COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

27,1 (1986)

DESTABILIZING EFFECT OF UNILATERAL CONDITIONS IN REACTION-DIFFUSION SYSTEMS

Milan KUČERA, Jiří NEUSTUPA

Abstract: Stationary solutions of reaction-diffusion systems with unilateral constraints are considered. It is shown in examples that the spatially homogeneous stationary solution of the system with unilateral constraints can be unstable even for diffusion coefficients for which it is stable as a solution of the classical problem with Neumann conditions. A general result of this type is announced.

Key words: Reaction-diffusion system, unilateral conditions, inequalities, destabilization, spatially homogeneous stationary solution, eigenvalue.

Classification: 35B30, 35B35, 35P30

Introduction. Consider a reaction- diffusion system of the type

$$\text{(RD)}\qquad \begin{aligned} \frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + f(u,v) \\ \frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + g(u,v) \end{aligned} \qquad \text{for } [t,x] \in \langle 0,\infty) \times \langle 0,1\rangle$$

where f, g are real functions on $\mathbb{R}^2$, d_1, d_2 are positive parameters (diffusion coefficients). First, consider Neumann boundary conditions

$$\text{(NC)}\qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = \frac{\partial v}{\partial x}(t,0) = \frac{\partial v}{\partial x}(t,1) = 0 \text{ on } \langle 0,\infty).$$

In some cases connected with chemical and biological models there exists a stationary spatially homogeneous (constant) solution $\bar{u},\bar{v}$ of (RD), (NC) (i.e. $f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0$) which is stable only for some parameters d_1, d_2 (lying in so called domain of stability). See e.g. [10],[11].

The aim of this paper is to show how this situation can change by introducing unilateral conditions. We give simple examples and announce one general result showing that $\bar{u}$, $\bar{v}$ can be an unstable solution of (RD) or at least of its linearization with unilateral conditions even for some d_1, d_2 lying in the domain of stability of (RD), (NC). One of the simplest examples of unilateral constraints we have on mind are boundary conditions

$$(1) \quad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \quad \frac{\partial v}{\partial x}(t,0) = 0,$$

$$v(t,1) \geq \bar{v}, \ \frac{\partial v}{\partial x}(t,1) \geq 0, \ (v(t,1)-\bar{v})\frac{\partial v}{\partial x}(t,1) = 0 \text{ on } \langle 0 \ +\infty).$$

The stability of a stationary solution of any initial value problem (e.g. of (RD), (NC) or (RD) with some unilateral conditions) is understood in the following sense: $\bar{u}$, $\bar{v}$ is stable with respect to a given norm $\|\cdot\|$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution u, v of the problem considered satisfying $\|u(t_0,\cdot)-\bar{u}\| < \delta$, $\|v(t_0,\cdot)-\bar{v}\| < \delta$ for some t_0 is defined on $\langle t_0,+\infty)$ and $\|u(t,\cdot)-\bar{u}\| < \varepsilon$, $\|v(t,\cdot)-\bar{v}\| < \varepsilon$ for all $t \in \langle t_0,+\infty)$; $\bar{u},\bar{v}$ is unstable if it is not stable. In all examples below, we shall show an instability with respect to any arbitrary norm.

In what follows we shall suppose $\bar{u} = \bar{v} = 0$ without loss of generality.

We shall study mainly a destabilizing influence of unilateral conditions for the linearization

$$(RD_L) \quad \begin{aligned} \frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + b_{11}u + b_{12}v \\ \frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + b_{21}u + b_{22}v \end{aligned}$$

of (RD), where $b_{11} = \frac{\partial f}{\partial u}(\bar{u},\bar{v})$, $b_{12} = \frac{\partial f}{\partial v}(\bar{u},\bar{v})$, $b_{21} = \frac{\partial g}{\partial u}(\bar{u},\bar{v})$, $b_{22} = \frac{\partial g}{\partial v}(\bar{u},\bar{v})$.

The main idea is to show the existence of a positive eigenvalue of the problem

$$(RD_\lambda) \qquad \begin{aligned} d_1 \frac{\partial^2 u}{\partial x^2} + b_{11}u + b_{12}v &= \lambda u \\ d_2 \frac{\partial^2 v}{\partial x^2} + b_{21}u + b_{22}v &= \lambda v \end{aligned}$$

with the corresponding unilateral conditions and with some d_1, d_2 from the domain of stability of (RD), (NC). The instability of the trivial solution of (RD_L) with the corresponding unilateral conditions is an easy consequence. In one situation we shall show that it means also the instability for the original nonlinear system (RD) with unilateral conditions.

