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ITERATED COUNTABLE PRODUCTS AND SUMS OF THE INFINITE CYCLIC GROUP

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Abstract: We prove a general non-isomorphism of iterated countable sums and products of the infinite cyclic group.

Key words: Infinite abelian groups, reduced groups, Specker group.

Classification: 20K25, 20K20

In this note we investigate the groups obtained from the group $\mathbb{Z}$ by repeated application of the functors $\prod_{i\in\mathbb{N}_0} -$, $\sum_{i\in\mathbb{N}_0} -$. We come to an infinite sequence of groups like

(+) $\sum_{i\in\mathbb{N}_0}\mathbb{Z},\ \prod_{i\in\mathbb{N}_0}\mathbb{Z},\ \sum_{i\in\mathbb{N}_0}\prod_{j\in\mathbb{N}_0}\mathbb{Z},\ \prod_{i\in\mathbb{N}_0}\sum_{j\in\mathbb{N}_0}\mathbb{Z},\ \sum_{i\in\mathbb{N}_0}\prod_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\mathbb{Z},\ \ldots$.

Our motivation for studying this sequence was the question of V. Bartík (closely connected with a fact from [1]) as to whether

$$\sum_{i\in\mathbb{N}_0}\prod_{j\in\mathbb{N}_0}\mathbb{Z} \not\cong \prod_{i\in\mathbb{N}_0}\sum_{j\in\mathbb{N}_0}\mathbb{Z} .$$

In this note we prove more than this inequality; namely, we will show that any two members of the sequence (+) are non-isomorphic.

1. Conventions and notation. Throughout this note, $\mathbb{N}_0$ designates the set of all nonnegative integers, while $\mathbb{Z}$ stands for the group of all integers. As usual, Hom(-,-) in abelian groups designates the hom-functor equipped with the obvious group structure. The symbols $\prod$, $\sum$ mean a direct product (resp. a direct sum) of abelian groups.

2. **Special symbols**. Define groups E_n, F_n ($n \in \mathbb{N}_0$) inductively by

$$E_0 = F_0 = \mathbb{Z}$$

$$E_{n+1} = \sum_{i \in \mathbb{N}_0} F_n, \quad F_{n+1} = \prod_{i \in \mathbb{N}_0} E_n,$$

Let for $n \geq 1$

$$\pi_i^n : F_n \longrightarrow E_{n-1}, \quad \rho_i^n : E_n \longrightarrow F_{n-1}$$

be projections onto the i-th component, and let

$$\iota_i^n : E_{n-i} \longrightarrow F_n, \quad \varkappa_i^n : F_{n-i} \longrightarrow E_n$$

be the corresponding injections.

3. Theorem: Let $n \geq 1$. Then for each homomorphism

$$h : F_n \longrightarrow \mathbb{Z}$$

there is some $m \in \mathbb{N}_0$ with the property that for any $u \in F_n$ satisfying

$$\pi_0^n(u) = \pi_1^n(u) = \ldots = \pi_m^n(u) = 0$$

we have

$$h(u) = 0.$$

Proof: We first prove the fact for $n = 1$ by contradiction. Suppose there were a homomorphism

$$h : F_1 \longrightarrow \mathbb{Z}$$

not satisfying our conclusion. Then we can construct a sequence of elements $u_n \in F_1$ and an increasing sequence of natural numbers k_n such that

$$\pi_i^1(u_n) = 0 \text{ for } i < k_n,$$

$$\pi_{k_n}^1(u_n) \neq 0$$

$$h(u_n) \neq 0$$

for any nonnegative integer n. Put $a_n = |h(u_n)|$. ($|\ \ |$ means the absolute value.) There is an element $u \in F_1$ with

$$(3.1)\quad \pi_m^1(u) = \sum_{n\in\mathbb{N}_0} 3^n.(\prod_{i=0}^{n-1} a_i).\pi_m^1(u_n)$$

for each $m \in \mathbb{N}_0$. (Realize that the right hand side contains only finitely many nonzero elements.) Thus, for each $n \in \mathbb{N}_0$ we have an $x_n \in F_1$ with

$$u = \sum_{i=0}^{n-1} 3^i.(\prod_{j=0}^{i-1} a_j).u_i + (3^n.\prod_{j=0}^{n-1} a_j).x_n,$$

