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ON THE SEPARATION OF CLOSED, CONVEX SETS

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Abstract: A generalized version of Dieudonné's theorem on the closedness of the sum of two convex sets is proved. Using that a new separation theorem for disjoint, closed, convex (possibly unbounded) sets is established. The tools used come from convex analysis.

Key words: Convex function, conjugate function, subdifferential, separation.

Classification: 46A50, 46A55

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1. Introduction. The purpose of this note is to extend the Dieudonné's well known result on the closedness of the sum of two convex sets and provide an extended separation principle for disjoint, closed, convex (possibly unbounded) sets. Our proof is based on techniques of convex analysis and in particular on the characterization of relative continuity points of convex functions in terms of local compactness properties of the conjugate functions.

2. Preliminaries. In this section we would like to introduce a few basic notions from convex analysis and develop to auxiliary lemmata that we will need in the proof of our main result.

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Let X be a set, $f:X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$ a function. The epigraph of $f(\cdot)$ is the set $\text{epi} f = \{(x, \lambda) \in X \times \mathbb{R} : f(x) \leq \lambda\}$. The effective domain of $f(\cdot)$ is the set $\text{dom } f = \{x \in X : f(x) < +\infty\}$. The function $f(\cdot)$ is said to be proper if and only if $f(\cdot)$ is not identically $+\infty$ and $f(x) > -\infty$ for every $x \in X$. The indicator function of a set $A \subseteq X$ is the map $\delta_A : X \to \overline{\mathbb{R}}$ defined by

$$\delta_A(x) = \begin{cases} 0 \text{ if } x \in A \\ +\infty \text{ if } x \notin A. \end{cases}$$

Let (X, X^*) be a dual pair of spaces with a separating (Hausdorff) bilinear functional $(\cdot,\cdot)$. From now on we will stay within this dual system. We say that in order to avoid distinguishing in the sequel between polars and prepolars of sets and conjugates and preconjugates of functions. Let $f:X \to \overline{\mathbb{R}}$ be a function. The conjugate of $f(\cdot)$ is defined by $f^*:X^* \to \mathbb{R}: x^* \to \sup_{x \in X} [(x^*,x) - f(x)]$. Similarly if $g:X^* \to \overline{\mathbb{R}}$ we define $g^*:X \to \overline{\mathbb{R}}: x \to \sup_{x^* \in X^*} [(x^*,x) - g(x^*)]$. Clearly $f^*(\cdot)$ is a convex and $w(X^*,X)$-lower semicontinuous since it is the supermum of the $w(X^*,X)$-continuous affine functions $x^* \to (x^*,x) - f(x)$ over all $x \in \text{dom } f$. Similarly, for $g(\cdot)$ it follows that $g^*(\cdot)$ is convex and $w(X,X^*)$-l.s.c. From the Hahn-Banach separation theorem it follows that when $\overline{\text{conv}}\, f$ = lower semicontinuous, convex hull of $f(\cdot)$ is proper, then $f^{**} = \overline{\text{conv}}\, f$.

We use the following notation. If $A \subseteq X$ then int A, cor A, ri A, r cor A, cl A, span A, aff A, conv A denote the interior of A, the algebraic interior or core of A, the relative interior of A, the relative algebraic interior or relative core of A, the closure of A, the span of A, the affine hull of A and the convex hull of A. By relative interior of A we mean the interior of A in the relative topology of X on aff A; that is $x \in \text{ri } A$ if and only if there is a 0-neighborhood U s.t. $(x + U) \cap \text{aff } A \subseteq A$. Similarly $x \in \text{r cor } A$ if and only if $x \in A$ and $A - x$ absorbs aff $A - x$ or equivalently if and only if $x + \mathbb{R}_+ \cdot A \supseteq A$ and $x \in A$. Recall that aff A = $= A + \text{span}(A-A) = x_0 + \text{span}(A-x_0)$ where $x_0 \in A$.

