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CHARACTERIZATION OF THE COLOMBEAU PRODUCT OF DISTRIBUTIONS

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Abstract. The distribution T is equal to the Colombeau product of distributions $R \tilde{\odot} S$ iff the distribution $1/2\ [\ R(x-y)S(x+y) + R(x+y)S(x-y)\ ]$ has for $y = 0$ the section equal to $T(x)$.

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The aim of this paper is to prove the following characterization.

Theorem 1. Let R,S,T be distributions on an open set $\Omega \subset \mathbf{R}^N$. Then $T = R \tilde{\odot} S$ (Colombeau product) iff the distribution

$$\frac{1}{2}\ [\ R(x-y)S(x+y) + R(x+y)S(x-y)\ ]$$

has a section for $y = 0$ (in the Lojasiewicz's sense [4]) and this section is equal to $T(x)$.

The proof will be done at the end of the paper.

Definition 1. If $q \in \mathbf{N} := \{0,1,2,\dots\}$ let $\mathcal{A}_q$ be the set of all functions $\varphi \in \mathcal{D}(\mathbf{R}^N)$ such that

(1) $\int \varphi = 1$

(2) $\int \varphi(x)\, x^i\, dx = 0$ for $1 \le |i| \le q$

($i = (i_1.i_2,\dots,i_N) \in \mathbb{N}^N$) . Let $\mathcal{A}_q^{(m)}$ be the set of all functions $\varphi \in \mathcal{D}^{(m)}(\mathbb{R}^N)$ (compactly supported and continuously differentiable up to order m) satisfying (1) and (2) and let $\mathcal{A}_q^{(m)}(K)$ resp. $\mathcal{A}_q(K)$ ($K \subset \mathbb{R}^N$) be the set of all φ for which moreover $\operatorname{supp} \varphi \subset K$.

<u>Remark</u>. If $p \geq q$ then $\mathcal{A}_p \subset \mathcal{A}_q$. If $\operatorname{int} K \neq \emptyset$ we can see that $\mathcal{A}_q \neq \emptyset$ for $q = 0,1,2,\dots$ (cf.[1] 3.3.1). In this case $\mathcal{A}_q(K) - \mathcal{A}_q(K)$ is the set of all $\varphi \in \mathcal{D}(K)$ for which

$$\int \varphi(x)\, x^i\, dx = 0 \qquad \text{for} \qquad |i| \leq q \quad .$$

If $\varphi \in \mathcal{D}$ and $|j| \geq 1$ then $D^j \varphi \in \mathcal{A}_{|j|-1} - \mathcal{A}_{|j|-1}$ ($j = (j_1,\dots,j_N)$, $D^j\varphi(x)$ signifies $(\frac{\partial}{\partial x})^j \varphi(x)$).

<u>Notation 1</u>. If $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and $\varepsilon > 0$, denote

$$\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(x/\varepsilon)$$

We have $(\varphi_{\varepsilon_1})_{\varepsilon_2} = \varphi_{\varepsilon_1\varepsilon_2}$, $\varphi_1 = \varphi$. If $\varphi \in \mathcal{A}_q$ then $\varphi_\varepsilon \in \mathcal{A}_q$.

We can immediately check the following proposition.

<u>Proposition 1</u>. If $K \subset \mathbb{R}^N$ is compact then ($\forall\, q,m$) the linear space

$$\operatorname{Sp} \mathcal{A}_q^{(m)}(K) = \mathbb{C}.\mathcal{A}_q^{(m)}(K) \cup (\mathcal{A}_q^{(m)}(K) - \mathcal{A}_q^{(m)}(K))$$

spanned by the set $\mathcal{A}_q^{(m)}(K)$, is the set of all $\varphi \in \mathcal{D}^{(m)}(K)$ for which (2) holds. It is a Banach space if it is equipped with the norm of the space $\mathcal{D}^{(m)}$

(3) $$\|\varphi\|_m = \max_{\substack{|j| \leq m \\ x \in \mathbb{R}^N}} |(\frac{\partial}{\partial x})^j \varphi(x)| \quad .$$

The space $Sp\ \mathcal{A}_q(K)$ with the topology induced by $\mathcal{D}$ is a Fréchet space.

Proposition 2. If $\rho \in \mathcal{A}_q$ and $\varphi \in \mathcal{A}_q^{(m)}$ then (the convolution) $\varphi * \rho \in \mathcal{A}_q$. If $K \subset \mathbb{R}^N$ is compact then the closure of the set $\mathcal{A}_q(K)$ in the space $\mathcal{D}^{(m)}(K)$ contains $\mathcal{A}_q^{(m)}(\text{int } K)$.

