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ON PSEUDO-RADIAL SPACES

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Abstract. A new cardinal invariant, the quasi-character, is introduced and some of its interesting properties are studied, particularly in the class of chain-net or pseudo-radial spaces. Main results are that the quasi-character coincides with the tightness for pseudo-radial monolithic spaces and, under GCH, for pseudo-radial spaces which are Hausdorff and compact or have cardinality not greater than $\aleph_\omega$. However still open is the problem if quasi-character and tightness are the same in the class of pseudo-radial spaces. Introducing the notion of tightness with respect to a family of subsets, upper bounds for the cardinality of the closure of a set are developed in a general topological space.

Key words and phrases: Cardinal invariant, pseudo-radial space, chain-net space.

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1. Introduction and basic definitions. Pseudo-radial or chain-net spaces were first introduced by Herrlich [9] in 1967. The same class of topological spaces was then considered by Meyer, Mrówka, Rajagopalan,... [14],[16], and systematically examined by Arhangel'skiĭ [1],[2]. Some questions presented there stimulated the publication of other papers [11],[17],[8],[10],[18].

In this section and in Section 2 all spaces are supposed to be

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T_1, if not otherwise stated. We shortly recall the basic equivalent definitions of a pseudo-radial or chain-net space:

Definition A. A topological space X is called pseudo-radial or chain-net if, for every non-closed subset A, there are a point $x \in \bar{A} \setminus A$ and a family $\mathcal{P}$ of subsets of X, such that the family $\mathcal{P}$ is linearly ordered by inclusion and

(i) $P \cap A \neq \emptyset$, for every P in $\mathcal{P}$;

(ii) for every neighbourhood U of x there is P in $\mathcal{P}$ such that $P \subset U$;

(iii) $\bigcap \mathcal{P} = \{x\}$.

Definition B. X is a chain-net or pseudo-radial space if for every non-closed subset A of X there are a point $x \in \bar{A} \setminus A$ and a (transfinite) λ-sequence $(x_\alpha : \alpha < \lambda)$ in A converging to x.

The following theorem furnishes a useful characterization of pseudo-radial spaces.

Theorem. X is a pseudo-radial space if and only if for any non-closed set A there exist a point $x \in \bar{A} \setminus A$ and a subset B of A of regular cardinality, such that for any neighbourhood U of x, $|B \setminus U| < |B|$ (see [1]).

Radial or Fréchet chain-net spaces are topological spaces such that every point in $\bar{A}$ satisfies the properties of Definitions A or B or that one expressed in the theorem (see [2]).
We now give the following

Definition 1.1. A subset B of a topological space X is said to be topologically directed (in X) if $|B|$ is a regular cardinal number and there exists a point x such that for every neighbourhood U of x, $|B \setminus U| < |B|$: In this case we also say that every neighbourhood U of x contains "almost all" points of B.

If X is a Hausdorff space, then the point x is unique. In case the point x is unique it will be called the <u>end</u> of B.

The following definition introduces a new cardinal invariant, which seems rather interesting.

<u>Definition 1.2</u>. Let X be a topological space and x a point of it. We define the <u>quasi-character</u> of X at x the least cardinal number τ such that, if A is any subset of X and $x \in \bar{A} \setminus A$, then there is a family γ of subsets of A such that $|\gamma| \leq \tau$, $x \notin \bar{B}$ for any B in γ but $x \in \overline{\bigcup \gamma}$. We denote the quasi-character of X at x by $q\chi(x,X)$. $q\chi(X) = \sup\{q\chi(x,X): x \in X\}$, is called the <u>quasi-character</u> of the space X.

It can be useful sometimes to think at the quasi-character as obtained from the following cardinal invariant.

<u>Definition 1.3</u>. Let X be a topological space and $x \in \bar{A} \setminus A$. The <u>primitive quasi-character</u> of the point x with respect to the subset A is the least cardinal number τ such that there exists a family γ of subsets of A with the properties as in Definition 1.2, i.e. $|\gamma| \leq \tau$, $x \notin \bar{P}$ for any P in γ but $x \in \overline{\bigcup \gamma}$.
It then follows that $q\chi(x,X) = \sup\{pq\chi(x,A): A \subset X$ and $x \in \bar{A} \setminus A\}$.