Notice that some abstract results of the mentioned type are proved in [5],[6]. An influence of unilateral constraints to the position of the first bifurcation point of the corresponding stationary problem is studied in [4],[9].

<u>Remark 1</u>. (Domains of stability and instability - classical case. See e.g. [10],[11].) Let us denote

$$B = \begin{pmatrix} b_{11}, b_{12} \\ b_{21}, b_{22} \end{pmatrix}, \quad D = \begin{pmatrix} d, 0 \\ 0, 1 \end{pmatrix}.$$

Suppose that the assumption

(2) $b_{11} > 0$, $b_{21} > 0$, $b_{12} < 0$, $b_{22} < 0$, $b_{11} + b_{22} < 0$, $\det B > 0$

is fulfilled. This is true in some models from chemistry and biology where u and v represents a density of an activator (or a prey) and a density of an inhibitor (or a predator), respectively. Remember that the trivial solution of (RD_L), (NC) is stable if all eigenvalues of (RD_λ), (NC) have nonpositive real parts and it is unstable if there is an eigenvalue of (RD_λ), (NC) with a positive real part. Any solution $U = [u,v]$ of (RD_λ), (NC) can be written as

$$U(x) = \sum_{n=0}^{\infty} C_n \cos n\pi x$$

(with some $C_n \in \mathbb{R}^2$, $n = 0,1,2,\ldots$) and therefore (RD_λ), (NC) is equivalent to the system

$$[(n\pi)^2 D - B + \lambda E] \; C_n = 0, \; n = 0,1,2,\ldots \; .$$

It follows that λ is an eigenvalue of (RD_λ), (NC) if and only if

$$\text{(3)} \qquad \lambda^2 - [(b_{11} + b_{22}) - (d_1 + d_2)(n\pi)^2]\lambda + \\ + [b_{11} - (n\pi)^2 d_1] \cdot [b_{22} - (n\pi)^2 d_2] - b_{12}b_{21} = 0$$

for some n.

In this case $U(x) = C_n \cos n\pi x$ is the corresponding eigenvector. Denote by Γ_n the hyperbolic curve in the quadrant d_1, $d_2 \geq 0$ given by

$$[b_{11} - (n\pi)^2 d_1] \cdot [b_{22} - (n\pi)^2 d_2] - b_{12}b_{21} = 0$$

($n = 1,2, \ldots$) and let Γ be the envelope of Γ_n, $n = 1,2,\ldots$ (see Fig. 1). Elementary investigation of the roots of (3) shows that for any $d_1, d_2 \geq 0$ all the complex eigenvalues of (RD_λ), (NC) have negative real parts; for d_1, d_2 on the right from Γ all the real eigenvalues are negative; for d_1, d_2 on the left from Γ there exists a positive real eigenvalue of (RD_λ), (NC). Hence, Γ divides the quadrant $d_1, d_2 \geq 0$ onto the domain of stability (on the right from Γ) and the domain of instability (on the left from Γ). See Fig. 1.

<u>Example 1</u>. Let (2) hold. Consider (RD_L) with unilateral conditions

$$u_x(t,0) = v_x(t,0) = 0,\ u(t,1) \geq 0,\ v(t,1) \geq 0,$$

(4)

$$u_x(t,1) \geq 0,\ v_x(t,1) \geq 0,\ u(t,1)u_x(t,1) = v(t,1)v_x(t,1) = 0.$$

The corresponding boundary conditions for the eigenvalue problem (RD_λ) read

$$u_x(0) = v_x(0) = 0,\ u(1) \geq 0,\ v(1) \geq 0,$$

(4´)

$$u_x(1) \geq 0,\ v_x(1) \geq 0,\ u(1)u_x(1) = v(1)v_x(1) = 0.$$

Let us search for nonnegative eigenvalues of (RD_λ), (4´) correspording only to eigenvectors of the form $[u_k, v_k]$ with

(5) $$v_k(x) = (-1)^{k+1} \cos(k + \tfrac{1}{2})\, \pi x.$$

It follows from the second equation in (RD_λ) that

(6) $$u_k(x) = (-1)^{k+1} \left[\frac{d_2}{b_{21}}(k + \tfrac{1}{2})^2 \pi^2 - \frac{b_{22}-\lambda}{b_{21}} \right] \cos(k + \tfrac{1}{2})\, \pi x.$$

The boundary conditions (4´) are fulfilled with respect to (2) for $\lambda \geq 0$. It is easy to see (by substituting into (RD_L)) that $\lambda \geq 0$ is an eigenvalue corresponding to u_k, v_k if and only if