Proof. I. If $1 \leq |i| \leq q$ then

$$\int [\varphi * \rho(x)]\ x^i\ dx = \int\int \varphi(x-y)\ \rho(z)\ x^i\ dzdx$$

$$= \int \varphi(x) \int \rho(z)\ (x+z)^i\ dzdx =$$

(if $\rho \in \mathcal{A}_q$)

$$\int \varphi(x)\ x^i\ dx = 0$$

(if $\varphi \in \mathcal{A}_q^{(m)}$).

II. Let us choose $\rho \in \mathcal{A}_q$. If $\varphi \in Sp\ \mathcal{A}_q^{(m)}(\text{int } K)$ then $\varphi = \lim_{\varepsilon \searrow 0} \varphi * \rho_\varepsilon$ in the space $\mathcal{D}^{(m)}(K)$, which proves the result.

In [2] a commutative algebra $\mathcal{G}(\Omega)$ is introduced as follows.

Definition 2. Any element $\langle g \rangle \in \mathcal{G}(\Omega)$ has as a representative the functional

$$g : \mathcal{A}_1 \times \Omega \rightarrow \mathbf{C} \qquad \text{(complex numbers)}$$
$$(\varphi, x) \mapsto g(\varphi, x)$$

which is $\mathcal{C}^\infty$ in x for any fixed $\varphi \in \mathcal{A}_1$ and which satisfies the following moderate growth condition: for every compact subset $K \subset \Omega$ and for every $j \in \mathbf{N}^N$ there are $n_1, n_2 \in \mathbf{N}$, $n_1 \geq 1$, such that $\forall \varphi \in \mathcal{A}_{n_1}$ $\exists\, c > 0$ $\exists\, \varepsilon_0 > 0$ such that $(\forall x, \varepsilon)$

$$x \in K,\ 0 < \varepsilon < \varepsilon_0 \implies |(\tfrac{\partial}{\partial x})^j g(\varphi_\varepsilon, x)| \leq c \cdot \varepsilon^{-n_2}.$$

The algebra $\mathcal{G}(\Omega)$ is defined by factorization as follows.

<u>Definition 3.</u> Two functionals g_1, g_2 satisfying the above definition are by definition representatives of the same element of $\mathcal{G}(\Omega)$, i.e. $\langle g_1 \rangle = \langle g_2 \rangle$, if for every compact subset $K \subset \Omega$ and for every $j \in \mathbf{N}^N$ there are $n_0 \in \mathbf{N}$ and numbers $\gamma_n \nearrow \infty$ ($n_0 \geq 1$, $n = n_0, n_0+1, n_0+2, \dots$) such that $\forall\, n \geq n_0$ $\forall \varphi \in \mathcal{A}_n$ $\exists\, c > 0$ $\exists\, \varepsilon_0 > 0$ such that $(\forall x, \varepsilon)$

$$x \in K,\ 0 < \varepsilon < \varepsilon_0 \implies$$
$$|(\tfrac{\partial}{\partial x})^j [g_1(\varphi_\varepsilon, x) - g_2(\varphi_\varepsilon, x)]| \leq c \cdot \varepsilon^{\gamma_n}.$$

The elements of $\mathcal{G}(\Omega)$ are called generalized functions.

<u>Definition 4</u> of the multiplication on $\mathcal{G}(\Omega)$. If $\langle f \rangle, \langle g \rangle \in \mathcal{G}(\Omega)$ we put $\langle f \rangle \odot \langle g \rangle = \langle f \cdot g \rangle$ where $(f \cdot g)(\varphi, x) = f(\varphi, x) \cdot g(\varphi, x)$ (pointvise product of functionals).

<u>Definition 5</u> of the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$. Any distribution $T \in \mathcal{D}'(\Omega)$ is identified with the generalized function representative of which is the functional

$$(\varphi, x) \mapsto \langle T(z), \varphi(z-x)\rangle .$$

According to the factorization by Definition 3 the representative need not be defined for all (φ, x) .

Due to the above identification we may consider that $\mathcal{D}'(\Omega)$ is contained in $\mathcal{G}(\Omega)$. In addition to that identification a weaker equivalence relation, that we are going to recall, between distributions and generalized functions is introduced.

