (\lambda_0, \lambda_{k-p})-\sigma(A)$ the inequality (9) has 0,1 or 2 solutions (more precisely see the proof of Lemma 8).

(ii) Let $\lambda_1 \geq \dots \geq \lambda_{k-1} > \lambda_k > 0$ be consecutive eigenvalues of the operator A, $\lambda_1 = \max_{\|u\| \leq 1} \langle Au,u \rangle$. Let $\mathrm{Ker}(\lambda_i I-A) \subset K$ for $i=1,\dots,k-1$ and $\mathrm{Ker}(\lambda_k I-A) \cap K^0 \neq \emptyset$. Then $\sigma_K(A) \cap (\lambda_k,+\infty) \subset \sigma(A)$ and each eigenvector of the inequality (3) with $\lambda_0 > \lambda_k$ is simultaneously the eigenvector of the operator A.
For $\lambda > \lambda_k$, $\lambda \notin \sigma(A)$ the inequality (9) has the unique solution for any $f \in H$.

Proof. Theorem 4 is a corollary of Lemmas 7,8,9,10 and Theorem 1.

In what follows we shall suppose $K = \{u \in H; \langle u,w_i \rangle \geq 0$ for $i=1,\dots,n\}$, where $w_i \neq 0$ $(i=1,\dots,n)$. Denote $N = \{1,2,\dots,n\}$ and for $M \subset N$ denote

$K_M = \{u \in K;\ \langle u,w_i \rangle = 0 \text{ for } i \in M,\ \langle u,w_i \rangle > 0 \text{ for } i \in N-M\}$,

$H_M = \{w_i ; i \in M\}^{\perp}$,

$P^M : H \longrightarrow H_M$ the orthogonal projection of H onto H_M,

$A_M = P^M A/H_M$, $\Sigma = \bigcup_{M \subset N} \sigma(A_M)$.

Obviously $K = \bigcup_{M \subset N} K_M$, where the union is disjoint.

Lemma 11. Let $u \in K_m$, $\langle \lambda u-w, v-u\rangle \geqq 0$ $\forall v \in K$. Then $\lambda u = P^M w$. Particularly, if $P_K w \in K_M$, then $P_K w = P^M w$.

Proof. Putting $v=u+z$, where $z \in H_M$ is arbitrary (but small), we get $P^M(\lambda u-w)=0$, i.e. $\lambda u = P^M w$. If $P_K w \in K_M$, put $u=P_K w$, $\lambda=1$.

Lemma 12. The set $\sigma_K(A)-\{0\}$ is isolated in $\mathbb{R}-\{0\}$.

Proof. Suppose $\lambda \in \sigma_K(A)$, i.e. there exists $0 \neq u \in K_M$ (for suitable $M \subset N$) such that $\langle \lambda u-Au, v-u\rangle \geqq 0$ $\forall v \in K$. According to Lemma 11, $\lambda u = P^M Au = A_M u$, hence $\lambda \in \sigma(A_M) \subset \Sigma$. Consequently $\sigma_K(A) \subset \Sigma$ and now it is sufficient to notice that the set $\Sigma - \{0\}$ is isolated in $\mathbb{R}-\{0\}$.

Lemma 13. Let $\lambda \in \mathbb{R}-\Sigma$, $f \in H$, $M \subset N$. Then there exists at most one solution of (9) in K_M. Consequently, the number of solutions of (9) is bounded by 2^n.

Proof. Let $u^1, u^2 \in K_M$ be solutions of (9). Using Lemma 11 we get $\lambda u^i = P^M(Au^i+f)$, i.e. $\lambda u^i - A_M u^i = P^M f$ $(i=1,2)$. Since $\lambda \notin \sigma(A_M)$, we have $u^1 = u^2$.

Definition. Let $\lambda > 0$, $T(\lambda, f, 0)u = 0$. We shall say that u is a singular solution of the equation $Tu=0$, if either T is not differentiable in any neighbourhood of u or $T'(u)$ is not isomorphism.

Lemma 14. Let $\lambda > 0$. Then $\{f \in H; (\exists u) T(\lambda, f, 0)u=0$ and u is singular$\} \subset S$, where S is a finite union of subspaces of codim $\geqq 1$ (in H).