2. <u>Fundamental properties of the quasi-character</u>. Before giving the first simple but important properties of the quasi-character we recall one more definition

<u>Definition 2.1</u>. Given a topological space X, a set $A \subset X$ and a point $x \in \bar{A}$, we call primitive tightness of x with respect to A, pt(x,A), the least infinite cardinality of a subset B of A, such that $x \in \bar{B}$ (see [5],[12]).

<u>Proposition 2.1</u>. For any T_1 space X, $q\chi(x,X) \leq t(x,X)$. More in general, the inequality $pq\chi(x,A) \leq pt(x,A)$ holds under the same

hypothesis. Also the inequality $q\chi(x,X) \leq \psi(x,X)$ should be noted.

Proof. Only the last inequality needs to be proved. Let $\{U_\alpha : \alpha < \lambda\}$ be a family of open sets with $\lambda = \psi(x,X)$, such that $\bigcap U_\alpha = \{x\}$. Consider $K_\alpha = (\bigcap_{\beta \leq \alpha} U_\beta) \setminus U_{\alpha+1}$. Then $x \notin \overline{K_\alpha}$, for any α; $\bigcup K_\alpha = U_0 \setminus \{x\}$ hence $x \in \overline{\bigcup K_\alpha}$. In fact for any $y \in U_0 \setminus \{x\}$, there is the least α such that $y \in U_\alpha \setminus U_{\alpha+1}$; then $y \in K_\alpha$.

Proposition 2.2. For any T_2 space X, $q\chi(X) \leq s(X)$.

Proof. Remember that s(X) coincides with the hereditary Suslin number of X. Let $x \in \overline{A} \setminus A$; for any $y \in A$ take the family $\mathcal{V}_y$ of all open neighbourhoods V_y of y such that $x \notin \overline{V_y}$. From $\mathcal{V} = \bigcup\{\mathcal{V}_y : y \in A\}$ extract a maximal disjoint subfamily γ. Then $|\gamma| \leq s(X)$, $x \notin \overline{P}$ for any P in γ but $x \in \overline{\bigcup \gamma}$.

The strict inequality can hold in the result of Proposition 2.2, as shown by the following

Example 1. Let $X = Y_0 \cup Y_1$, where $Y_i = I \times \{i\}$, i=0,1 and I is the unit interval. Let Y_1 be discrete, while the neighbourhoods V of a point $(x,0) \in Y_0$ are $V = U \times \{0\} \cup (U \setminus \{x\}) \times \{1\}$, and U is a neighbourhood of x in the usual topology of I. Then it is easy to see that X is a T_2 compact 1^{st} countable space such that $q\chi(X)$ is countable and $s(X) = 2^{\aleph_0}$.

In what follows, given a set A in a topological space X, we shall denote by $cl_\lambda(A)$ the following

$cl_\lambda(A) = \bigcup\{\overline{B} : B \subset A \text{ and } |B| \leq \lambda\}$.

Theorem 2.3. Let X be a T_1 radial space. Then $pq\chi(x,A) = pt(x,A)$ for any subset A of X such that $x \in \overline{A} \setminus A$,

Proof. Let $\tau = pq\chi(x,A) < pt(x,A)$. Then there is a family γ of subsets of A such that $|\gamma| \leq \tau$, $x \notin \overline{P}$ for any P in γ but

$x \in \overline{\bigcup \gamma}$. Since X is radial, there is $C \subset \bigcup \gamma$, with C topologically directed towards x of regular cardinality not less than $pt(x,A)$. Take $P' = P \cap C$ for any P in γ and let γ' be the family of all the P's. We have $x \notin \overline{P'}$ for any P' in γ' and therefore $P' \subset C \setminus U$ for some neighbourhood U of x. Since C is topologically directed, then $|P'| < |C|$. From the regularity of $|C|$ since $|P'| < |C|$ and $|\gamma'| \leq \tau < |C|$ it follows $|\bigcup \gamma'| < |C|$, which contradicts $\bigcup \gamma' = C$.

As easy corollaries one finds

Corollary 1. Let X be a T_1 radial space. Then $q\chi(x,X) = t(x,X)$

Corollary 2. If X is a T_1 radial space then $q\chi(X) = t(X)$.