(7)
$$\lambda^2 - [(b_{11} + b_{22}) - (d_1 + d_2)(k + \tfrac{1}{2})^2 \pi^2]\, \lambda + [b_{11} - (k + \tfrac{1}{2})^2 \pi^2 d_1] \cdot [b_{22} - (k + \tfrac{1}{2})^2 \pi^2 d_2] - b_{12}b_{21} = 0.$$

Denote by $\tilde{\Gamma}_k$ $(k = 0,1,\ldots)$ the hyperbolic curve in the quadrant $d_1, d_2 \geq 0$ defined by

$$[b_{11} - (k + \tfrac{1}{2})^2 \pi^2 d_1] \cdot [b_{22} - (k + \tfrac{1}{2})^2 \pi^2 d_2] - b_{12}b_{21} = 0$$

and let $\tilde{\Gamma}$ be the envelope of $\tilde{\Gamma}_k$, $k = 0,1,2,\ldots$ (see Fig. 2). It follows from an elementary investigation of the roots of (7) that for any d_1, d_2 on the left from $\tilde{\Gamma}$ there exists a positive eigenvalue λ of (RD_λ), (4´) with the corresponding eigenvector u_k, v_k from (5),(6) for some k. The couple $u(t,x) = \exp(\lambda t) \cdot$

$\cdot \varepsilon u_k(x)$, $v(t,x) = \exp(\lambda t) \cdot \varepsilon v_k(x)$ with an arbitrary small $\varepsilon > 0$ is a classical solution of (RD_L),(4) satisfying $\lim_{t\to+\infty} \|u(t,\cdot)\| = \lim_{t\to+\infty} \exp(\lambda t)\cdot\varepsilon\|u_k\| = +\infty$, $\lim_{t\to+\infty}\|v(t,\cdot)\| = +\infty$ for any arbitrary norm $\|\cdot\|$. Hence, for d_1, d_2 on the left from $\tilde{\Gamma}$, the trivial solution of (RD_L),(4) is unstable. A comparison of $\tilde{\Gamma}$ with Γ from Remark 1 shows that there is a nonempty intersection of the domain of instability of (RD_L),(4) with the domain of stability of the classical problem (see Fig. 2).

Example 2.(A simple free boundary problem.) Consider the problem

(8) $$u_t = d_1 u_{xx} + b_{11}u + b_{12}v \text{ on } \langle 0,\infty) \times \langle 0,1\rangle,$$

(9) $$\begin{cases} v_t - d_2 v_{xx} - b_{21}u - b_{22}v \geq 0, \quad v \geq 0, \\ (v_t - d_2 v_{xx} - b_{21}u - b_{22}v)v = 0 \quad \text{a.e. on } \langle 0,\infty) \times \langle 0,1\rangle, \end{cases}$$

(10) v_x is continuous on $\langle 0,+\infty) \times \langle 0,1\rangle$, (NC) holds,

where (8) can be understood in the classical sense, (9) should be fulfilled a.e. on $\langle 0,\infty) \times \langle 0,1\rangle$ with the derivative v_t existing a.e. on $\langle 0,\infty) \times \langle 0,1\rangle$. It is another formulation of the free boundary problem

(11) $$\begin{aligned} &u_t = d_1 u_{xx} + b_{11}u + b_{12}v \text{ on } \langle 0,\infty) \times \langle 0,1\rangle, \\ &v_t = d_2 v_{xx} + b_{21}u + b_{22}v \text{ on } Q_+, \\ &v = 0 \text{ on } \langle 0,\infty) \times \langle 0,1\rangle \setminus Q_+, \; v_x \text{ is continuous on} \\ &\qquad \langle 0,\infty) \times \langle 0,1\rangle, \text{ (NC) holds,} \end{aligned}$$

where the domain $Q_+ = \{[t,x] \in \langle 0,\infty) \times \langle 0,1\rangle;\ v(t,x) > 0\}$ is unknown. We shall neither give here a precise definition of the solution (this will be given in a more general setting in Remark 5) nor discuss its existence and properties. We want to show only that under the assumption

(12) $b_{11} > 0,\ b_{21} \geq 0$

spatially homogeneous classical solutions tending to infinity (as $t \to +\infty$) start in any neighbourhood of the origin. For this, it is sufficient to realize that for any $\xi < 0$ the couple $u(t,x) = \exp(b_{11}t)\xi$, $v(t,x) = 0$ satisfies (8)-(10) classically. It follows that the trivial solution of (8)-(10) (i.e. of (11)) is unstable for any d_1, d_2 (even for those from the domain of stability of (RD_L), (NC) under the assumption (2) - see Remark 1). Moreover, this is an instability even with respect to spatially constant solutions only.