Let $A \subseteq X$. We set

$A^+ = \{x^* \in X^* : (x^*,x) \geq 0$ for all $x \in A\}$

$A^- = \{x^* \in X^* : (x^*,x) \leq 0$ for all $x \in A\}$

$A^\perp = A^+ \cap A^-$

Similarly for $B \subseteq X^*$ the sets B^+, B^-, $B^\perp$ are defined in X in the same way. If $A \neq \emptyset$ then $(A^+)^+ = \text{cl}\, \mathbb{R}_+ \cdot \text{conv}\, A$, $(A^\perp)^\perp = \text{cl span}\, A$ and $A + ((A-A)^\perp)^\perp = \text{cl aff}\, A$. Again for $A \subseteq X$ nonempty we define its recession cone to be the set A_∞ of all half-lines contained in cl conv A. So $x \in A_\infty$ if and only if for any fixed point $a \in A$ the half-line $a + \mathbb{R}_+ x$ starting at a and passing through x is entirely contained in cl conv A. Clearly then A_∞ is a closed, convex cone with vertex at the origin. The recession function $f_\infty(\cdot)$ of $f: X \to \bar{\mathbb{R}}$ is that function whose epigraph is the recession cone of epif i.e. $\text{epi} f_\infty = (\text{epi} f)_\infty$. It can be shown that $f_\infty(x) = \sup_{x^* \in \text{dom} f^*} (x^*,x) = \sigma_{\text{dom} f^*}(x)$ (see Laurent [3] and Rockafellar [4]).

Finally the subdifferential of $f: X \to \bar{\mathbb{R}}$ at x is defined to be

$\partial f(x) = \{x^* \in X^* : (x^*, y-x) \leq f(y) - f(x)$ for all $y \in X\}$

Observe that $\partial f(x) = \emptyset$ when $f(x) = +\infty$ (assuming $f \not\equiv +\infty$). The subdifferential at a point is always convex and $w(X^*,X)$-closed.

Now we are ready for the auxiliary results that we will need in the proof of the main theorem in Section 3. In both we prove more than we will actually need and so they are also interesting in their own.

<u>Lemma 1</u>. <u>If</u> (X,X^*) is a dual pair, $f: X \to \bar{\mathbb{R}}$ is convex and ri epi$f \neq \emptyset$ <u>then</u> $f(\cdot)$ is a continuous on r cor dom f and the following are equivalent for $x_0 \in X$.

1) $f(\cdot)$ is relatively continuous at $x_0 \in$ dom f

2) $\{x^* \in X^*: (f^*)_\infty (x^*) - (x^*, x_0) \le 0\}$ is a subspace.

Proof. From Rockafellar [5] we know that 1) is equivalent to saying that $x_0 \in$ ri dom f. Now we claim that this is equivalent to saying that $[\text{dom } f - x_0]^- = [\text{dom } f - x_0]^\perp = \{x^* \in X^*: x^* = \text{constant}$ on dom f$\}$. To see that let $D = \text{dom } f - x_0$. Clearly D is convex and has a nonempty relative interior. Hence by the Hahn-Banach separation and extension theorems, we have that $0 \notin$ ri D if and only if there exists $x^* \in X^*$ such that x^* is not constant on aff D = = aff dom f - x_0 and $\sup_{x \in D} (x^*, x) \le 0$; equivalently $x^* \in D^- = [\text{dom } f - x_0]^-$ and $x^* \notin D^\perp = [\text{dom } f - x_0]^\perp$. Thus we deduce that 1) is equivalent to saying that $[\text{dom } f - x_0]^-$ is a subspace. Next note that:

$$\{x^* \in X^*: (f^*)_\infty (x^*) - (x^*, x_0) \le 0\}$$
$$= \{x^* \in X^*: \sup_{x \in \text{dom } f^{**}} (x^*, x) - (x^*, x_0) \le 0\}$$
$$= [\text{dom } f^{**} - x_0]^-$$

Now dom $f^{**} \subseteq$ cl dom f, since $f^{**}(\cdot) + \delta(\cdot)_{\text{cl dom } f}$ is a convex lower semicontinuous function dominated by f and so $f^{**}(\cdot) + \delta(\cdot)_{\text{cl dom } f} \le f^{**}(\cdot)$. Recall that $f^{**}(\cdot) \le f(\cdot)$, hence dom f $\subseteq$ dom f^{**}. Therefore

$$\text{dom } f - x_0 \subseteq \text{dom } f^{**} - x_0 \subseteq \text{cl dom } f - x_0$$

and so $[\text{dom } f - x_0]^- \supseteq [\text{dom } f^{**} - x_0]^- \supseteq [\text{cl dom } f - x_0]^-$. But $[\text{dom } f - x_0]^- = [\text{cl dom } f - x_0]^-$. So $[\text{dom } f - x_0]^- = [\text{dom } f^{**} - x_0]^- = \{x^* \in X^*: (f^*)_\infty (x^*) - (x^*, x_0) \le 0\}$ = subspace. The continuity of $f(\cdot)$ on r cor dom f is a well known result.