Proof. Suppose $T(\lambda, f, 0)u = 0$, u singular, $u \in K_M$. According to Lemma 11 $\lambda u = P_K(Au+f) = P^M(Au+f)$.

(i) Let there exist $v_n \to u$ such that $P_K(Av_n+f) \neq P^M(Av_n+f)$.

Then (by Lemma 11), $P_K(Av_n+f) \notin K_M$ and we may suppose $P_K(Av_n+f) \in$ $\in K_L$, where $L \subset N$ is fixed, $L \neq M$. Since $P_K(Av_n+f) \longrightarrow P_K(Au+f) =$ $= \lambda u \in K_M$, we get $L \subset M$. Moreover, for any $i \in M-L$ the corresponding vector w_i does not belong to the linear hull of the set $\{w_j\}_{j \in L}$ (since $K_L \neq \emptyset$). Consequently $H_M \subsetneq H_L$. Since $P^L(Av_n+f) =$ $= P_K(Av_n+f) \longrightarrow P_K(Au+f) = \lambda u$ and $P^L(Av_n+f) \longrightarrow P^L(Au+f)$, we have $\lambda u = P^L(Au+f)$, $P^L(\lambda u-Au-f) = 0$,

$$f \in H_M^L \equiv (\lambda I-A)H_M + H_L^{\perp},$$

where H_M^L is a subspace of codim $\geqq 1$.

(ii) Let the assumption of (i) fail, i.e. $P_K(Av+f) =$ $= P^M(Av+f)$ for all v sufficiently close to u. Then $Tv = v -$ $- \frac{1}{\lambda} P_K(Av+f) = v - \frac{1}{\lambda} P^M(Av+f)$, thus T is differentiable at u. Since u is singular, the mapping $T'(u) = I - \frac{1}{\lambda} P^M A$ is not isomorphism, i.e. $\lambda \in \sigma(A_M)$. Thus the range R_M of the operator $\lambda I-A_M$ has codim $\geqq 1$ in H_M and from $P^M(\lambda u-Au-f)=0$ it follows

$$f \in R_M + H_M^{\perp}.$$

Obviously it is sufficient to put $S = (\bigcup_{H_M \subsetneq H_L} H_M^L) \cup (\bigcup_{\lambda \in \sigma(A_M)} (R_M+H_M^{\perp}))$.

<u>Theorem 5</u>. Let $\lambda \in \mathbb{R}^+ - \sigma_K(A)$, $f \notin S = S(\lambda)$ (see Lemma 14). Then the number of solutions of the inequality (9) is finite (bounded by 2^n), locally constant (with respect to $\lambda \in \mathbb{R}^+ - \sigma_K(A)$ and $f \in H-S(\lambda)$) and odd resp. even if $d(\lambda)$ is odd resp. even. All these solutions depend analytically on f and λ. If $\lambda \in \mathbb{R} - \Sigma$, then the number of solutions of (9) has an upper bound 2^n for any $f \in H$.

Proof. For $f \notin S$ each solution u of (9) is regular and is unique in K_M for any $M \subset N$ (see the proof of Lemma 13 and the definition of the set S). Using well-known properties of Leray-Schauder degree one can easilv orove that the parity of the number of solutions of (9) depends only on the parity of $d(\lambda)$. Using implicit function theorem we get analytical dependence of solutions of (9) on f and λ. Moreover, if $T(\lambda,f,0)^{-1}(0) = \{u^1,\dots,u^p\}$ and $\varepsilon > 0$ is sufficiently small, then $\text{card}(T(\tilde{\lambda},\tilde{f},0)^{-1}(0) \cap B_\varepsilon(u^i)) = 1$ for any $i=1,\dots,p$ and $(\tilde{\lambda},\tilde{f})$ sufficiently close to (λ,f), so that the function $\text{card}(T(\lambda,f,0)^{-1}(0))$ is lower-semicontinuous. We shall prove that it is also upper-semicontinuous. Suppose the contrary, i.e. there exist λ_n, f_n, u_n such that $\lambda_n \to \lambda \in \mathbb{R}^+ - \sigma_K(A)$, $f_n \to f \notin S$,

$$(25) \qquad T(\lambda_n, f_n 0)\, u_n = 0$$

and $u_n \notin B = \bigcup_{i=1}^{p} B_\varepsilon(u^i)$.

