Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 2, 229--234

Persistent URL: http://dml.cz/dmlcz/106446

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27,2 (1986)

EXTREMAL SUBOBJECTS OF COMPLETE POSETS

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Abstract: In Pasztor [82] a structural characterization of the extremal subobjects is given for the category of Z-complete posets together with all Z-continuous maps. Unfortunately, this characterization is incorrect. The aim of the present note is to give a correct characterization, and, beyond this, to draw attention upon the very interesting concept of stable dominion. We prove for arbitrary categories a characterization theorem of strong monos by means of the stable dominion - the same way regular monos were characterized by dominions in Herrlich-Strecker [73].

Key words: Complete posets, extremal subobject, stable dominion.

Classification: 06A10, 18A32, 18A20, 68C99

In Pasztor [82] I gave a structural characterization of the epis of POS(Z) - the category of Z-complete posets together with all Z-continuous maps - where Z is an arbitrary subset system. As a consequence, in Corollary 1 on pg. 296, I gave a structural characterization of the extremal subobjects in POS(Z). Unfortunately, this characterization is incorrect.

The aim of the present note is to replace the incorrect characterization by the correct one. But beyond this aim, I would like to draw attention upon the very interesting concept of stable dominion introduced first in Isbell [67] (cf. also Bacsich [74]) and prove a characterization theorem of strong monos (which in POS(Z) coincide with the extremal monos) by means of the stable dominion - the same way regular monos are characterized by dominions in Herrlich-Strecker [73].

In order to recall and then correct Corollary 1 (on pg. 296) of Pasztor [82], we will recall some definitions and results. However, we will not use the notation of Pasztor [82], but a strongly improved one (cf. Pasztor [82a]).

First let us recall from Herrlich-Strecker [73] 34H the following: Let C be an arbitrary well-powered and complete category. Then any morphism $f: X \longrightarrow Y$ in C has a factorization $X \xrightarrow{g} D \xrightarrow{d} Y$, where d is a regular mono having the property that for every morphism r and s, if $f \cdot r = f \cdot s$ then $d \cdot r = d \cdot s$. (Warning: composition is written in the diagrammatic order.) The regular subobject (D,d) of Y is called the dominion of f and is denoted by Dmi f. The factorization $f = g \cdot d$ is called the dominion factorization of f.

The following is easy to prove:

Proposition 1: A morphism $f: X \longrightarrow Y$ of C is an epi iff Dmi f is isomorphic to $(Y, 1_Y)$, and is a regular mono iff Dmi f is isomorphic to (X,f).

Now let us turn to the category POS(Z), Z being an arbitrary, but fixed subset system. It is again easy to see that the dominion of a Z-continuous map $f: X \longrightarrow Y$ is (D,d), where $D = \{y \in Y : r(y) = s(y)$ whenever r and s are morphisms of POS(Z) with $r \restriction f(X) = s \restriction f(X)\}$ and d is the full embedding of D into Y. A map $f: X \to Y$ in POS(Z) is full iff $f(x) \leq f(x')$ implies $x \leq x'$ for all $x, x' \in X$. Note that the above characterization of dominions holds also in e.g. $Alg_{\Sigma}(Z)$ - the category of all Z-continuous Σ-algebras with all Z-continuous homomorphisms.

From Proposition 1 we know that in order to give a structural characterization of epis of a category C, it is enough to give a structural characterization of the dominions (of morphisms) in C. From Pasztor [82] we will now recall the structural characterization of dominions of POS(Z) (using basically the notations of Pasztor [82a]).

Let Y be a Z-complete poset and X a subset of Y. We define $<_X$ to be the least binary relation on Y satisfying the following three conditions: for every a,b,c and d $\in$ Y

(A) if $a = b \in X$, then $a <_X b$

(B) if $a \leq_Y b <_X c \leq_Y d$, then $a <_X d$

(C) if a is the supremum of a Z-set $A \subseteq Y$ and if for every $b \in A$ $b <_X c$, then $a <_X c$.

Now let $CL(X,Y) := \{a \in Y : a <_X a\}$. Notice that CL(X,Y) with the ordering of Y is also in POS(Z).

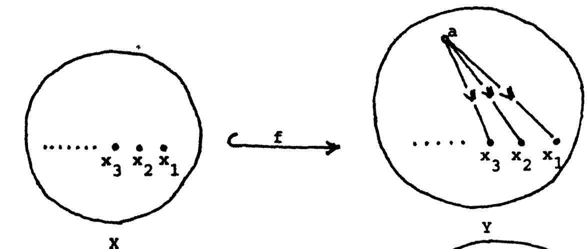
The main theorem of Pasztor [82] essentially proves the following (for a nice proof see Pasztor [82a]):

Proposition 2: Let $f: X \to Y$ be a morphism in POS(Z). The dominion of f in POS(Z) is (CL(f(X),Y),d), where d is the (identical) full embedding.

Corollary 1 on page 296 of Pasztor [82] states the following: A morphism $f: X \to Y$ of POS(Z) is an extremal mono iff it is full and CL(f(X),Y) = f(X). Compared with Proposition 1, this says that extremal monos and regular monos coincide in POS(Z). But this is not the case as proved in Lehmann-Pasztor [82] Theorem 4. So the statement of Corollary 1 is false. By Proposition 1 however, the following is true:

<u>Proposition 3</u>: A morphism $f:X \rightarrow Y$ of POS(Z) is a <u>regular mono</u> iff it is full and $CL(f(X),Y) = f(X)$.