Definition 6. We say that a distribution $T \in \mathcal{D}'(\Omega)$ is associated to a generalized function $\langle g\rangle \in \mathcal{G}(\Omega)$ if for every $\omega \in \mathcal{D}(\Omega)$ $\exists q$ such that $\forall \varphi \in \mathcal{A}_q$

$$\langle T, \omega\rangle = \lim_{\varepsilon \searrow 0} \int g(\varphi_\varepsilon, x)\, \omega(x)\, dx \quad .$$

The distribution associated to $G = \langle g\rangle$, provided it exists, is uniquely defined by G and denoted by $\tilde{G}$.

In this paper we investigate the relation $T = R \tilde{\odot} S$ on Ω which means: $T, R, S \in \mathcal{D}'(\Omega)$ and the distribution T is associated to the generalized function $R \odot S \in \mathcal{G}(\Omega)$.

We are going to deduce the following lemma directly from the above definitions.

Lemma 1. $T = R \tilde{\odot} S$ on Ω iff for every $\omega \in \mathcal{D}(\Omega)$ $\exists q$ such that $\forall \varphi \in \mathcal{A}_q$

$$\langle T, \omega\rangle = \lim_{\varepsilon \searrow 0} \langle R(x-y)S(x+y) , \varsigma_\varepsilon(x,y)\rangle$$

where

$$\varsigma_\varepsilon(x,y) = \varepsilon^{-N} \int \varphi(z - \frac{y}{2\varepsilon})\varphi(z + \frac{y}{2\varepsilon})\omega(x - 2\varepsilon z)\, dz \quad .$$

Proof. From Definitions 4,5,6 and Notation 1 we obtain: $T = R \tilde{\odot} S$ on Ω iff for every $\omega \in \mathcal{D}(\Omega)$ $\exists q$ such that $\forall \varphi \in \mathcal{A}_q$ $\langle T, \omega \rangle =$

$$\lim_{\varepsilon \searrow 0} \int \langle R(x), \varphi_\varepsilon(x-z) \rangle_x \cdot \langle S(y), \varphi_\varepsilon(y-z) \rangle_y \cdot \omega(z)\, dz$$

$$\lim_{\varepsilon \searrow 0} \langle R(x) \times S(y) \ , \ \varepsilon^{-2N} \int \varphi(\frac{x-z}{\varepsilon})\, \varphi(\frac{y-z}{\varepsilon})\, \omega(z)\, dz \rangle_{x,y} \quad .$$

The substitution $(x-y,x+y)$ instead of (x,y) (with the jacobian $= 2^N$) gives

$$= \lim_{\varepsilon \searrow 0} \langle R(x-y)S(x+y) \ , \ \varepsilon^{-2N} \cdot 2^N \int \varphi(\frac{x-y-z}{\varepsilon})\, \varphi(\frac{x+y-z}{\varepsilon})\, \omega(z)\, dz \rangle \quad ;$$

the substitution $x - \varepsilon z$ instead of z and then 2ε instead of ε prove the result.

Definition 7. Let F be a distribution on a neighborhood of zero in $\mathbf{R}^N$. We say that F admits a value at the point $y = 0$ (in the Lojasiewicz's sense) and this value equals to $a \in \mathbf{C}$ if for every $\varphi \in \mathcal{A}_0$ (i.e. $\varphi \in \mathcal{D}$ and satisfies (1)) we have

$$\lim_{\varepsilon \searrow 0} \langle F, \varphi_\varepsilon \rangle = a \quad .$$

Theorem 2 ([4] 4.2 Th.2). Let $\varepsilon_n \searrow 0$ and let $\liminf_{n \to \infty} \varepsilon_{n+1} / \varepsilon_n > 0$. F has at $y = 0$ the value equal to $a \in \mathbf{C}$ iff $\forall \varphi \in \mathcal{A}_0$

$$\lim_{n \to \infty} \langle F, \varphi_{\varepsilon_n} \rangle = a \quad .$$

<u>Definition 8</u>. Let $F(x,y)$ ($x \in \mathbf{R}^N$, $y \in \mathbf{R}^M$) be a distribution on a neighborhood of $\Omega \times \{0\}$ (zero in $\mathbf{R}^M$) We say that F admits a section at $y = 0$ and this sectio is equal to $T(x) \in \mathcal{D}'(\Omega)$ if for every $\omega \in \mathcal{D}(\Omega)$ the distribution

$$\langle F(x,y) , \omega(x) \rangle_x \in (\mathcal{D}')_y$$

has at $y = 0$ the value equal to $\langle T, \omega \rangle$.