Q.E.D.

In [5] Rockafellar has shown that continuity of a convex function at a given point is equivalent to equicontinuity of certain level sets of the conjugate function. These results can be easily extended to show the equivalence between relative continuity of a convex function with respect to a closed affine set of finite codimension (usually this set is aff dom f) and local equicontinuity of the level sets of the conjugate function. Finally if aff dom f is not of finite codimension then the level sets of the conjugate will contain the infinite dimensional subspace $(\text{dom } f - \text{dom } f)^{\perp}$ and so we cannot hope for local equicontinuity. However, by characterizing the level sets of f^* modulo their behavior on $(\text{dom } f - \text{dom } f)^{\perp}$ i.e. by considering the duality between aff dom f and $X^*/_{(\text{dom } f-\text{dom} f)^{\perp}}$ we obtain in a straightforward manner a further extension of the results of Rockafellar [5].

Following this chain of arguments we can have the following extended version of Rockafellar's results. Let $f:X \to \mathbb{R}$ be convex and $M \subseteq X$ affine and containing dom f. The case of interest is when $M = \text{aff dom } f$ or $M = \text{dom } f + (\text{dom } f - \text{dom } f)^{\perp} = \text{cl aff dom } f$.

Lemma 2. The following two statements are equivalent

1) ri epif $\neq \emptyset$ and aff dom f is closed with finite codimension in M.

2) $f^* \equiv +\infty$ or there exists $x_0 \in X$, $r_0 > -f(x_0)$ s.t $\{x^* \in X^* : f^*(x^*) - (x^*, x_0) \leq r_0\}$ is $w(X^*,X)/M^{\perp}$ - locally bounded.

Remark: Clearly 1) is equivalent to the following:

1') r cor conv dom f $\neq \emptyset$, conv $f|_{\text{r cor conv dom } f}$ is continuous and aff dom f is closed with finite codimension in M.

3. Main result. In this section we develop a general criterion for the sum of two closed, convex sets to be closed, extending in this context Dieudonné's theorem (see [2]) and we also obtain a new separation principle. In what follows let

$B_\varepsilon = \{x \in X : d_B(x) = \inf_{y \in B} \|x - y\| < \varepsilon\} = B + \varepsilon \mathring{B}_X$ where $\mathring{B}_X$ is the open unit ball in X.

So assume that X is a reflexive Banach space.

<u>Theorem</u>. <u>If</u> A, B are closed, convex subsets of X satisfying

1) $A_\infty \cap B_\infty$ is a subspace M

2) For some $\varepsilon > 0$ $A \cap B_\varepsilon$ is nonempty and $w(x,x^*)/M$-locally bounded .

<u>Then</u> A - B is closed. In particular if A and B are disjoint then they can be strongly separated i.e. there exists $x^* \in X^*$ s.t.

$\inf_{a \in A} (x^*,a) > \sup_{b \in B} (x^*,b)$

<u>Proof</u>. Assume that A, B are nonempty. Let $z \notin A - B$. We will show that $z \notin$ cl (A-B) or equivalently $d_{A-B}(z) = \inf_{\substack{a \in A \\ b \in B}} \|z - (a-b)\| > 0$. By translation we may assume that z = 0. Define $f(\cdot):X \longrightarrow \overline{\mathbb{R}}$ by

$f(x) = d_B(x) + \delta_A(x) = \inf_{b \in B} \|x-b\| + \delta_A(x)$

Recall that $d_B(\cdot)$ is Lipschitz and convex since B is convex. Also since A is closed, convex, $\delta_A(\cdot)$ is a convex, lower semicontinuous function. Thus $f(\cdot)$ is proper convex and lower semicontinuous. We have

$f^*(\cdot) = [d_B(\cdot) + \delta_A(\cdot)]^*$

Using Theorem 6.5.8 of Laurent [3] we have that

$[d_B(\cdot) + \delta_A(\cdot)]^* = (d_B^* \,\square\, \delta_A^*)(\cdot)$

where $\square$ indicates the operation of infimal convolution.