Remark 2. (A motivation for inequalities in $\mathbb{R}^2$.) A couple of functions $u(t)$, $v(t)$ is a spatially constant solution of (8)-(10) if and only if

$$(13)\qquad \begin{aligned} & u_t - b_{11}u - b_{12}v = 0 \quad \text{on } \langle 0,\infty), \\ & v \geq 0,\ (v_t - b_{21}u - b_{22}v)(\psi - v) \geq 0 \text{ for all } \psi \in \langle 0,\infty),\ \text{a.a. } t \in \langle 0,\infty). \end{aligned}$$

This system can be written also in the form

$$(14)\qquad \begin{aligned} & U(t) \in K \quad \text{for all } t \in \langle 0,\infty), \\ & \langle U_t(t) - BU(t), \Psi - U(t)\rangle \geq 0 \text{ for all } \Psi \in K, \text{a.a. } t \in \langle 0,\infty), \end{aligned}$$

where $U = [u,v]$, $K = K_v^+ = \{\Psi = [\varphi, \psi] \in \mathbb{R}^2; \psi \geq 0\}$, $\langle \cdot,\cdot \rangle$ is the scalar product in $\mathbb{R}^2$ Hence, the study of spatially constant solutions of (8)-(10) can be a motivation for an investigation of the inequality (14) which has a good sense for any closed convex cone K in $\mathbb{R}^2$ with its vertex at the origin.

Remark 3. (Trajectories of inequalities in $\mathbb{R}^2$.) Let K be a closed convex cone in $\mathbb{R}^2$ with its vertex at the origin determined by half-lines k_1, k_2 (Fig. 3). Denote by K^0 and ∂K the interior and the boundary of K. For any $V \in \mathbb{R}^2$ let $U_V(t)$ be the solution of the system of equations

$$(15) \qquad U_t = BU$$

satisfying $U(0) = V$. If $V \in K$, $V \neq 0$ then the following cases are possible:

(a) $V + \tau BV \in K^0$ for all $\tau > 0$ sufficiently small;

(b) $V + \tau BV \notin K$ for all $\tau > 0$;

(c) $V + \tau BV \in \partial K$ for all $\tau > 0$ sufficiently small.

The conditions (a),(b) and (c) mean that the trajectory of (15) at V tends into K^0, outside of K and along ∂K, respectively. In the case (a) we have $U_V(t) \in K$ for $t \in \langle 0, t_0)$ with some $t_0 > 0$ and it follows that $U_V(t)$ is simultaneously a solution of (14) on $\langle 0, t_0)$. The cases (b),(c) can occur for $V \in \partial K$ only. If (b) holds then there exists λ_0 such that

$$(16) \qquad \langle \lambda_0 V - BV, \Phi - V \rangle \geq 0 \quad \text{for all } \Phi \in K.$$

Indeed, this condition means that the vector $\lambda_0 V - BV$ shifted to V tends into K and is perpendicular to the half-line k_i containing V (see Fig. 3). It is clear from Fig. 3 that this is fulfilled with some λ_0. In other words, in the case (b), V is an eigenvector corresponding to some eigenvalue λ_0 of the inequality (16). It follows (by substituting into (14) and using (16)) that $U_I(t) = \exp(\lambda_0 t)V$ is a solution of (14) on $\langle 0, +\infty)$. Moreover, $\lambda_0 > 0$ or $\lambda_0 < 0$ if $\langle BV, V \rangle > 0$ or $\langle BV, V \rangle < 0$, i.e. if the trajectory of (15) and the oriented half-line k_i enclose an acute or an obtuse angle, respectively; see Fig. 3, 4; it is $\lambda_0 = 0$ if $\langle BV, V \rangle = 0$, i.e. if the trajectory of (15) is perpendicular to ∂K at V.

In the case (c) we have $\lambda_0 V = BV$ for some λ_0, i.e. V is an eigenvector corresponding to some eigenvalue of B (and simultaneously of (16)). The function $U_V(t) = \exp(\lambda_0 t)V$ is simultaneously a solution of (14) and (15) on $\langle 0,\infty)$.

Now, for any $V \in K$, $V \neq 0$ we can set $t_V = \inf\{t \geq 0; U_V(t) \notin K\}$. (The cases $t_V = 0$, $t_V = +\infty$ are possible.) It follows from our considerations that the function

$$U_{K,V}(t) = U_V(t) \quad \text{for } t \in \langle 0,t_V),$$
$$\exp \lambda_0(t-t_V) \cdot U_V(t_V) \quad \text{for } t \geq t_V$$

is a solution of (14) on $\langle 0,\infty)$, where λ_0 (if $t_V < \infty$) is an eigenvalue of (16) corresponding to the eigenvector $V_0 = U_V(t_V)$. It is easy to see that it is a unique solution of (14) satisfying $U_{K,V}(0) = V$.