which implies

$$(3.2)\qquad h(u) = \sum_{i=0}^{n-1} 3^i(\prod_{j=0}^{i-1} a_j).h(u_i) + t_n.(3^n \prod_{j=0}^{n-1} a_j)$$

for some integer t_n. We compute

$$\left|\sum_{i=0}^{n-1} 3^i \prod_{j=0}^{i-1} a_j.h(u_i)\right| \le \sum_{i=0}^{n-1} 3^i \prod_{j=0}^{i} a_j =$$

$$= \prod_{j=0}^{n-1} a_j(3^{n-1} + \frac{3^{n-2}}{a_{n-1}} + \frac{3^{n-3}}{a_{n-1}\cdot a_{n-2}} + \ldots + \frac{3^0}{a_{n-1}\cdots a_0} \le$$

$$\le \prod_{j=0}^{n-1} a_j.(\sum_{i=0}^{n-1} 3^i) = \prod_{j=0}^{n-1} a_j \frac{3^{n-1}}{2} < \frac{1}{2}(3^n \prod_{j=0}^{n-1} a_j).$$

Comparing this computation with (3.2) we conclude that

$$(3.3)\qquad |h(u)| \ge \frac{1}{2} 3^n \prod_{j=0}^{n-1} a_j$$

whenever $t_n \neq 0$. Since (3.3) obviously does not hold for n greater than certain n_0 (the right hand side increases arbitrarily), we have

$$t_n = 0 \text{ for } n > n_0$$

and hence

$$(3.4)\qquad h(u) = \sum_{i=0}^{n-1} 3^i \prod_{j=0}^{i-1} a_j h(u_i) \text{ for } n > n_0.$$

In (3,4), the right hand side formally varies with n, while the left hand one does not. Thus, we must have

$$3^n \prod_{j=0}^{n-1} a_j h(u_n) = 0 \text{ for } n > n_0,$$

contradicting our assumptions.

Assume now $n > 1$. Let $h \in \mathrm{Hom}(F_n, \mathbb{Z})$ not satisfy the conclusion. Then we have a sequence u_k $(k \in \mathbb{N}_0)$ of elements of F_n with the property that

$$\pi_0^n(u_k) = \pi_1^n(u_k) = \ldots = \pi_k^n(u_k) = 0,$$

while $h(u_k) \neq 0$.

Let D_k designate the subgroup of E_{n-1} generated by all the elements $\pi_k^n(u_i)$, i nonnegative. Then the groups D_k are finitely generated and hence free abelian. Thus, the group

$$D = \{u \in F_n \mid (\forall k \in \mathbb{N}_0)(\pi_k^n(u) \in D_k)\}$$

satisfies

$$D \cong \prod_{k \in \mathbb{N}_0} D_k \cong \prod_{k \in \mathbb{N}_0} \mathbb{Z},$$

while the homomorphism $h|D$ clearly contradicts our theorem for $n = 1$. □

4. Corollary: For each $n \in \mathbb{N}_0$ we have

$$\mathrm{Hom}(E_n, \mathbb{Z}) \cong F_n, \quad \mathrm{Hom}(F_n, \mathbb{Z}) \cong E_n.$$

Proof: We have

$$\mathrm{Hom}(E_n, \mathbb{Z}) \cong \prod_{i \in \mathbb{N}_0} \mathrm{Hom}(F_{n-1}, \mathbb{Z}).$$

On the other hand, Theorem 3 yields

$$\mathrm{Hom}(F_n, \mathbb{Z}) = \prod_{i \in \mathbb{N}_0} \mathrm{Hom}(E_{n-1}, \mathbb{Z}),$$

since the left hand group is generated by the homomorphisms $h \circ \pi_k^n$, where $h \in \mathrm{Hom}(E_{n-1}, \mathbb{Z})$ and $k \in \mathbb{N}_0$. The proof is concluded by an obvious induction. □

5. Theorem: For $n > 0$, we have

(i) F_n is not isomorphic to any direct component of E_n,

(ii) E_n is not isomorphic to any direct component of F_n.