Observe that $d_B(\cdot) = (\|\cdot\| \,\square\, \delta_B)(\cdot)$. So $d_B^*(\cdot) = (\|\cdot\| \,\square\, \delta_B)^*(\cdot) = \|\cdot\|^* + \delta_B^*(\cdot)$ (Theorem 6.5.4 of Laurent [3]). It is easy to check that $\|\cdot\|^* = \delta_{B_{X^*}}(\cdot)$ where B_{X^*} is the unit ball in X^* . Also recall that $\delta_A^*(\cdot) = \sigma_A(\cdot)$ and $\delta_B^*(\cdot) = \sigma_B(\cdot)$ where $\sigma_\bullet(\cdot)$ denotes the support function of the corresponding set. So finally we have:

$$f^*(x^*) = [(\sigma_{B_{X^*}} + \sigma_B) \,\Box\, \sigma_A](x^*) \Rightarrow f^*(x^*) =$$

$$= \inf_{y^* \in X^*} [\sigma_{B_{X^*}}(y^*) + \sigma_B(y^*) + \sigma_A(x^* - y^*)] = \inf_{y^* \in B_{X^*}} [\sigma_B(y^*) +$$

$$+ \sigma_A(x^* - y^*)] = \inf_{y^* \in B_{X^*}} [\sup_{b \in B} (y^*, b) + \sup_{a \in A} (x^* - y^*, a)] =$$

$$= \inf_{y^* \in B_{X^*}} \sup_{\substack{a \in A \\ b \in B}} [(x^*, a) + (y^*, b-a)]$$

Recall that B_{X^*} is w-compact (Alaoglus theorem plus the reflexivity of X). So we can apply Nikaido's minsup theorem (see Aubin [1] p. 217) and get that

$$f^*(x^*) = \sup_{\substack{a \in A \\ b \in B}} \inf_{y^* \in B_{X^*}} [(x^*, a) + (y^*, b-a)] = \sup_{\substack{a \in A \\ b \in B}} [(x^*, a) - \| b-a \|].$$

We will now show that hypotheses 1) and 2) of the theorem are sufficient to prove that $f^*(\cdot)$ is relatively continuous at 0. By Lemma 1 ($2 \rightarrow 1$) and Lemma 2 ($2 \rightarrow 1$) it suffices to show that a level set of $f(\cdot)$ is locally bounded in the topology $w(X,X^*)/M$ where $M = A_\infty \cap B_\infty = \{x \in X : f_\infty(x) \leq 0\}$ = a subspace. But the level sets of $f(\cdot)$ are precisely $\{x \in X : f(x) \leq \varepsilon\} = A \cap B_\varepsilon$ for $\varepsilon > 0$ which by hypothesis 2) is $w(X,X^*)/M$-locally bounded for some $\varepsilon > 0$. Hence 1) and 2) are indeed the required conditions.

So $f^*(\cdot)$ is relatively continuous at 0 and this implies that $\partial f^*(0)\ \emptyset$. This in turn, means that there exists $\hat{x} \in X$ s.t. $\hat{x} \in \partial f^*(0)$ $\Longleftrightarrow 0 \in \partial f^{**}(\hat{x}) = \partial f(\hat{x})$ (since $f(\cdot)$ is proper, convex and lower semicontinuous). But we know from convex analysis that $0 \in \partial f(\hat{x})$ means that $f(\cdot)$ achieves its minimum at $\hat{x}$. So $\inf_{\substack{x \in A \\ b \in B}} \|x-b\| = \inf_{b \in B} \|\hat{x}-b\| > 0$, where the last inequality holds since $\hat{x} \notin B$ (recall that since $0 \notin A - B$, $A \cap B = \emptyset$) and B is closed. Therefore we have shown that A - B is closed. Finally since $0 \notin \text{cl}\,(A-B)$, A, B can be strictly separated.

<u>Q.E.D.</u>

Remark: If $A_\infty \cap B_\infty$ is a subspace and A is locally bounded then conditions 1) and 2) follow immediately. In Dieudonné's theorem (see [2]) $A_\infty \cap B_\infty$ is $\{0\}$ and A is locally bounded.

The idea of determining continuity points of convex functions through the local equicontinuity of the level sets of its conjugate was applied by the author successfully to problems of optimal control and approximation theory (minimum norm extremals and spline problems). These applications will appear in a forthcoming paper.

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