So for radial spaces $q\chi(X)$ coincides with $t(X)$. It is not clear if the same equality holds in general also for pseudo-radial spaces. The remaining part of this section is dedicated to the investigation of this problem, and several partial answers are given. For non pseudo-radial spaces $q\chi(X)$ can be strictly less than $t(X)$, as shown by the following

Example 2. Let $W_0 = \{\eta : \eta \leq \omega_0\}$ and $W_1 = \{\eta : \eta \leq \omega_1\}$ and define $X = (W_0 \setminus \{\omega_0\}) \times (W_1 \setminus \{\omega_1\}) \cup \{(\omega_0, \omega_1)\}$ as a subspace of $W_0 \times W_1$ with the product topology. Then $q\chi(x,X) = \aleph_0$ for any $x \in X$ but $t(X) = \aleph_1$. The first claim needs to be verified only for (ω_0, ω_1). If such a point is in $\overline{A} \setminus A$ for some subset A of X, we can consider the countable family γ, whose elements are $P_n = A \cap (\{n\} \times (W_1 \setminus \{\omega_1\}))$.

However, the following is an example of a pseudo-radial compact T_2 space X for which $pq\chi(x,A) < pt(x,A)$ for some subset A of X with $x \in \overline{A} \setminus A$. The example holds under Martin's Axiom and the negation of the Continuum Hypothesis.

Example 3. Under $(MA + \neg CH)$ it was shown in [2] that $X = D^{\aleph_1}$ is a compact T_2 pseudo-radial space. Let A be a Σ-product contained in X and $x \in X \setminus A$. Then $pq\chi(x,A) = \aleph_0$, since $c(A) \leq \aleph_0$ (see [1], paragraph 5), but $pt(x,A) = \aleph_1$, since $x \notin A$ and $cl_{\aleph_0}(A) = A$.

Proposition 2.4. Let X be a T_1 pseudo-radial space such that $q\chi(X) \leq \lambda$. Let $A \subset X$ be such that $\psi(x, A \cup \{x\}) \leq \lambda^+$ for every $x \in X$ and $cl_\lambda A = A$. Then $\overline{A} = A$.

Proof. By contradiction, suppose $\overline{A} \neq A$. Then there are $z \in \overline{A} \setminus A$ and $B \subset A$ topologically directed towards z. Since $\{z\} = \cap\{U_\alpha : \alpha < \lambda^+\}$ with U_α open in $A \cup \{z\}$, we have $B = \cup\{B_\alpha : \alpha < \lambda^+\}$, with $|B_\alpha| < |B|$. In fact $\psi(x, B \cup \{x\}) \leq \psi(x, A \cup \{x\}) \leq \lambda^+$; it is enough to take $B_\alpha = B \setminus U_\alpha$. Then $cf(|B|) \leq \lambda^+$, since $|B|$ is regular, $|B| = cf(|B|) \leq \lambda^+$.

Now $q\chi(X) \leq \lambda$ implies the existence of a family γ of subsets of B such that $|\gamma| \leq \lambda$, $|P| < |B|$ for any P in γ (since $z \notin \overline{P}$), and $z \in \overline{\cup\gamma}$. But from $|P| < |B| \leq \lambda^+$ it follows that $|P| \leq \lambda$ for any $P \in \gamma$ and then for $C = \cup\gamma$, $|C| \leq \lambda \cdot \lambda = \lambda$. But $z \in \overline{C}$ gives a contradiction since we supposed $z \notin A = cl_\lambda A$.

So we have proved, under the above conditions, that $cl_\lambda(A) = \overline{A}$.

Definition 2.2. Let X be a topological space. We say that the pseudo-network weight of X, pnw(X), is not greater than τ if there exists a family $\mathcal{P}$ of closed subsets of X such that for every $x \in X$, $\{x\} = \cap\{F \in \mathcal{P} : x \in F\}$, and $|\mathcal{P}| \leq \tau$ The family $\mathcal{P}$ will be called a pseudo-network of X.
pnw(X) is the least infinite cardinal number τ fulfilling the above property.

Proposition 2.5. Let X be a Hausdorff topological space. If

$A \subset X$ and $nw(A) \leq \lambda$ (for example $|A| \leq \lambda$), then $pnw(cl_{\tau}A) \leq \lambda^{\tau}$
(For T_3 spaces $nw(cl_{\tau}A) \leq \lambda^{\tau}$.)