<u>Proposition 3</u>. Let Y be a continuous function on $\mathbf{R}^N$, $q \in \mathbf{N}$. Then there is a function $\beta \in \mathcal{D}$ equal to 1 on so neighborhood of zero and such that

$$\int Y(x) \ \beta(x) \ x^i \ dx = 0$$

provided $|i| \leq q$.

<u>Proof</u>. If Y is not identically zero, choose a point $x_0 \neq 0$ with $Y(x_0) \neq 0$ and put

$$B = \{ x \ ; \ |x - x_0| \leq \frac{|x_0|}{2} \} \quad .$$

Since on B the distribution $x^i \ Y(x)$ is not a linear combi tion of the distributions $x^j \ Y(x)$ ($j \neq i$, $|j| \leq q$) , there is a function $\beta_i \in \mathcal{D}(B)$ such that ([5],II.3,lemma5

$$\int x^i \ Y(x) \ \beta_i(x) \ dx = 1$$

and

$$\int x^j \, Y(x) \, \beta_i(x) \, dx = 0$$

provided $j \neq i$, $|j| \leq q$. Choose $\alpha \in \mathcal{D}$, $\alpha = 1$ on some neighborhood of zero; then putting

$$\beta = \alpha - \sum_{|j| \leq q} \left(\int x^j \, Y(x) \, \alpha(x) \, dx \right) \beta_j$$

proves the result.

Lemma 2. Let K be a compact symmetric neighborhood of zero in $\mathbb{R}^N$, $q \in \mathbb{N}$; let $\{T_a\}_{a \in A}$ be a set of distributions such that for every two functions $\varphi, \psi \in \mathcal{A}_q(K)$ the set of numbers

$$\{ \langle T_a , \varphi * \psi \rangle \}_{a \in A}$$

is bounded. Then the set $\{T_a\}_{a \in A}$ is equicontinuous on $\mathrm{Sp}\, \mathcal{A}_q(K)$.

Proof. Since $\mathrm{Sp}\, \mathcal{A}_q(K)$ is a Fréchet space (Proposition 1), it suffices to prove that $\forall \psi \in \mathcal{A}_q(K)$ the set of numbers $\{ \langle T_a, \psi \rangle \}_a$ is bounded. By the assumption of this lemma for a fixed $\varphi \in \mathrm{Sp}\, \mathcal{A}_q(K)$ the set of linear forms

$$\{ \psi \mapsto \langle T_a , \varphi * \psi \rangle \}_{a \in A} \subset (\mathrm{Sp}\, \mathcal{A}_q(K))'$$

(ψ ranges in $\mathrm{Sp}\, \mathcal{A}_q(K)$) is pointvise bounded; hence by Banach Steinhaus Theorem ([5] IV.2,Th.3) it is equicontinuous on the Fréchet space $\mathrm{Sp}\, \mathcal{A}_q(K)$. It means that the bilinear mapping

(4) $$\operatorname{Sp} \mathcal{A}_q(K) \times \operatorname{Sp} \mathcal{A}_q(K) \longrightarrow \ell_A^\infty$$

$$(\varphi, \psi) \longmapsto \{\langle T_a, \varphi * \psi \rangle\}_{a \in A}$$

is separately continuous. Since $\operatorname{Sp} \mathcal{A}_q(K)$ is a Fréchet space, this mapping is continuous ([5] VII.2,prop.11). It means that there are numbers m, m', c such that $\forall \varphi, \psi \in \operatorname{Sp} \mathcal{A}_q(K)$ and $\forall a \in A$ we have

(5) $$\|\varphi\|_m \leq 1 \,,\ \|\psi\|_{m'} \leq 1 \ \Rightarrow\ |\langle T_a \,,\ \varphi * \psi \rangle| \leq c$$

It is known that for any $\psi \in \mathcal{D}$ the mapping $\varphi \longmapsto \langle T_a, \varphi * \psi \rangle$ is continuous on $\mathcal{D}^{(m)}$ and hence the relation (5) holds even for $\varphi \in \overline{\operatorname{Sp} \mathcal{A}_q(K)}$ (closure in $\mathcal{D}^{(m)}$), $\psi \in \operatorname{Sp} \mathcal{A}_q(K)$. We put for φ a fix function βY satisfying the following conditions. Namely, choose a number $n \in \mathbb{N}$ such that

