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ON THE UNIQUE SOLVABILITY OF NONRESONANT ELLIPTIC EQUATIONS

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Abstract: We obtain a result on the unique solvability of the class of nonlinear Dirichlet problems satisfying a certain nonresonance condition.

Key words: Elliptic equations, weak solutions, eigenvalue, fixed point.

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1. Introduction. In this note we shall consider the equations of the following type:

$$(1) \qquad \begin{aligned} -\Delta u &= g(x,u,\nabla u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Throughout the paper we assume that Ω is a bounded domain in R^n, $n \geq 1$. We do not impose any regularity conditions on the boundary of Ω. In turn, we shall seek only weak solutions of (1).

There exists an enormous literature about solvability of such equations and its analogues involving higher order elliptic operators. See for instance [4] and a long list of references cited therein.

To our best knowledge the question of unique solvability for nonresonant elliptic equations of the form (1) has not yet been studied, at least not in the noncoercive case. It is our intention in this note to give a result about existence and uniqueness, which is rather close to the existence result obtained in [3] for more general elliptic equations. It also represents an improvement of the theorem 3 in [5]. The method of the proof which we use here is constructive and is analogous to that in [5], but the estimates are carried out in a much more precise way.

2. Preliminaries. Let us first introduce some notation. By R we shall denote the set of all reals. For $p = (p_1, \dots, p_n) \in R^n$ we define $|p| = (\sum_{i=1}^{n} |p_i|^2)^{1/2}$.

The Lebesgue space $L_2(\Omega)$ will be equipped with the usual scalar product, which we denote by $(\cdot,\cdot)_0$, while the Sobolev space $H_0^1(\Omega)$ will be equipped with the scalar product defined by

$$(u,v)_1 = \int_\Omega \nabla u \cdot \nabla v \, dx$$

The corresponding norms will be denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively.

Let (λ_i) be the sequence of eigenvalues of the operator $-\Delta$ with the Dirichlet boundary condition on Ω. As it is well known, we have

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$
$$\lambda_i \longrightarrow \infty$$

From this we see that there exist infinitely many distinct consecutive pairs of eigenvalues. Let (φ_i) be the corresponding sequence of eigenfunctions, which we assume to be normalized in $L_2(\Omega)$, i.e. $\|\varphi_i\|_0 = 1$. Thus it forms the orthonormal base in $L_2(\Omega)$ and is also a complete orthogonal set in $H_0^1(\Omega)$ such that

(2) $$\|\varphi_i\|_1^2 = \lambda_i .$$

Recall that a mapping $g: \Omega \times R \times R^n \longrightarrow R$ is said to be of the Carathéodory type if $g(\cdot,u,p)$ is measurable for all (u,p) and

$g(x,\cdot,\cdot)$ is continuous for almost all x.

3. The unique solvability result. Let us now formulate the main result of this paper.

Theorem. Let $g: \Omega \times R \times R^n \longrightarrow R$ be a Carathéodory mapping such that $g(\cdot,0,0) \in L_2(\Omega)$ and let there exist $\varepsilon > 0$, $k \geq 1$ such that

(3) $\quad \lambda_k + \varepsilon \leq (g(x,u_1,p)-g(x,u_2,p))/(u_1-u_2) \leq \lambda_{k+1} - \varepsilon$

for all (x,u_i,p), $i = 1,2$, $u_1 \neq u_2$, where λ_k, λ_{k+1} are two distinct consecutive eigenvalues of $-\Delta$. Furthermore, assume that

(4) $$|g(x,u,p_1)-g(x,u,p_2)| \leq C|p_1-p_2|$$

for all (x,u,p_i)., $i = 1,2$. If

(5) $$C < \varepsilon/\lambda_{k+1}^{1/2}$$

then there exists a unique weak solution of (1).

In the proof we shall use the following classical result about the unique solvability of nonlinear elliptic equations (see [2], or [1] for the much more general setting), which can be proved using the Banach fixed point theorem.

Lemma. Let $f: \Omega \times R \longrightarrow R$ be a Carathéodory mapping such that $f(\cdot,0) \in L_2(\Omega)$. Let $a,b \in R$ be such that

$$a \leq (f(x,u_1)-f(x,u_2))/(u_1-u_2) \leq b$$

for all x, u_1, u_2, $u_1 \neq u_2$.