If $\|u_n\| \to \infty$, then passing to the limit in (25) divided by $\|u_n\|$ we get $T(\lambda,0,0)w=0$ for some $w \neq 0$, thus $\lambda \in \sigma_K(A)$, a contradiction. Hence we may suppose that $\{u_n\}$ is bounded, $u_n \rightharpoonup u$. Passing to the limit in (25) we get $u_n \to u$, $T(\lambda,f,0)u = 0$, which gives us a contradiction, since $u_n \notin B$.

Exercise 2. Let $K = \{u \in H;\ \langle u,w_i\rangle \geqq 0 \text{ for } i=1,2\}$. Let w_1, w_2 be linearly independent, $\lambda \in \mathbb{R}^+ - \sigma_K(A)$. Prove that there exists $f \notin S(\lambda)$ such that $\text{card}(T(\lambda,f,0)^{-1}(0)) \leqq 1$. Consequently, if $d(\lambda)=0$, then the inequality (9) is not solvable for some $f \in H$.
Hint: For $M \subset \{1,2\}$ put $T_M = \{f; T(\lambda,f,0)^{-1} \cap K_M \neq \emptyset\}$. If $\lambda \in \sigma(A_M)$, then T_M is contained in a subspace of codim $\geqq 1$. If $\lambda \notin \sigma(A_M)$, then $\overline{T_M}$ is a closed convex cone which is strictly less than halfspace in H and $\text{card}(T(\lambda,f,0)^{-1} \cap K_M) = 1$ for $f \in T_M$. Now observe that card(exp N) = 4.

4. Examples

Example 1. In this example we shall show that the set

$\sigma_K^-(A)$ need not be closed in $\mathbb{R}^- = \{t \in \mathbb{R}; t < 0\}$ and, consequently, a negative bifurcation point of (2) need not be the eigenvalue of (3).

Let $A: H \longrightarrow H$ be a symmetric, completely continuous, linear operator with simple eigenvalues $\lambda_1 = -2$, $\lambda_k = \frac{2}{k}$ $(k=2,3,\dots)$ and corresponding eigenvectors u_1, u_k $(k \geq 2)$. We suppose that $\{u_k\}_{k=1}^{\infty}$ form an orthonormal basis in H. Put $K = \{u \in H, \langle u, u_1 - u_k \rangle \geq 0$ for $k=2,3,\dots\}$. Then $\lambda^k = -1 + \frac{1}{k}$ is an eigenvalue of (3) with an eigenvector $u^k = u_1 + u_k$, since $\lambda^k u^k - Au^k = (1+\frac{1}{k})(u_1 - u_k)$, $\langle \lambda^k u^k - Au^k, u^k \rangle = 0$ and $\langle \lambda^k u^k - Au^k, v \rangle \geq 0 \quad \forall v \in K$. Suppose $-1 = \lim \lambda^k \in \sigma_K(A)$. Then there exists $w \in K$, $\|w\| = 1$, such that

(26) $\langle -w - Aw, v - w \rangle \geq 0 \quad \forall v \in K.$

We can write $w = \sum_{k=1}^{\infty} c_k u_k$, where $\sum_{k=1}^{\infty} c_k^2 = 1$.
From (26) it follows $\langle -w - Aw, w \rangle = 0$, hence $\langle Aw, w \rangle = -\|w\|^2 = -1$, so that

$$-2 \cdot c_1^2 + 2 \sum_{k=2}^{\infty} \frac{c_k^2}{k} = -1, \quad c_1^2 = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{c_k^2}{k}.$$

Suppose $c_j \neq 0$ for some fixed $j \geq 2$. Then $c_1^2 \geq \frac{1}{2} + \frac{c_j^2}{j} > \frac{1}{2}$,

$c_k^2 \leq 1 - c_1^2 \leq \frac{1}{2} - \frac{c_j^2}{j} < \frac{1}{2}$ for any $k \geq 2$. Thus $c_1^2 > c_k^2$ and since

$0 \leq \langle w, u_1 - u_k \rangle = c_1 - c_k$, we have $c_1 > 0$ and

$\langle w, u_1 - u_k \rangle = c_1 - c_k \geq \sqrt{\frac{1}{2} + \frac{c_j^2}{j}} - \sqrt{\frac{1}{2} - \frac{c_j^2}{j}} > 0$ for any $k \geq 2$.