(6) $$n > \frac{q}{2}$$

and $n > (N+m)/2$ so that there exists a function Y continuously derivable up to order m and satisfying the equation

$$\Delta^n Y = \delta$$

([3], formulae (II,3;16) and (II,3;18)). Y is $\mathcal{C}^\infty$ on $\mathbb{R}^N \setminus \{0\}$. By Proposition 3 we choose a function $\beta \in \mathcal{D}$ (int K) equal to 1 on some neighborhood of zero and such that $\beta Y \in \mathcal{A}_q^{(m)} - \mathcal{A}_q^{(m)}$. It follows from Proposition 2 that $\beta Y \in \overline{\operatorname{Sp} \mathcal{A}_q(K)}$. By (6) and the remark following Definition 1 we have $\Delta^n \psi \in \operatorname{Sp} \mathcal{A}_q(K)$. We obtain from (5)

(7) $\langle T_a , \beta Y * \Delta^n \psi \rangle \leq c \, \|\beta Y\|_m \, \|\Delta^n \psi\|_{m'}$

and we compute

(8) $\beta Y * \Delta^n \psi = \Delta^n(\beta Y) * \psi = (\delta + \xi) * \psi$

where $\xi = \Delta^n(\beta Y)$ on $\mathbf{R}^N \setminus \{0\}$, $\xi(0) = 0$, $\xi \in \mathcal{D}$.
If $0 \leq |i| \leq q < 2n$ (by (6)) we have

$$\langle \delta(x) + \xi(x) , x^i \rangle = \langle \Delta^n [\beta(x)Y(x)] , x^i \rangle$$

$$= \langle \beta(x)Y(x) , \Delta^n x^i \rangle = 0 ,$$

so $\xi \in -\mathcal{A}_q$. We obtain from (7) and (8)

$$c \, \|\beta Y\|_m \, \|\Delta^n \psi\|_{m'} \geq \langle T_a , \psi \rangle + \langle T_a , \xi * \psi \rangle$$

and therefore if $\psi \in \mathrm{Sp}\, \mathcal{A}_q(K)$ the set of numbers $\{\langle T_a , \psi \rangle\}_a$ is bounded.

<u>Theorem 3</u>. Let B be an open neighborhood of zero in $\mathbf{R}^N$, $F \in \mathcal{D}'(B)$, $q \in \mathbf{N}$, $a \in \mathbf{C}$. Then the following are equivalent.

(i) F has at zero the value $= a$ (in the Lojasiewicz's sense)

(ii) $\forall \eta \in \mathcal{A}_q$ we have (according to Notation 1)

(9) $$\lim_{\substack{n \in \mathbb{N} \\ n \to \infty}} \langle F , \eta_{2^{-n}} \rangle = a .$$

(iii) $\forall \varphi \in \mathcal{A}_q$ if $\eta = \varphi * \varphi$ (9) holds.

<u>Proof</u>. (i) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii) : We write (9) equivalently

(10) $$\lim_{n \to \infty} \langle F(2^{-n}x) , \eta(x) \rangle = a .$$

If (iii) holds then for every $\varphi, \psi \in \mathrm{Sp}\, \mathcal{A}_q$

$$\lim_{n \to \infty} \langle F(2^{-n}x) , (\varphi(x) + \psi(x)) * (\varphi(x) + \psi(x)) \rangle$$

$$= a \cdot \int (\varphi + \psi) * (\varphi + \psi) \quad .$$

We deduce from it

$$(11) \qquad \lim_{n \to \infty} \langle F(2^{-n}x) , \varphi(x) * \psi(x) \rangle = a \int \varphi * \psi .$$

For any compact symmetric neighborhood K of zero in $\mathbf{R}^N$ the distributions $F(2^{-n}x)$ are defined on K for n large enough and by Lemma 2 they form an equicontinuous set on $\mathrm{Sp}\, \mathcal{A}_q(K)$. Since the functions $\varphi * \psi$ form a dense set in $\mathrm{Sp}\, \mathcal{A}_q$, we deduce (10) from (11) ($\forall \eta \in \mathcal{A}_q$).

(ii) $\Rightarrow$ (i) : By Theorem 2 we need to prove the relation (9) for every $\eta \in \mathcal{A}_0$ and we are going to do it by induction. Let $r \in \mathbf{N}$, $r \geq 1$. From the assumption: (9) holds for every function $\eta \in \mathcal{A}_r$, we are going to deduce:

$$\lim_{n \to \infty} \langle F , \varphi_{2^{-n}} \rangle = a$$

for every $\varphi \in \mathcal{A}_{r-1}$. Indeed, if φ is such a function, then the function

$$\eta := \frac{2^r \varphi_{1/2} - \varphi}{2^r - 1}$$

belongs to $\mathcal{A}_r$ and by the induction assumption it satisfies (9). We have (for $k = 1,2,\ldots,n$)

$$\eta_{2^{k-n}} = \frac{2^r \varphi_{2^{k-n-1}} - \varphi_{2^{k-n}}}{2^r - 1}$$

and therefore

$$\sum_{k=1}^{n} \frac{2^r - 1}{2^{kr}} \eta_{2^{k-n}} = \varphi_{2^{-n}} - 2^{-nr} \varphi \quad .$$