<u>Example 3</u>. (Destabilization for inequalities in $\mathbb{R}^2$ and a free boundary problem.) Suppose that $b_{11} > 0$, $b_{21} > 0$ and B has a pair of complex conjugate eigenvalues with negative real parts. Let $K = K_V^+ = \{[u,v] \in \mathbb{R}^2; v \geq 0\}$. In this case the character of trajectories of (14) and (15) is shown on Fig. 5. Any solution of (15) tends exponentially to the origin. Any solution of (14) touches the line $k_1 = \{[u,0]; u < 0\}$ and then tends to infinity in the direction of the eigenvector $[-1,0]$ corresponding to a positive eigenvalue of (16). (See Remark 3, where (b) with $\langle BV,V\rangle > 0$ holds for any $V \in k_1$.) It follows that any spatially constant solution of (8)-(10) tends to infinity in spite of that any spatially constant solution of (RD_L), (NC) tends to zero. Particularly, the trivial solution of (8)-(10) is unstable for any d_1, d_2 as we have known already from Example 2.

<u>Example 4</u>. (Stabilization for inequalities in $\mathbb{R}^2$ and a free boundary problem.) Suppose that $b_{12} < 0$, $b_{22} < 0$ and B has a pair of complex conjugate eigenvalues with positive real parts. Fig.6

shows the character of trajectories of (14) and (15) with $K = K_u^+ = \{[u,v] \in \mathbb{R}^2; u \geq 0\}$. Any solution of (14) tends to zero along the half-line $k_1 = \{[0,v]; v > 0\}$ in spite of that any nontrivial solution of (15) tends exponentially to infinity. Notice that $[0,1]$ is an eigenvector corresponding to a negative eigenvalue of (16). (See Remark 3, where (b) with $\langle BV,V\rangle < 0$ holds for any $V \in k_1$.) Now, consider the unilateral problem

$$(17) \quad \begin{cases} u_t - d_1 u_{xx} - b_{11}u - b_{12}v \geq 0, \ u \geq 0, \\ (u_t - d_1 u_{xx} - b_{11}u - b_{12}v)u = 0, \end{cases}$$

$$(18) \quad v_t = d_1 v_{xx} + b_{11}u + b_{12}v$$

(19) u_x are continuous on $\langle 0,\infty) \times \langle 0,1\rangle$, (NC) holds (cf. Example 2). Using considerations analogous to those from Remark 2 we obtain that any spatially constant solution of (17)-(19) tends to zero, i.e. the trivial solution of (17)-(19) is stable with respect to constant perturbations. Simultaneously, the trivial solution of (15) is unstable even with respect to spatially constant perturbations because any constant solution of (15) tends to infinity. Of course, we do not obtain any result about the stability for the inequality (with respect to nonhomogeneous perturbations).

Remark 4. (Eigenvalues and stability for inequalities in $\mathbb{R}^2$.) Using Remark 3 and the fact that the eigenvectors of the inequality (16) lying in K^0 coincide with those of the matrix B, it is easy to determine all the eigenvalues and eigenvectors of (16) for a given 2x2 matrix B. By a detailed examination of all possibilities of the behaviour of trajectories of (15) and using the considerations from Remark 3 it is possible to prove that the trivial solution of (14) is stable if and only if there is no positive eigenvalue of (16).

This assertion does not hold for inequalities in higher-dimensional spaces. (It is not difficult to find a counterexample in $\mathbb{R}^3$.) We have no analogue of a complex eigenvalue for an inequality.

Example 5. (Destabilization in a nonlinear case.) Consider the problem

(20) $u_t = d_1 u_{xx} + f(u,v)$ on $\langle 0,\infty) \times \langle 0,1\rangle$,

(21) $\begin{cases} v_t = d_2 v_{xx} - g(u,v) \geq 0, \; v \geq 0 \\ [v_t - d_2 v_{xx} - g(u,v)] \cdot v = 0 \text{ a.e. on } \langle 0,\infty) \times \langle 0,1\rangle \end{cases}$

(22) v_x are continuous on $\langle 0,\infty) \times \langle 0,1\rangle$, (NC) holds

representing a free boundary problem (cf. Example 2). If $f(0,0) = g(0,0) = 0$ (i.e. $\bar{u} = \bar{v} = 0$) and f, g are two-times differentiable, then spatially constant solutions of (20)-(22) are solutions of the inequality

