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FULL EMBEDDINGS INTO THE CATEGORIES OF BOOLEAN ALGEBRAS

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Abstract: We prove that every small thin category can be fully embedded in the category of Boolean algebras and all one-one homomorphisms and also in the category of Boolean algebras and all surjective homomorphisms.

Key words: Boolean algebra, full embeddings of categories.

Classification: 18B15, 06E99

A category is called s-universal if every small category can be fully embedded in it. If $\mathcal{K}$ is s-universal, then every monoid can be represented as the monoid of all endomorphisms of an object of $\mathcal{K}$ (this is the result of a full embedding of a one-object category with the morphism part formed by the given monoid). Many current categories are known to be s-universal. This field of problems is extensively investigated in the monograph [PT].

If $\mathcal{K}$ is an s-universal category, then also every group can be represented as the group of all automorphisms of an object of $\mathcal{K}$. This fact implies that neither the category

$\mathcal{B}$(1-1) of all Boolean algebras and all one-one homomorphisms

nor the category

$\mathcal{B}$(onto) of all Boolean algebras and all surjective homomorphisms

is s-universal. In fact, there are groups not representable as the groups of all automorphisms of a Boolean algebra, e.g. Z_3, see [MM]. On the other hand, there are large Boolean algebras with the trivial group of automorphisms,[K]. By [M], there are arbitrarily large Boolean algebras such that the identities are their unique one-one endomorphisms. By [LR], for every uncountable cardinal λ , there is a Boolean algebra of the cardinality λ such that the identity is its unique surjective endomorphism. In the final remark of [LR], the existence of a full embedding of every small discrete category (i.e. having only the unit morphisms) into $\mathcal{B}$(onto) is stated. Here, we investigate full embeddings of small thin categories into the above categories of Boolean algebras (let us recall that a category k is thin if, for every couple of dbjects A, B of k, there is at most one morphism from A into B). Let us state explicitly that in the proof of the theorem stated below, the constructions of [LR] and [M] are essential and only a small reasoning is added to them. However, the embedding theorem seems to be of some interest in connection with the field of problems investigated in [PT].

Theorem. Every small thin category can be fully embedded into $\mathcal{B}$(1-1) and into $\mathcal{B}$(onto).

Proof.

A) Full embeddings into $\mathcal{B}$(1-1).

1. Let $\varkappa$ be a cardinal. Following [S] and [M], let us say that a Boolean algebra B is $\varkappa$-complicated if, for every collection $\{(b_\alpha, a_\alpha) \mid \alpha < \varkappa\}$ of pairs of non-zero elements of B such that

a) $b_\alpha \wedge b_{\alpha'} = 0$ and $a_\alpha \wedge a_{\alpha'} = 0$ whenever $\alpha \neq \alpha'$ and

b) $a_\alpha \nleq b_\alpha$ for all $\alpha < \varkappa$,

there exists $S \subseteq \varkappa$ such that

α) there exists $\underline{\text{no}}$ $w \in B$ with $a_\alpha \leq w$ for all $\alpha \in S$

$a_\alpha \wedge w = 0$ for all $\alpha \in \varkappa \setminus S$,

β) but there exists $u \in B$ with $b_\alpha \leq u$ for all $\alpha \in S$

$b_\alpha \wedge u = 0$ for all $\alpha \in \varkappa \setminus S$.

By [M], if $\varkappa$ is an infinite cardinal such that $2^\varkappa = \varkappa^{\aleph_0}$, then there exists an atomless $\varkappa$-complicated Boolean algebra C (of the cardinality $2^\varkappa$) such that for any non-zero $a \in C$ there is a system $\{b_\alpha \mid \alpha < \varkappa\}$ of non-zero pairwise disjoint elements of C such that $b_\alpha \leq a$ for all $\alpha < \varkappa$.

2. Let C be as above. Let us verify the following assertion: if $c, d \in C$ and there is a one-one homomorphism h of $C \restriction c$ onto $C \restriction d$, then $c = d$ and h is the identity. In fact, suppose that h is not the identity. Then there is a non-zero $a \in C$, $a \leq c$, such that $a \wedge h(a) = 0$. Let $\{b_\alpha \mid \alpha < \varkappa\}$ be a system of non-zero pairwise disjoint elements of C with $b_\alpha \leq a$ for all $\alpha < \varkappa$. Then $\{(b_\alpha, h(b_\alpha)) \mid \alpha < \varkappa\}$ fulfils the above a) b), hence there exists $S \subseteq \varkappa$ and $u \in C$ such that the above α), β) are fulfilled (with a_α replaced by $h(b_\alpha)$). However, $w = h(u \wedge a)$ fulfils

$h(b_\alpha) \leq w$ for all $\alpha \in S$ and

$h(b_\alpha) \wedge w = 0$ for all $\alpha \in \varkappa \setminus S$,

which is a contradiction (this verification is analogous to the proof of Theorem 10 in [M]).