By (9) it gives $\lim_{n \to \infty} \langle F , \varphi_{2^{-n}} \rangle =$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{2^r - 1}{2^{kr}} \cdot \langle F , \eta_{2^{k-n}} \rangle = a$$

since

$$\sum_{k=1}^{\infty} \frac{2^r - 1}{2^{kr}} = 1$$

Lemma 3. For the remainder of the Taylor development of any function $\omega \in \mathcal{D}(\Omega)$

$$(14) \qquad \omega(x+h) = \sum_{|j| \leq m} \left(\frac{\partial}{\partial x}\right)^j \omega(x) \frac{h^j}{j!} + \omega_m(x,h)$$

we have estimates

$$\left| \left(\frac{\partial}{\partial x}\right)^k \omega_m(x,h) \right| \leq c_k \, |h|^m$$

with numbers $c_k \geq 0$ independent from x and h .

Proof. For $k = 0$ it is a well known estimate. For the other k's the estimate follows from the fact that the derivative of (14) is the Taylor development of the derivative of ω .

<u>Lemma 4</u>. For $\omega \in \mathcal{D}$, $\varphi \in \mathcal{D}(\{z\}; |z| \leq r)$ denote (see (14))

$$(15) \qquad \rho_{\varepsilon,m}(x,y) =$$

$$\varepsilon^{-N} \int \varphi(z - \frac{y}{2\varepsilon})\, \varphi(z + \frac{y}{2\varepsilon})\, \omega_m(x\ , -2\varepsilon z)\, dz \quad .$$

Then.

$$\operatorname{supp} \rho_{\varepsilon,m}(x,y) \subset \{\operatorname{dist}(x, \operatorname{supp} \omega) \leq 2\varepsilon r\ ,\ |y| \leq 2\varepsilon r\} \quad .$$

If $|z| \geq r$ we have

$$(16) \qquad \varphi(z - \frac{y}{2\varepsilon})\, \varphi(z + \frac{y}{2\varepsilon}) = 0$$

and therefore in the formula (15) it suffices to integrate over the set $\{|z| < r\}$.

<u>Proof</u>. If $|z| \geq r$ we have either $|z - y/2\varepsilon| \geq r$ or $|z + y/2\varepsilon| \geq r$ which gives (16).

If $|y| > 2\varepsilon r$ then for any z the points $z - y/2\varepsilon$, $z + y/2\varepsilon$ have the distance greater than $2r$. So they do not both belong to $\operatorname{supp} \varphi \subset \{|z| \leq r\}$ which gives (16) for all z and consequently $\rho_{\varepsilon,m}(x,y) = 0$.

If $\operatorname{dist}(x, \operatorname{supp} \omega) > 2\varepsilon r$ with $2\varepsilon r > 2\varepsilon|z|$ (according to the last part of Lemma) it follows that neither x nor $x - 2\varepsilon z$ belong to $\operatorname{supp} \omega$ and by (14) $\omega_m(x\ , -2\varepsilon z) = 0$ which gives $\rho_{\varepsilon,m} = 0$.

<u>Lemma 5</u>. Let $R,S \in \mathcal{D}'(\Omega)$ and $\omega \in \mathcal{D}(\Omega)$, $\varphi \in \mathcal{D}$ be given and let o be the order of the distribution $R(x-y)S(x+y)$ on some neighborhood of the set $\operatorname{supp} \omega(x) \times 0$ (zero in $(\mathbf{R}^N)_y$). Then if $m > N + o$ ($m \in \mathbf{N}$) we have (see (15))

$$(17) \qquad \lim_{\varepsilon \searrow 0} \langle R(x-y)S(x+y) , \varrho_{\varepsilon,m}(x,y) \rangle = 0$$

and if $|i| > N + o$ we have

$$(18) \qquad \lim_{\varepsilon \searrow 0} \langle R(x-y)S(x+y) , \varepsilon^{|i|-N} \omega(x)$$

$$\cdot \int \varphi(z - \tfrac{y}{2\varepsilon}) \varphi(z + \tfrac{y}{2\varepsilon}) z^i \, dz \rangle = 0$$