Let us look at the proof of Theorem 4 of Lehmann-Pasztor [82]. It says that in POS(ω) both the embeddings



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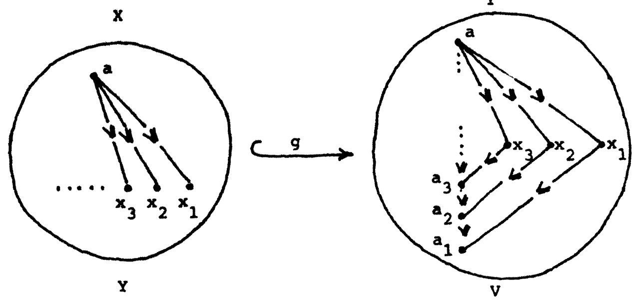

are regular monos, but their composition $f \cdot g$ is not. Indeed, the dominion of f is $(f(X),d)$ with d the full embedding of $f(X)$ into Y and the dominion of g is $(g(Y),d')$ with d' the full embedding of $g(Y)$ into V. But the dominion of $f \cdot g$ is $(g(Y),d')$ and $g(Y) \neq g(f(X))$. Since regular monos are also extremal and extremal monos are closed under composition, $f \cdot g$ must be an extremal mono. So at this point the problem of finding a structural characterization of the extremal subobjects of POS(Z) is still open. Notice also that in POS(Z) (and in well-powered and complete categories like our category C in the beginning in general) extremal and strong subobjects coincide (for the definitions see Herrlich-Strecker [73]).

Following Isbell [67] and Bacsich [74], we define the following: Let C again be a well-powered and complete category. Let $f:X\to Y$ be a morphism in C and $X\xrightarrow{g}D\xrightarrow{d}Y$ its dominion factorization. Denote g by g^1, D by D^1 and d by d^1. For any ordinal α we define a factorization $X\xrightarrow{g^{\alpha+1}}D^{\alpha+1}\xrightarrow{d^{\alpha+1}}Y$ of f as follows: $X\xrightarrow{g^{\alpha+1}}D^{\alpha+1}\xrightarrow{h}D^{\alpha}$ is the dominion factorization of g^{α} and $d^{\alpha+1}:=$ $:=h\cdot d^{\alpha}$. For any limit ordinal α we define a factorization $X\xrightarrow{g^{\alpha}}D^{\alpha}\xrightarrow{d^{\alpha}}Y$ of f as follows: (D^{α},d^{α}) is the intersection of $(D^{\beta},d^{\beta})_{\beta<\alpha}$ and g^{α} is the unique morphism with the property that $g^{\alpha}\cdot d^{\alpha}=f$. For every α we call (D^{α},d^{α}) the α th dominion of f and denote it by $\overset{\alpha}{\mathrm{Dmi}}\ f$. Since C is well-powered, there is a least ordinal α such that (D^{α},d^{α}) is isomorphic to (D^{β},d^{β}) for each $\beta>\alpha$. Let us denote this (D^{α},d^{α}) by (D^{∞},d^{∞}) and call it the stable dominion $(\overset{\infty}{\mathrm{Dmi}}\ f)$ of f. We denote g^{α} by g^{∞} and call $g^{\infty}\cdot d^{\infty}$ the stable dominion factorization of f. We can now prove the following

<u>Proposition 4</u>: Let C be as above. A morphism $f:X\longrightarrow Y$ of C is an extremal mono iff (D^{∞},d^{∞}) is isomorphic to (X,f).

<u>Proof</u>: 1) Suppose (D^{∞},d^{∞}) is isomorphic to (X,f). Let $f=$ $=e\cdot h$ with e epi. By Herrlich-Strecker [73] 34H/(e), $\overset{\infty}{\mathrm{Dmi}}\ h$ is isomorphic to (D^{∞},d^{∞}), hence, if we denote the stable dominion factorization of h by $g_h^{\infty}\cdot d_h^{\infty}$, we obtain that $e\cdot g_h^{\infty}$ is an isomorphism. This proves that e is also an isomorphism.

2) Suppose now that $f:X\to Y$ is an extremal mono. Let $g^{\infty}\cdot d^{\infty}$ be the stable factorization of f. By the definition of d^{∞} Dmi g^{∞} is isomorphic to $(D^{\infty},1_{D^{\infty}})$, hence by Proposition 1 g^{∞} is an epimorphism. So g^{∞} is an isomorphism and (X,f) is isomorphic to (D^{∞},d^{∞}). □

We know from Herrlich-Strecker [73] 34.5 that every complete and well-powered category C is an (epi, extremal mono) category. The above proof on the other hand, also proves the following

Corollary 5: Let C be a complete and well-powered category. Then the stable dominion factorization of its morphisms coincides with their epi-extremal mono factorization.

Coming back to POS(Z), for any Z-complete poset Y and subset $X \subseteq Y$, let us denote CL(X,Y) by $CL^1(X,Y)$. For any ordinal α let $CL^{\alpha+1}(X,Y) := CL(X, CL^{\alpha}(X,Y))$ and for any limit ordinal let $CL^{\alpha}(X,Y) := \bigcap_{\beta<\alpha} CL^{\beta}(X,Y)$. Obviously there is a least ordinal α such that $CL^{\beta}(X,Y) = CL^{\alpha}(X,Y)$ for all $\beta > \alpha$. Let us denote this ordinal by ∞. In view of Proposition 2 we now obtain the structural characterization of the extremal subobjects in POS(Z).

Proposition 6: A morphism $f: X \rightarrow Y$ in POS(Z) is an extremal mono iff it is full (i.e. $f(x) \leq f(y)$ implies $x \leq y$ for all $x, y \in X$) and $CL^{\infty}(f(X),Y) = f(X)$. □

R e f e r e n c e s

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(Oblatum 16.7. 1985)