Proof: First of all, note that the functor $Hom(-,\mathbb{Z})$ preserves direct components and hence, for each n, (i) and (ii) are equivalent by Corollary 4. Another observation is that for $n = 1$, (i) follows from an easy cardinality argument.

Thus, it suffices to prove (ii) for $n>1$. This will be done by induction on n. Choose the least n for which (ii) does not hold and let $r:F_n \longrightarrow E_n$, $i:E_n \longrightarrow F_n$ satisfy $r \circ i = Id$. Put

$$G_k^n = \{u \in F_n \mid \wp_0^n(u) = \ldots = \wp_k^n(u) = 0\}$$

$$H_k^n = \{u \in F_n \mid \pi_0^n(u) = \ldots = \pi_k^n(u) = 0\}$$

$$\bar{H}_k^n = \{u \in F_n \mid (\forall t > k)\}.$$

Define projections

$$p_k^n:F_n \rightarrow \bar{H}_k^n,\ t_k^n:E_n \longrightarrow G_k^n$$

by

$$p_k^n(u) = (\pi_0^n(z), \pi_1^n(u), \ldots, \pi_k^n(u), 0, 0, \ldots)$$

$$t(u) = (\underbrace{0,\ldots,0}_{k\ \text{times}}, \wp_{k+1}^n(u), \wp_{k+2}^n(u), \ldots).$$

Now let $r_k = (t_k^n \circ r)|H_k^n$. Our first aim is to show that at least one of the homomorphisms $r_k:H_k^n \longrightarrow G_k^n$ is trivial.

Suppose the contrary. Then one can choose $a_k \in H_k^n$ with $r_k(a_k) \neq 0$ for each $k \in \mathbb{N}_0$. Put

$$m_k = \max\{j \mid \wp_j^n \circ r(a_k) \neq 0\}.$$

Note that since obviously $m_k \geq k$ and $r_k(u) \neq 0$ implies $r_\ell(u) \neq 0$ for all $\ell \leq k$, we may assume

$$m_\ell < m_k \text{ whenever } \ell < k.$$

Now fix homomorphisms $e_k:F_{n-1} \longrightarrow \mathbb{Z}$ with the property that

$$(5.1) \qquad e_k \circ \wp_{m_k}^n \circ r(a_k) \neq 0.$$

Define $e:E_n \longrightarrow E_1$ by

$$e(u) = \sum_{n \in \mathbb{N}_0} \imath_k^1 \circ e_k \circ \wp_{m_k}^n(u).$$

Since the elements $er(a_k)$ form a triangle matrix (see (5.1)), we can construct a homomorphism $f: E_1 \longrightarrow \mathbb{Z}$ with

$$fer(a_k) \neq 0$$

for any $k \in \mathbb{N}_0$. Comparing this conclusion with the assumption of $a_k \in H_k$, we obtain a contradiction to Theorem 3 by putting $h := fer$.

Thus, at least one of the homomorphisms r_k is trivial. Take the restriction

$$p_k^n \circ i \mid G_k^n \longrightarrow \bar{H}_k^n$$

and the restriction

$$\bar{r}_k = (t_k^n \circ r) : \bar{H}_k^n \longrightarrow G_k^n .$$

For any $u \in E_n$ we have $p_k^n \circ i(u) = i(u) + v$ with some $v \in H_k^n$. Thus, for any $u \in G_k^n$ we can compute

$$\bar{r}_k \circ p_k^n \circ i(u) = t_k^n \circ r \circ p_k^n \circ i(u) = t_k^n \circ r(i(u) + v) =$$
$$= t_k^n \circ r \circ i(u) + t_k^n \circ r(v) = t_k^n \circ r \circ i(u) + r_k(v) = t_k^n \circ r \circ i(u) =$$
$$= t_k^n(u) = u .$$

Thus, the group $G_k^n \cong E_n$ is isomorphic to a direct component of $\bar{H}_k^n = E_{n-1}$. Taking into account that F_{n-1} is a direct component of E_n we come to a contradiction with the induction hypothesis. □

6. Corollary: The only two isomorphic groups of the type E_n, F_m with different symbols are $E_0 = F_0 = \mathbb{Z}$.

Proof: Since for each k both E_k, F_k are isomorphic to direct components of E_n, F_n with any $n > k$, the fact follows easily from Theorem 5. □

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