Proof. We will prove (17) only, the proof of (18) being similar. According to Lemma 4 we have to estimate the derivatives of order $\leq o$ of the functions $\varrho_{\varepsilon,m}$. By Lemma 4 we have

$$(\tfrac{\partial}{\partial x})^l (\tfrac{\partial}{\partial y})^j \varrho_{\varepsilon,m}(x,y) =$$

$$\varepsilon^{-N} \int \sum_{k \leq j} \binom{j}{k} (\tfrac{\partial}{\partial y})^k \varphi(z - \tfrac{y}{2\varepsilon}) \cdot (\tfrac{\partial}{\partial y})^{j-k} \varphi(z + \tfrac{y}{2\varepsilon})$$

$$\cdot (\tfrac{\partial}{\partial x})^l \omega_m(x , -2\varepsilon z) \, dz =$$

$$2^{-|j|} \varepsilon^{-N-|j|} \int_{|z|<n} \sum_k \binom{j}{k} (-1)^{|k|} D^k \varphi(z - \tfrac{y}{2\varepsilon}) D^{j-k} \varphi(z + \tfrac{y}{2\varepsilon})$$

$$\cdot (\tfrac{\partial}{\partial x})^l \omega_m(x , -2\varepsilon z) \, dz \qquad .$$

If we admit $|j + l| \leq o$ only we obtain from Lemma 3

$$| (\tfrac{\partial}{\partial x})^l (\tfrac{\partial}{\partial y})^j \varrho_{\varepsilon,m}(x,y) | \leq c \, \varepsilon^{m-N-|j|}$$

where the constant c depends on o, φ, m, ω but does not depend on x, y, ε . Since $m > N + o \geq N + |j|$ we obtain (17).

<u>Lemma 6</u>. If $T = R \mathring{\otimes} S$ on Ω then $\forall \omega \in \mathcal{D}(\Omega)$ $\exists q$ such that the relation (18) holds for every $i \neq 0$ provided $\varphi \in \mathcal{A}_q$.

<u>Proof</u>. Let K be a compact set in Ω. We are going to prove inductively the lemma for any $\omega \in \mathcal{D}(K)$. Suppose a number $p \in \mathbb{N}$, $p \geq 1$, satisfies the following induction assumption:

> $\forall \omega \in \mathcal{D}(K)$ $\exists q'$ such that the relation (18) holds for every i with $|i| > p$ provided $\varphi \in \mathcal{A}_{q'}$.

By Lemma 5 if o is the order of $R(x-y)S(x+y)$ on some neighborhood of the set $\{(x,0) ; x \in K\}$ then the number $p = N + o$ satisfies the above assumption even for every q'. From the above assumption we are going to deduce:

> $\forall \omega \in \mathcal{D}(K)$ $\exists q''$ such that the relation (18) holds for every i with $|i| \geq p$ provided $\varphi \in \mathcal{A}_{q''}$.

Thus the lemma will be inductively proved. So, let $\omega \in \mathcal{D}(K)$, $|i| = p$. In Lemma 1 we replace the function $\omega(x - 2\varepsilon z)$ by its Taylor development from Lemma 3 ($h = -2\varepsilon z$). If $m > N + o$ (17) gives

$$(19) \qquad \langle T, \omega \rangle =$$

$$\sum_{|j| < m} \lim_{\varepsilon \searrow 0} \frac{(-2)^{|j|}}{j!} \varepsilon^{|j|-N} . \langle R(x-y)S(x+y) ,$$

$$\left(\frac{\partial}{\partial y}\right)^j \omega(x) \int \varphi\left(z - \frac{y}{2\varepsilon}\right) \varphi\left(z + \frac{y}{2\varepsilon}\right) z^j \, dz \rangle$$

Let us denote by $n_1, n_2, \ldots, n_p \in \{1,2,\ldots,N\}$ indices for which

$$(20) \qquad z_{n_1} \cdot z_{n_2} \cdot \ldots \cdot z_{n_p} = z^i$$

($z = (z_1,\dots,z_N)$). For any complex numbers $t_1,\dots,t_p$, from the relation $\psi \in \mathcal{A}_{q+p}$ it follows easily

$$(21) \qquad \varphi(z) := \psi(z) \cdot \prod_{k=1}^{p} (1 + t_k z_{n_k}) \in \mathcal{A}_q$$