Proof. Let S be a network for A such that $|S| \leq \lambda$. For every $x \in cl_{\tau}A$ and every O_x, open neighbourhood of x, there is B such that $B \subset A \cap O_x$, $x \in \overline{B}$ and $|B| \leq \tau$. There is also a family $\xi \subset S$, with $|\xi| \leq \tau$ such that $B \subset \cup\xi \subset O_x$. Then $x \in \overline{B} \subset \overline{\cup\xi} \subset \overline{O_x}$. Consider $\mathcal{P} = \{\overline{\cup\xi} : \xi \subset S, |\xi| \leq \tau\}$. $\mathcal{P}$ is then a pseudo-network for $cl_{\tau}(A)$ and $|\mathcal{P}| \leq \lambda^{\tau}$. Hence $pnw(cl_{\tau}A) \leq |\mathcal{P}| \leq \lambda^{\tau}$.

Proposition 2.6. Let X be a Hausdorff topological space, $M \subset X$ and $|M| \leq 2^{\lambda}$. Then $pnw(cl_{\lambda}M) \leq 2^{\lambda}$.

Proof. Since $nw(M) \leq |M| \leq 2^{\lambda}$, it follows from Proposition 2.5 that $pnw(cl_{\lambda}M) \leq (2^{\lambda})^{\lambda} = 2^{\lambda}$.

Proposition 2.7. Let X be a T_2 pseudo-radial space such that $q\chi(X) \leq \lambda$ and take a subset $M \subset X$ with $|M| \leq \lambda^{+}$. If we assume the generalized continuum hypothesis (GCH), then $cl_{\lambda}M = \overline{M}$.

Proof. Put $A = cl_{\lambda}M$. Then $cl_{\lambda}(A) = cl_{\lambda}(cl_{\lambda}(M)) = cl_{\lambda}(M) = A$. For any $x \in X$, $A \cup \{x\} \subset cl_{\lambda}(M \cup \{x\})$. Hence, by Proposition 2.6,

$$pnw(A \cup \{x\}) \leq pnw(cl_{\lambda}(M \cup \{x\})) \leq 2^{\lambda} .$$

Since we assume GCH, $2^{\lambda} = \lambda^{+}$ holds and we have

$$\psi(x, A \cup \{x\}) \leq pnw(A \cup \{x\}) \leq \lambda^{+} = 2^{\lambda} .$$

From Proposition 2.4 then $\overline{A} = A$. But from $M \subset cl_{\lambda}(M) = A$ and $A \subset \overline{M}$ it follows $\overline{M} = \overline{A} = A = cl_{\lambda}(M)$.

Theorem 2.8. Let X be a pseudo-radial T_2 space, such that $t(X) = \lambda^{+}$. Under GCH we have

$$q\chi(X) = t(X).$$

Proof. We have $q\chi(X) \leq t(X) = \lambda^{+}$. Let us assume that $q\chi(X) < t(X)$, so that $q\chi(X) \leq \lambda$. As $t(X) = \lambda^{+}$ there must be a set $M \subset X$

such that $|M| = \lambda^+$ and $cl_\lambda(M) \neq \overline{M}$. But this is impossible for Proposition 2.7. By contradiction, $q\chi(X) = t(X)$.

Theorem 2.9. If GCH holds, then $q\chi(X) = t(X)$ for every pseudo-radial compact Hausdorff space X.

Proof. Put $\tau = q\chi(X)$ and assume that $\tau < t(X)$. From the characterization of the tightness in compact T_2 spaces, as the supremum of lengths of free sequences [1],[4], it follows that for any $\tau \leq \varkappa < t(X)$ there is in X a free sequence whose length is $\varkappa^+$, $S = \{x_\alpha : \alpha < \varkappa^+\}$. Let $Y = \overline{S}$. Since X is Hausdorff we have (assuming GCH) that $|Y| \leq ((\varkappa^+)^+)^+$. In Y we then have $\varkappa^+ \leq t(Y) \leq \varkappa^{+++}$, so that t(Y) is one of the following isolated cardinal numbers: $\varkappa^+$, $\varkappa^{++}$ or $\varkappa^{+++}$. Y is obviously pseudo-radial and so, from Theorem 2.8, we have $q\chi(Y) = t(Y)$, i.e. $q\chi(Y) \geq \varkappa^+$. But obviously we also have $q\chi(Y) \leq q\chi(X)$ so that $q\chi(Y) \leq \tau \leq \varkappa < \varkappa^+$, which is in contradiction with the previous result.