(23) $$U(t) \in K \text{ for all } t \in \langle 0,\infty),$$
$$\langle U_t(t) - BU(t) - N(U,t)), \Psi - U(t)\rangle \geq 0 \text{ for all } \Psi \in K, \text{ a.a. } t \geq 0$$

with $K = K_V^+$ (cf. Remark 2) and with a mapping $N: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfying

(24) $$\lim_{\|U\| \to 0} \frac{N(U)}{\|U\|} = 0.$$

The assumption (2) together with (24) ensure that the trajectory of the system $U_t = BU + N(U)$ tends outside of K at any $V = [u,0]$, $u \in \langle -\varepsilon, 0)$ (with some $\varepsilon > 0$) and $\langle BV + N(V), V\rangle > 0$ for these V. Considerations analogous to those from Remark 3 imply that for any $V = [u,0]$, $u \in (-\varepsilon, 0)$ there is a solution U of (23) such that $U(0) = V$, $U(t) = [u(t),0]$ with $u(t) \in (-\varepsilon, 0)$ for $t \in (0,t_0)$ and

$U(t_0) = [-\varepsilon, 0]$ for some $t_0 > 0$. This gives the instability of the trivial solution of (23) and also the instability of the trivial solution of (20)-(22) (even with respect to the spatially homogeneous perturbations) for all d_1, d_2 (even for those from the domain of stability from Remark 1).

Remark 5. (A general formulation of unilateral problems.) Let $W_2^1 = W_2^1(0,1)$ be the usual Sobolev space with the norm $\|\cdot\|_{1,2}$ (see e.g. [3]), K a closed convex cone in W_2^1 with its vertex at the origin. Consider the inequality

$$(25)\quad \begin{cases} \int_0^1 \{u_t(t,x)\varphi(x) + d_1 u_x(t,x)\varphi_x(x) - \\ \qquad - [b_{11}u(t,x) + b_{12}v(t,x)]\varphi(x)\}\, dx = 0, \\ v(t,\cdot) \in K, \\ \int_0^1 \{v_t(t,x)[\psi(x) - v(t,x)] + d_2 v_x(t,x)[\psi_x(x) - v_x(t,x)] - \\ \qquad - [b_{21}u(t,x) + b_{22}v(t,x)][\psi(x) - v(t,x)]\}\, dx \ge 0 \\ \qquad \text{for all } \varphi \in W_2^1,\ \psi \in K,\ \text{a.a. } t \in \langle 0, \infty). \end{cases}$$

We do not need a general definition of a solution in fact because our aim will be to show only the existence of a smooth in T solution starting arbitrarily close to the origin and tending to infinity; for such a solution it will be clear in which sense (25) is fulfilled. In general, the solution on $\langle 0,T)$ (eventually with $T = \infty$) could be defined as a couple $u, v \in L_2(0,T;W_2^1)$ such that $u_t, v_t \in L_2(0,T;(W_2^1)^*)$ and (25) is fulfilled for a.a. $t \in (0,T)$ (cf. e.g. [2]). A function from $L_2(0.T;W_2^1)$ has a derivative in the sense of distributions with values in W_2^1, i.e. also with values in $L_2(0,1)$. By $u_t \in L_2(0,T;(W_2^1)^*)$ we mean that this distribution can be represented by a function u_t with values in $L_2(0,1)$ such that $\sup_{\|\varphi\|_{1,2} \le 1} \int_0^T |\int_0^1 u_t(t,x)\varphi(x)\, dx|^2\, dt$ is finite.

Notice that any function $u \in L_2(0,T;W_2^1)$ with $u_t \in L_2(0,T;(W_2^1)^*)$ is continuous as an abstract function with values in $L_2(0,1)$. Hence, initial conditions $u_0, v_0 \in L_2$ for (25) can be considered.

Remark 6. If we set for instance $K = \{v \in W_2^1; v(1) \geq 0\}$ then u, v satisfies (25) if and only if (RD_L),(1) is fulfilled in the classical sense. It is not hard to show by using integration by parts, boundary conditions (1) and elementary considerations about the regularity of the solution (cf. e.g. [3]).

If we set $K = \{v \in W_2^1; v \geq 0 \text{ on } \langle 0,1\rangle\}$ then (25) is another formulation of the free boundary problem from Example 2. This follows by integrating by parts again.

In general, we can say that (25) is an abstract formulation of (RD_L) with the constraints given by the cone K.