Hence $w \in K^0$, $-w - Aw = 0$, a contradiction.

Thus $c_j = 0$ for $j \geq 2$, $w = u_1$, which gives us again a contradiction.

In [6] there is given an abstract example of a symmetric operator A and a cone K in an infinite dimensional Hilbert space H such that the set $\sigma_K(A)$ has exactly n elements, where n is an arbitrary natural number (this example is a direct generalization of an example of M. Čadek, where $\sigma_{\nu}(A)$ is a one-point set). The following example shows that such example can be constructed also for operators and cones which have a physical interpretation.

Example 2 (V. Šverák). Let $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$, $M = \Omega - (0,\frac{1}{2})\times(0,\frac{1}{2})$, $H = W_0^{1,2}(\Omega)$ (the Sobolev space), $K = \{u \in H; u \geq 0 \text{ on } M\}$,

$$\langle u,v\rangle = \int_\Omega \left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) dx, \quad \langle Au,v\rangle = \int_\Omega uv\, dx.$$

Then $\sigma_K(A) = \left\{\frac{1}{2\pi^2}, \frac{1}{8\pi^2}\right\}$.

Idea of the proof. Let $\lambda \in \sigma_K(A)$, let u be the corresponding eigenvector. Then $\lambda > 0$,

$$\int_\Omega (-\lambda\Delta u - u)\varphi\, dx \geq 0 \qquad \forall \varphi \in \mathcal{D}^+(\Omega).$$

Thus $-\lambda\Delta u - u = \mu$, where μ is a nonnegative measure with its support in M. Further $u = \frac{1}{\lambda} G(u+\mu)$, where G is Green function for Ω. Using potential theory, we get that u is continuous in Ω (since $\lambda u = P_K Gu$) and superharmonic in M^0 (since $-\Delta u \geq 0$ in M^0). From the minimum principle it follows $u \equiv 0$ in M or $u > 0$ in M^0.

Let $u > 0$ in M^0 and denote $\lambda_1(M^0)$ the first eigenvalue of $-\Delta$ on M^0 (with the corresponding eigenfunction $w > 0$). Then $-\lambda\Delta u - u = 0$ in M^0, thus

$$0 < \int_{M^0} uw\, dx = -\lambda \int_{M^0} (\Delta u)w\, dx = \lambda\left(\int_{\partial M^0} u\frac{\partial w}{\partial n}\, dS - \int_{M^0} u(\Delta w)dx\right) =$$

$$= \lambda \int_{\partial M^0} u\frac{\partial w}{\partial n}\, dS + \frac{\lambda}{\lambda_1(M^0)}\int_{M^0} uw\, dx \leq$$

$$\leq \frac{\lambda}{\lambda_1(M^0)}\int_{M^0} uw\, dx,$$

since $\frac{\partial w}{\partial n} \leq 0$ and $u \geq 0$ on ∂M^0. Hence $\lambda \geq \lambda_1(M^0)$.

If $u(x) < 0$ for some $x \in \Omega - M$, then $\lambda \leq \lambda_1(\Omega - M)$, since λ is the first eigenvalue of $-\Delta$ on a subdomain of $\Omega - M$.

Under our assumptions we have $\lambda_1(M^o) > \lambda_1(\Omega - M)$, thus either $u \equiv 0$ on M or $u \geqq 0$ on Ω.

If $u \equiv 0$ on M, then $\lambda = \lambda_1(\Omega - M)$ and u is the first eigenfunction of $-\Delta$ on $\Omega - M$; if $u \geqq 0$ on Ω, then using the minimum principle, we obtain $u > 0$ on Ω, $\lambda = \lambda_1(\Omega)$.