Theorem 2.10. If GCH holds and X is a pseudo-radial T_2 space then

$t(X) \leq \max\{q\chi(x), d(X)\}$.

Proof. If $\varkappa = \max\{q\chi(X), d(X)\}$, then $\psi(X) \leq pnw(X) \leq 2^{d(X)} \leq$ $\leq 2^{\varkappa} = \varkappa^+$, since we assume GCH. Also $q\chi(X) \leq \varkappa$. From Proposition 2.4 it follows that if $cl_\varkappa(A) = A$, then $\overline{A} = A$. So $cl_\varkappa(A) = \overline{A}$ for any $A \subset X$, which implies $t(X) \leq \varkappa$.

The following are straightforward consequences

Corollary 1. For a separable pseudo-radial Hausdorff space X, if GCH holds, we have

$$t(X) = q\chi(X).$$

Corollary 2. For a pseudo-radial T_2 space, if GCH holds and

$q\chi(X) < t(X)$, then $t(X) \leq d(X)$.

Corollary 3. If X is a T_1 pseudo-radial space, $q\chi(X) = \varkappa$ and $\psi(X) \leq \varkappa^+$, then

$$t(X) = q\chi(X).$$

The proof is given by the same argument of Theorem 2.10.

Proposition 2.11. Let X be a pseudo-radial Hausdorff space, and let $x \in X$, $A \subset X$ be such that $pt(x,A) = \varkappa^+$. If GCH holds, then $q\chi(X) \geq \varkappa^+$.

Proof. Without any loss of generality we can assume that $|A| = \varkappa^+$ (otherwise we denote by A the appropriate subset of A). Put $Y = \overline{A}$, so that from the T_2 axiom and GCH it follows that $|Y| \leq \varkappa^{+++} = \mu$ and $\varkappa^+ = pt(x,A) \leq t(Y) \leq |\overline{A}| \leq \mu = \varkappa^{+++}$. Hence $t(Y)$ is one of the following isolated cardinal numbers: $\varkappa^+$, $\varkappa^{++}$ or $\varkappa^{+++}$. Since Y is pseudo-radial, as a closed subspace of the pseudo-radial space X, for Theorem 2.8, we have $q\chi(Y) = t(Y) \geq \varkappa^+$. Obviously $q\chi(X) \geq q\chi(Y)$. Thus $q\chi(X) \geq \varkappa^+$.

We recall (see [3]) that a space X is monolithic if for every $A \subset X$ we have

$$nw(\overline{A}) \leq |A|,$$

Lemma 2.12. For every monolithic space X and for each $A \subset X$, $nw(\overline{A}) = nw(A)$.

Proof. Obviously $d(A) \leq nw(A) \leq nw(\overline{A})$. But, since X is monolithic, we also have $nw(\overline{A}) \leq d(A)$. Thus $d(A) = nw(A) = nw(\overline{A})$.

Theorem 2.13. Let X be a pseudo-radial Hausdorff, monolithic space. Then

$$t(X) = q\chi(X)$$

Proof. Let X be a pseudo-radial Hausdorff monolithic space and $\tau = q\chi(X)$. By contradiction suppose $t(X) > \tau$. Put $\mathcal{C} = \{A \subset X : cl_\tau(A) = A \neq \overline{A}\}$. Then $\mathcal{C} \neq \emptyset$ in our hypothesis. Let $\mu = \min\{nw(A) : A \in \mathcal{C}\}$ and fix $A \in \mathcal{C}$ such that $nw(A) = \mu$. As X is pseudo-radial and $\overline{A} \neq A$, there exist $z \in \overline{A}$ and $B \subset A$ such that $|B|$ is regular and B is topologically directed towards z. Since $z \notin cl_\tau(A) = A$ and $z \in \overline{B}$, $B \subset A$, we have $|B| > \tau$. As $q\chi(X) \leq \tau$, there exist a family $\gamma \subset Exp(B)$ such that $|\gamma| \leq \tau$, $z \in \overline{\bigcup \gamma}$ and $z \notin P$ for every P in γ. It follows that $|P| < |B|$ for every $P \in \gamma$. From the regularity of $|B|$ and $\tau < |B|$ it follows $|\bigcup \gamma| < |B|$. But, as was observed in Proposition 2.4, from B being topologically directed towards z and the regularity of $|B|$, we conclude that $\psi(z, B \cup \{z\}) \geq |B|$. Hence $nw(B) \geq |B|$, which implies $nw(B) = |B|$: In fact, if the inequality did not hold, using T_2, one could produce a family of open sets whose intersection is z, but of cardinality less than $|B|$, which is impossible.