Remark 7. (An eigenvalue problem for inequalities in W_2^1 and stability.) The unilateral eigenvalue problem corresponding to (25) is

$$(26)\begin{cases} \int_0^1 [d_1 u_x \varphi_x - (b_{11}u + b_{12}v - \lambda u)\varphi]\, dx = 0 \text{ for all } \varphi \in W_2^1, \\ v \in K, \\ \int_0^1 [d_2 v_x(\psi_x - v_x) - (b_{21}u + b_{22}v - \lambda v)(\psi - v)]\, dx \geq 0 \text{ for all } \psi \in K. \end{cases}$$

A real λ_0 is called an eigenvalue of (26) if there exists a nontrivial couple $u_0, v_0 \in W_2^1$ satisfying (26). In this case, $[u_0, v_0]$ is called the corresponding eigenvector. It is easy to see (by substituting into (25) and using (26)) that then for any $\tau > 0$ fixed the couple $u(t,x) = \exp(\lambda_0 t)\cdot \tau u_0(x)$, $v(t,x) = \exp(\lambda_0 t)\cdot \tau v_0(x)$ satisfies (25) on $\langle 0,\infty)$. If $\lambda_0 > 0$ then $\|u(t,\cdot)\| + \|v(t,\cdot)\| \to \infty$ for $t \to \infty$ (for any reasonable norm $\|\cdot\|$) and this implies the instability of the trivial solution of (25) for such d_1, d_2.

Set $\Gamma_\delta = \{[d_1,d_2]; [d_1 - \delta, d_2] \in \Gamma\}$ for any $\delta > 0$, where Γ is the curve from Remark 1 (see Fig. 1).

<u>Theorem 1</u>. Let (2) be fulfilled and let $0 < \xi < \eta$. Then there is $\delta > 0$ such that for any $[d_1,d_2]$ lying between Γ, Γ_δ and satisfying $\xi \le d_2 \le \eta$, $d_1 \ge 0$, there exists a positive eigenvalue of (26).

<u>Consequence 1</u>. The trivial solution of (RD_L) with unilateral constraints given by K (see Remark 6) is unstable for all $[d_1,d_2]$ between Γ, Γ_δ, $\xi \le d_2 \le \eta$, $d_1 \ge 0$. It follows directly from Theorem 1 and Remark 7.

<u>Remark 8</u>. Theorem 1 is a special case of a more general result proved in [6]. The proof is based on a continuation theorem related to the known Dancer's global bifurcation result [1] and on an investigation of branches of solutions of the corresponding penalty equation (cf. also [7],[8],[5]).

<u>Remark 9</u>. Theorem 1 together with Consequence 1 say that the domain of instability of the problem with unilateral constraints intersects the domain of stability of the classical problem. For any $0 < \xi < \eta$ there is a δ-strip $G_\xi^\eta(\delta)$ (see Fig. 1) such that for $[d_1,d_2] \in G_\xi^\eta(\delta)$ the trivial solution of (25) is unstable in spite of that the trivial solution of (RD_L), (NC) is stable (see Remark 1). In the case of chemical or biological models, u and v represent the density of an activator (or prey) and of an inhibitor (or predator), respectively, under the assumption (2). Hence, unilateral conditions for the inhibitor (or predator) have a destabilizing effect. On the other hand, analogous unilateral condi-

tions for the activator (or prey) have a stabilizing effect in a certain sense (see [9]). Notice that in applications the instability of a spatially constant solution signals that the corresponding spatially homogeneous equilibrium state will not occur and, eventually, some spatially nonhomogeneous patterns can arise.

Remark 10. The problem (RD_L),(1) (i.e. (25) with $K = \{v \in W_2^1; v(1) \geq 0\}$) looks like that from Example 1 at the first sight, but an analogy of the considerations from Example 1 gives no result. (The eigenvectors of (RD_λ),(1) cannot be expressed elementarily; this is a consequence of the fact that different conditions are prescribed for u and v.) A destabilizing effect of the conditions (1) follows from Theorem 1 and Consequence 1.

In the case of the problem (8)-(10) (i.e. (25) with $K = \{v \in W_2^1; v \geq 0 \text{ on } \langle 0,1\rangle\}$), Theorem 1 gives a weaker information than the elementary considerations in Example 2.

Theorem 1 cannot be used for the problem from Example 1 because (4) represents unilateral conditions for both u and v. Even the more general theory given in [6] (cf. also [5]) ensures a destabilizing effect of unilateral conditions prescribed for v only under the assumptions of the type (2) (i.e. for an inhibitor or a predator).