If $[a,b] \cap \sigma(-\Delta) = \emptyset$ (a nonresonance condition), then there exists a unique weak solution of

$$-\Delta u = f(x,u) \text{ in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega$$

Proof of the theorem. As a consequence of the lemma and (3) we obtain that the mapping $T:H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ defined by $u = Tq$ and

$$-\Delta u = g(x,u,\nabla q) \text{ in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega$$

is well defined. We show that T is a contraction, so that the theorem will follow from the Banach fixed point theorem. So choose arbitrary $q_1,q_2 \in H_0^1(\Omega)$, and let $u_i = Tq_1$, $i = 1,2$. Denote $h = q_1-q_2$, $z = u_1-u_2$ and let

$$H^- = \text{span } \{\varphi_1,\dots,\varphi_k\}$$
$$H^+ = \overline{\text{span } \{\varphi_i : i > k\}}$$

We thus have the orthogonal decomposition

$$H_0^1(\Omega) = H^- \oplus H^+$$
$$z = z^- + z^+$$

where $z^{\pm} \in H^{\pm}$ respectively. Define also $\tilde{z} = -z^- + z^+$. From $u_i = Tq_i$, $i = 1,2$, we get

$$(6) \qquad -\Delta z - rz = [g(x,u_1,\nabla q_1)-g(x,u_2,\nabla q_1)-rz] + [g(x,u_2,\nabla q_1)-g(x,u_2,\nabla q_2)]$$

where $r = (\lambda_k+\lambda_{k+1})/2$. Denoting the first term in the square brackets on the right hand side by A and the second by B, we obtain using (3),(4) that

$$(7) \qquad |A| \le ((\lambda_{k+1}-\lambda_k)/2-\varepsilon)|z|$$

$$(8) \qquad |B| \le C|\nabla h|$$

Let $z = \sum_{i=1}^{\infty} z_i\varphi_i$, $z_i \in R$, be a Fourier series of z with respect to the basis (φ_i) in $L_2(\Omega)$. Testing the left hand side of (6) by $\tilde{z}$ and using (2), we get

$$(9) \qquad \langle -\Delta z - rz, \tilde{z}\rangle_{H^{-1},H_0^1} = -\|z^-\|_1^2 + \|z^+\|_1^2 + r(\|z^-\|_0^2 - \|z^+\|_0^2) = \sum_{i=1}^{\infty} |\lambda_i - (\lambda_{k+i}+\lambda_k)/2|z_i^2$$

On the other hand, testing the right hand side of (6) by z and using (7),(8), $\|z\|_0 = \|\tilde{z}\|_0$, we get that the resulting expression is less than or equal to

$$(10)\quad ((\lambda_{k+1}-\lambda_k)/2 - \varepsilon)\,\|z\|_0^2 + C\,\|h\|_1\,\|z\|_0 \leq$$
$$\leq ((\lambda_{k+1}-\lambda_k)/2 - \varepsilon + s)\sum_{i=1}^{\infty} z_i^2 + (C^2/4s)\,\|h\|_1^2$$

Here we used the inequality $ab \leq (a^2/4s) + sb^2$ with $a = C\,\|h\|_1$, $b = \|z\|_0$ and $0 < s < \varepsilon$ to be chosen later.

From (9) and (10) we finally obtain

$$(11)\quad \sum_{i=1}^{\infty} [\,|1-(\lambda_{k+1}+\lambda_k)/2\lambda_i| - (\lambda_{k+1}-\lambda_k)/2\lambda_i + (\varepsilon-s)/\lambda_i]\lambda_i z_i^2 \leq$$
$$\leq (C^2/4s)\,\|h\|_1^2$$

Denoting the expression in the square brackets in (11) by a_i, we get that

(12) if $i \leq k$ then $a_i = (\lambda_k + \varepsilon - s)/\lambda_i - 1 \geq (\varepsilon - s)/\lambda_k$

(13) if $i > k$ then $a_i = 1 - (\lambda_{k+1} - \varepsilon + s)/\lambda_i \geq (\varepsilon - s)/\lambda_{k+1}$

Thus from (11) we have

$$(\varepsilon - s)\,\|z\|_1^2/\lambda_{k+1} \leq (C^2/4s)\,\|h\|_1^2$$

and we see that for the contractivity of T it suffices to check the condition

$$0 \leq C^2\lambda_{k+1}/4s(\varepsilon - s) < 1$$

But this is fulfilled in view of (5), by putting $s = \varepsilon/2$. Note that this choice for s is optimal.

Q.E.D.

Remark. It is easy to see that the constraint (5) on the constant C represents an improvement of the corresponding one given in [5, Theorem 3]. We also note that we do not impose any constraint on ε from below, as was the case in [5].

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