3. The conclusion of 1 and 2: For every cardinal $\varkappa$ there exists a Boolean space (= compact Hausdorff 0-dim) X such that

(i) every nonvoid clopen (= closed-and-open) subset D of X contains a pairwise disjoint collection $\{D_\alpha \mid \alpha < \varkappa\}$ of nonvoid clopen subsets of X;

(ii) if D, E are clopen subsets of X and f is a continuous map of D onto E, then D = E and f is the identity.

4. Let a small thin category k be given. We may suppose

that it is skeletical (i.e. no two distinct objects of k are isomorphic). First, we fully embed k into the thin category of all infinite subsets of a set T, i.e. $\Psi: k \longrightarrow \exp T$ is such that each $\Psi(A)$ is infinite and $k(A,B) \neq \emptyset$ iff $\Psi(A) \subseteq \Psi(B)$. Choose $\varkappa \geq$ $\geq \operatorname{card} T$, let X be a Boolean space which fulfils (i) and (ii) in 3. Choose $\{D_t \mid t \in T\}$ a pairwise disjoint collection of nonvoid clopen subsets of X. For every object A of k, denote by $\Phi(A)$ a one-point compactification of the subset $\bigcup_{t \in \Psi(A)} D_t$ of X, the added point is denoted by ξ_A. If $\Psi(B) \supseteq \Psi(A)$, let us define $\varphi_B^A: \Phi(B) \longrightarrow \Phi(A)$ by

$\varphi_B^A(x) = x$ for all $x \in \bigcup_{t \in \Psi(A)} D_t$

$\varphi_B^A(x) = \xi_A$ else.

5. Let A, B be objects of k, let $f: \Phi(B) \longrightarrow \Phi(A)$ be a surjective continuous mapping. We want to prove that $\Psi(B) \supseteq \Psi(A)$ and $f = \varphi_B^A$. For every clopen subset D of X such that

$$D \subseteq G = \bigcup_{t \in \Psi(A)} D_t \setminus \{f(\xi_B)\},$$

put $E = f^{-1}(D)$. Then, by (ii), E = D and f(x) = x for all $x \in E$. This implies that $f^{-1}(G) = G$ and f(x) = x for all $x \in G$. Consequently, for every $t \in \Psi(A)$, we have $f(\xi_B) \notin D_t$. Thus $\Psi(B) \supseteq \Psi(A)$ and f(x) = x for all $x \in \bigcup_{t \in \Psi(A)} D_t = G$. The continuity of f implies $f(\xi_B) = \xi_A$ and f sends each D_t with $t \notin \Psi(A)$ on ξ_A because $f^{-1}(G) = G$. Thus $f = \varphi_B^A$. Consequently, the map

$$A \longmapsto \text{all clopen subsets of } \Phi(A)$$

defines a full embedding of k into $\mathcal{B}$(1-1).

B) <u>Full embeddings into</u> $\mathcal{B}$(onto).

1. The construction of [LR] will be used; let us recall some facts and notation. If λ is an uncountable regular cardinal, $I(\lambda)$ is the ideal of all **subsets of λ** disjoint from some closed

unbounded subset of λ and $D(\lambda) = P(\lambda)/_{I(\lambda)}$ is the Boolean algebra of all stationary subsets of λ. If X is a topological space, $x \in X$, then $Cf(x,X)$ denotes the set of all regular infinite cardinals μ such that there is a sequence $\{x_i | i < n\}$ in X with

$$x = \lim_{i < \mu} x_i, \text{ and, for every } \alpha < \mu,$$

$$\lim_{i < \alpha} x_i \text{ exists and is distinct from } x.$$

We say that $x \in X$ is λ-<u>special</u> if $\lambda \in Cf(x,X)$ and, for every $\{x_i | i < \lambda\}$ and $\{y_i | i < \lambda\}$ as in the definition of $Cf(x,X)$, the set $\{\alpha | \lim_{i<\alpha} x_i = \lim_{i<\alpha} y_i\}$ is closed and unbounded in λ. If x is λ-special, then S_x^X is the element of $D(\lambda)$ defined as follows: $S_x^X = S'/_{I(\lambda)}$, where

$$S' = \{\alpha | \lambda \in Cf(\lim_{i<\alpha} x_i, X)\},$$

with $\{x_i | i < \lambda\}$ as in the definition of $Cf(x,X)$ (since x is λ-special, S_x^X is independent of the choice of $\{x_i | i < \lambda\}$).