(q is chosen by Lemma 1). We have

$$(22) \qquad \int \varphi\left(z - \frac{y}{2\varepsilon}\right) \varphi\left(z + \frac{y}{2\varepsilon}\right) z^j \, dz$$

$$= \int \psi\left(z - \frac{y}{2\varepsilon}\right) \psi\left(z + \frac{y}{2\varepsilon}\right) z^j$$

$$\prod_{k=1}^{p} \left[1 + 2t_k z_{n_k} + t_k^2 \left(z_{n_k}^2 - \frac{y_{n_k}^2}{4\varepsilon^2}\right)\right] dz$$

Substituting $\varphi(z)$ by (21) into (19) gives in the second member of the equality (19) a polynom of variables $t_1,\dots,t_p$. As the equality holds for every $t_1,\dots,t_p$, the coefficient of the power $t^{\mathbf{1}} = t_1 \cdot \ldots \cdot t_p$ of the polynom in question must equal to zero. By (22) and (20) it means

$$\sum_{|j| < m} \lim_{\varepsilon \searrow 0} \frac{(-2)^{|j|}}{j!} \varepsilon^{|j|-N} \left\langle R(x-y)S(x+y) , \right.$$

$$\left. \left(\frac{\partial}{\partial x}\right)^j \omega(x) \int \psi\left(z - \frac{y}{2\varepsilon}\right) \psi\left(z + \frac{y}{2\varepsilon}\right) z^{j+i} \, dz \right\rangle = 0$$

By the induction assumption all the terms of this sum with $j \neq 0$ equal to zero (provided $\psi \in \mathcal{A}_{q'}$ where $q' \geq q + p$ is large enough) and therefore the term with $j = 0$ equals to zero, too. Thus the induction is proved.

Proof of Theorem 1. I. Suppose $T = R \tilde{\otimes} S$ on Ω. In the sum (19) all the terms with $j \neq 0$ equal to zero due to Lemma 6. So we have: $\forall \omega \in \mathcal{D}(\Omega)$ $\exists q$ such that $\forall \varphi \in \mathcal{A}_q$

$$(23) \qquad \langle T, \omega \rangle = \lim_{\varepsilon \searrow 0} \langle R(x-y)S(x+y) \, , \, \omega(x) \, \eta_\varepsilon(y) \rangle$$

(see Notation 1) where

$$(24) \qquad \eta(y) = \int \varphi(z - \tfrac{y}{2}) \, \varphi(z + \tfrac{y}{2}) \, dz = \check{\varphi} * \varphi(y)$$

($\check{\varphi}(z) = \varphi(-z)$). In (23) we substitute instead of η the function

$$\eta'' := \frac{\varphi + \check{\varphi}}{2} * \frac{\check{\varphi} + \varphi}{2}$$

$$= \tfrac{1}{2}\eta + \tfrac{1}{4}\eta' + \tfrac{1}{4}\check{\eta}'$$

where $\eta' = \varphi * \varphi$. From it and from (23) we deduce

$$\langle T, \omega \rangle =$$

$$\lim_{\varepsilon \searrow 0} \langle R(x-y)S(x+y) \, , \, \omega(x) \cdot \tfrac{1}{2} \, [\eta'_\varepsilon(y) + \check{\eta}'_\varepsilon(y)] \rangle =$$

$$\lim_{\varepsilon \searrow 0} \langle \tfrac{1}{2} [R(x-y)S(x+y) + R(x+y)S(x-y)] \, , \, \omega(x) \, \eta'_\varepsilon(y) \rangle$$

Now Theorem 3 says that the distribution $\langle \tfrac{1}{2} \, [R(x-y)S(x+y) + R(x+y)S(x-y)] \, , \, \omega(x) \rangle_x$ has for $y = 0$ the value equal to $\langle T, \omega \rangle$.

II. Suppose the distribution

$$\frac{1}{2}\,[\,R(x-y)S(x+y) + R(x+y)S(x-y)\,]$$

has for $y = 0$ the section equal to $T(x)$. Then for any even function $\eta = \check{\eta} \in \mathcal{D}$ we have

$$\lim_{\varepsilon \searrow 0} \langle R(x-y)S(x+y)\,,\ \omega(x)\ \eta_\varepsilon(y) \rangle = \langle T, \omega \rangle \cdot \int \eta \quad .$$

Consequently (18) holds for every $i \neq 0$ and for every $\omega \in \mathcal{D}(\Omega)$ and (23) holds for the function η defined by (24) With $\varphi \in \mathcal{A}_0$. By Lemma 5 also (17) holds for $m > N + o$. Now the Taylor development of $\omega(x - 2\varepsilon z)$ by Lemma 3 gives the condition in Lemma 1.

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