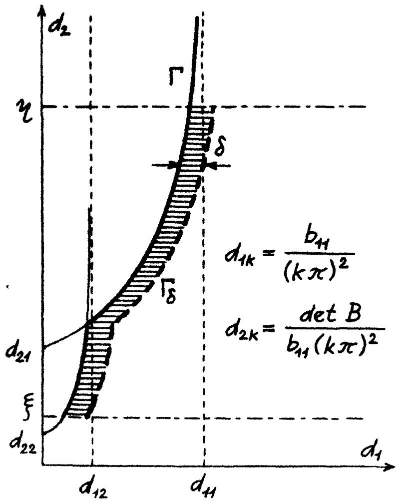

Fig. 1

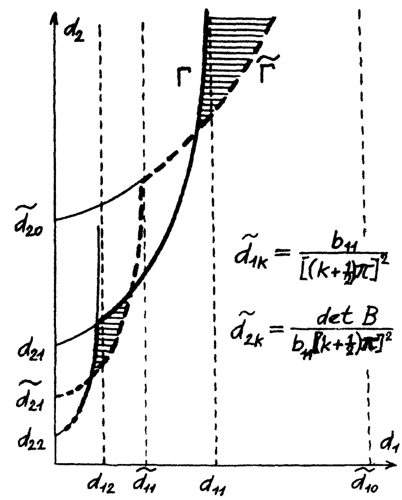

Fig. 2

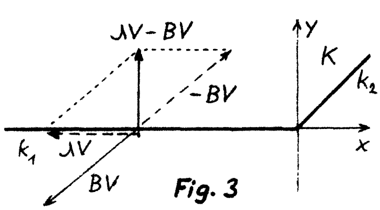

Fig. 3

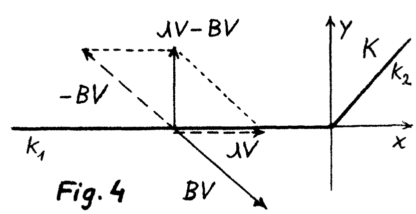

Fig. 4

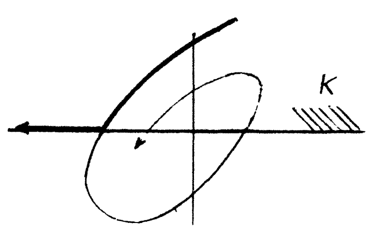

Fig. 5

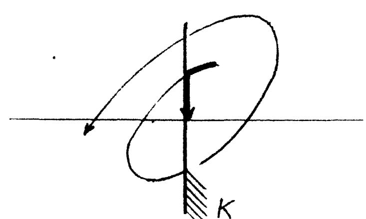

Fig. 6

References

[1] E.N. DANCER: On the structure of solutions of non-linear eigenvalue problems. Ind. Univ. Math. J. 23(1974), 1069-1076.

[2] G. DUVANT, J.-L. LIONS: Les inéquations en méchanique et en physique. Dunod, Paris 1972.

[3] S. FUČÍK, A. KUFNER: Nonlinear differencial equations , Elsevier, Scient. Publ. Comp., Amsterdam-Oxford-New York 1980.

[4] P. DRÁBEK, M. KUČERA, M. MÍKOVÁ: Bifurcation points of reaction-diffusion systems with unilateral conditions, Czechoslovak Math. J. 35(1985), 639-660.

[5] P. DRÁBEK, M. KUČERA: Eigenvalues of inequalities of reaction -diffusion type and destabilizing effect of unilateral conditions. Czechoslovak Math. J. 36(1986).

[6] P. DRÁBEK, M. KUČERA: Reaction-diffusion systems: Destabilizing effect of unilateral conditions. To appear.

[7] M. KUČERA: A new method for obtaining eigenvalues of variational inequalities based on bifurcation theory. Čas. pěst. mat. 104(1979), 389-411.

[8] M. KUČERA: A new method for obtaining eigenvalues of variational inequalities. Operators with multiple eigenvalues. Czechoslovak Math. J. 32(107)(1982), 197-207.

[9] M. KUČERA: Bifurcation points of inequalities of reaction-diffusion type. To appear.

[10] M. MIMURA, Y. NISHIURA, M. YAMAGUTI:: Some diffusive prey and predator systems and their bifurcation problems, Ann. New York Acad. Sci 316(1979), 490-510.

[11] Y. NISHIURA: Global structure of bifurcating solutions of some reaction-diffusion systems. SIAM J. Math. Anal. Vol. 13, No. 4, July 1982, 555-593.

M. Kučera: Mathematical Institute of the Czechoslovak Academy of Sciences, Žitná 25, 115 67 Prague 1, Czechoslovakia

J. Neustupa: Faculty of Mechanical Engineering, Czech Technical University, Suchbátarova 4, 166 07 Prague 6, Czechoslovakia

(Oblatum 4.9. 1985)