Such an example can be constructed also for general domains in R^n $(n \leq 5)$. Another possible generalization is given in the followihg example:

Let $\Omega = \Omega_0 = (0,4) \times (0,4)$, $M = \Omega - \bigcup_{i=1}^{5} \Omega_i$, where $\Omega_1 = (0,2-\varepsilon) \times (0,2-\varepsilon)$, $\Omega_2 = (2,3-\varepsilon) \times (0,1)$, $\Omega_3 = (3,4) \times (0,2-\varepsilon)$, $\Omega_4 = (0,3-\varepsilon) \times (2,4)$, $\Omega_5 = (3,4) \times (2,4)$, $\varepsilon > 0$.

Then card $\sigma_K(A) = 6$ and each eigenfunction of the variational inequality is the first eigenfunction of the operator $-\Delta$ on some Ω_i $(i = 0,1,\dots,5)$.

Idea of the proof. As before we get $u \equiv 0$ on M or $u > 0$ on M^o. If $u > 0$ on M^o, then $u \geqq 0$ on Ω_2 (since $\lambda_1(M^o) > \lambda_1(\Omega_2)$), so that $u > 0$ on $(M \cup \Omega_2)^o$ (since u is superharmonic on this set). Analogously we obtain $u > 0$ on $(M \cup \Omega_2 \cup \Omega_3)^o$, $u > 0$ on $(M \cup \Omega_2 \cup \Omega_3 \cup \Omega_1)^o$ etc.

Example 3. In this example we shall show that the set $\sigma_K(A)$ can contain an interval.

Put $H = \mathbb{R}^3$, $A = \begin{pmatrix} 1,0,0 \\ 1,1,0 \\ 0,0,1 \end{pmatrix}$, $K = \{x; x_1^2 + x_3^2 \leq x_2^2, x_2 \leq 0\}$.

Choose $t \in \langle 0,1 \rangle$ and put $u = \begin{pmatrix} t \\ -1 \\ \sqrt{1-t^2} \end{pmatrix} \in \partial K$, $\lambda = 1 - \frac{t}{2}$.

Then $\lambda u - Au = -\frac{t}{2} \begin{pmatrix} t \\ \frac{1}{\sqrt{1-t^2}} \end{pmatrix}$, so that $\lambda u - Au \perp \dot{u}$

and one can easily prove $\langle \lambda u - Au, v \rangle \geq 0 \quad \forall v \in K$.
Thus $\langle \frac{1}{2}, 1 \rangle \subset \sigma_K(A)$.

Example 4. Let $H = \mathbb{R}^2$, $A = \begin{pmatrix} 2,1 \\ 0,1 \end{pmatrix}$, $K = \{u \in H; \langle u, w_1 \rangle \geq 0\}$,
where $w_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.
Then $\sigma(A) = \{1,2\}$; $u_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the corresponding eigenvectors for A, A^* lying in K^0 (see Theorems 2, 3 and Lemma 8 for notation). Further $\langle u_1, v_1 \rangle < 0$, $\langle u_2, v_2 \rangle > 0$. We are able to compute $F(\lambda) = \langle R(\lambda, A) w_1, w_1 \rangle = \frac{5\lambda - 11}{(\lambda - 1)(\lambda - 2)}$. Using the results of Section 3, we get $\sigma_K(A) = \{1, 2, \frac{11}{5}\}$. Moreover, for $\lambda \notin \sigma_K(A)$ the inequality (9) is solvable for any $f \in H$ iff $\lambda \in (1,2) \cup (\frac{11}{5}, +\infty)$. Some of these results can be derived also using Theorems 2, 3.

Example 5. Let $H = W_0^{1,2}(0,1)$, $K = \{u \in H; u(\frac{1}{2}) \geq 0\}$, $\langle u, v \rangle =$ $= \int_0^1 u'v' dx$, $\langle Au, v \rangle = \int_0^1 uv\, dx$. Using Theorem 4 we get $\sigma_K(A) =$ $= \sigma(A) - \{0\}$. For $\lambda \in \mathbb{R}^+ - \sigma_K(A)$ the inequality (9) is solvable for any $f \in H$ iff $\lambda \in (\lambda_{2k+1}, \lambda_{2k})$ $(k=1,2,\dots)$ or $\lambda > \lambda_1$.

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