2. Let λ be an uncountable regular cardinal. In [LR], a complete linear ordering I is constructed such that the set of all its elements having a successor is dense in it and

(a) for every $x \in I$, either $\lambda \notin Cf(x,I)$ or x is λ-special;

(b) the set P of all λ-special $x \in I$ with $S_x^I \neq 0$ is dense in I;

(c) if $x, y \in P$, $x \neq y$, then $S_x^I \cap S_y^I = 0$.

By [LR], I (with the order topology) is a Boolean space such that the identity is the unique one-one continuous map of I into itself. Since $D(\lambda)$ contains λ pairwise disjoint non-zero elements, we can obtain, by the same construction, a collection $\{I_\gamma | \gamma < \lambda\}$ of linear orderings such that each of them has all the above properties and, moreover,

(d) if $x \in I_\alpha$, $y \in I_\beta$ are λ-special and $\alpha \neq \beta$, then $S_x^{I_\alpha} \cap S_y^{I_\beta} = 0$.

Then the following statement is fulfilled:

(*) if $\alpha, \beta < \lambda$, K is a clopen nonvoid subset of I_α and there is a one-one continuous map $f:K \longrightarrow I_\beta$, then $\alpha = \beta$ and f is the inclusion (i.e. $f(x) = x$ for all $x \in K$).

In fact, let us suppose that f is not the inclusion. Since $P_\alpha \cap K$ is dense in K (where P_α is the subset of I_α as in (b)), there exists $x \in P_\alpha \cap K$ such that $f(x) \neq x$. By (b), x is λ-special in I_α with $S_x^K = S_x^{I_\alpha} \neq 0$. Since $\lambda \in Cf(x, I_\alpha)$, λ is also in $Cf(f(x), I_\beta)$. By (a), f(x) is λ-special in I_β and, clearly, $S_x^K \subseteq S_{f(x)}^{I_\beta}$. However, by (b),(c) and (d), $S_{f(x)}^{I_\beta}$ is either 0 or disjoint from $S_x^{I_\alpha}$, which is a contradiction.

3. Let a small thin skeletical category be given. We embed fully its dual category k^* into the thin category of all infinite subsets of a set T; denote by $\Psi: k^* \longrightarrow \exp T$ the embedding. Find a pairwise disjoint collection $\{I_t \mid t \in T\}$ of Boolean spaces with the above properties and $\lambda = \operatorname{card} T$ (we may suppose that λ is an uncountable regular cardinal). For every object A of k^*, put again

$$\Phi(A) = \{\xi_A\} \cup \bigcup_{t \in \Psi(A)} I_t,$$

where I_t are clopen in $\Phi(A)$ and ξ_A makes a one-point compactification of the union. If $\Psi(A) \subseteq \Psi(B)$, then $\varphi_A^B: \Phi(A) \longrightarrow \Phi(B)$, defined by

$$\varphi_A^B(\xi_A) = \xi_B, \quad \varphi_A^B(x) = x \text{ for all } x \in \bigcup_{t \in \Psi(A)} I_t,$$

is a one-one continuous map. Now, let A, B be arbitrary objects of k^* and $f: \Phi(A) \longrightarrow \Phi(B)$ be a one-one continuous map. We want to show that then $\Psi(A) \subseteq \Psi(B)$ and $f = \varphi_A^B$. For every $a \in \Psi(A)$, put

$$B_a = \{b \in \Psi(B) \mid f(I_a) \cap I_b \neq \emptyset\}.$$

Since $f(I_a) \setminus \{\xi_B\} \neq \emptyset$, the set B_a is not empty. For every $b \in B_a$, put $\sigma_{a,b} = I_a \cap f^{-1}(I_b)$. Then $\sigma_{a,b}$ is a nonvoid clopen subset of I_a and f defines a one-one continuous map of $\sigma_{a,b}$ into I_b. By (*), b = a and f(x) = x for all $x \in \sigma_{a,b}$. Hence $B_a = \{a\}$ and $\sigma_{a,a}$ is clopen in I_a. Since I_a has no isolated points (see (b)), necessarily $\sigma_{a,a} = I_a$. Consequently $\Psi(A) \subseteq \Psi(B)$ and f(x) = x for all $x \in \bigcup_{a \in \Psi(A)} I_a$. By the continuity of f, $f(\xi_A) = \xi_B$, consequently $f = \varphi_A^B$. Thus, Φ is a full embedding of k^* into the category of Boolean spaces and one-one continuous maps, hence it determines a full embedding of k into $\mathcal{B}$(onto).

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