Put $M = \bigcup \gamma$ and $L = cl_\tau(M)$. From Lemma 2.12 it follows that $nw(M) = nw(\overline{M}) = nw(\overline{L}) = nw(L)$. As $nw(B) = |B|$, we have

$$nw(M) \leq |M| < |B| = nw(B) \leq nw(A) = \mu,$$

and thus $nw(L) = nw(M) < \mu$.

On the other hand $cl_\tau(L) = cl_\tau(cl_\tau(M)) = cl_\tau(M) = L$; $L = cl_\tau(M) \subset cl_\tau(B) \subset cl_\tau(A) = A$, and $z \in \overline{M} \subset \overline{L}$ but $z \notin A$. It follows that $z \in \overline{L} \setminus L$, i.e. $\overline{L} \neq L$, hence $L \in \mathcal{C}$. But then

$$\mu = \min\{nw(A) : A \in \mathcal{C}\} \leq nw(L) < \mu.$$

A contradiction.

<u>Theorem 2.14</u>. Let X be a pseudo-radial Hausdorff space such that $|X| \leq \aleph_\omega$. If GCH holds, then $t(X) = q\chi(X)$.

<u>Proof</u>. Let $\tau = q\chi(X)$ and suppose $\tau < t(X)$. Then there exists a set $A \subset X$ such that $cl_\tau(A) = A \neq \overline{A}$. Since X is pseudo-radial then there exists $B \subset A$ topologically directed towards some

$z \in \overline{A} \setminus A$, with $|B|$ regular. Then $|B| < \aleph_\omega$. Now $z \notin cl_\tau(B) \subset cl_\tau(A) =$ $= A$, so that $pt(z,B) > \tau$; $pt(z,B)$ is, however, not greater than $|B|$ and it is then an isolated cardinal, so that from Proposition 2.11 $q\chi(X) \geq pt(z,B)$. We have then obtained a contradiction $\tau =$ $= q\chi(X) > \tau$.

Theorem 2.15. Under GCH for a Hausdorff pseudo-radial k-space X, we have $t(X) = q\chi(X)$.

Proof. For any compact subspace K of X (K is closed and hence pseudo-radial) we have $t(K) = q\chi(K)$, for Theorem 2.9. Hence $t(K) \leq q\chi(X)$ for any compact subspace K of X. From the definition of a k-space and recalling that the weak tightness and the tightness coincide, it then follows that $t(X) \leq q\chi(X)$, i.e. that they are equal.

3. Tightness with respect to a family of subsets. Let $\mathcal{F}$ be a family of subsets of a topological space X, and let r be a relation between points and subsets of X, given in such a way that for any $x \in X$ and $A \subset X$ it can be decided if $x\ r\ A$ is true or not. Consider the following

Definition 3.1. Given the topological space X and a family $\mathcal{F}$ of subsets of X, we say that the topology of the space is Fréchet-generated by $\mathcal{F}$ or that X is Fréchet-tight with respect to $\mathcal{F}$, if for every set $A \subset X$ and point $x \in \overline{A}$ there is a subset $F \in \mathcal{F}$ such that $F \subset A$ and $x \in \overline{F}$.

Definition 3.2. Given the space X and a family $\mathcal{F}$, we say that the topology of X is gradually generated by $\mathcal{F}$ or that X is tight with respect to $\mathcal{F}$ if for every non-closed set A there is

$F \in \mathcal{F}$ such that $F \subset A$ and $\overline{F} \setminus A \neq \emptyset$.

The preceding definitions can be profitably generalized as follows.

Definition 3.1'. X is said to be Fréchet generated by the family $\mathcal{F}$ and the relation r or to be Fréchet-tight with respect to $\mathcal{F}$ and r, if for any $A \subset X$ and any $x \in \overline{A}$ there is $F \in \mathcal{F}$ such that $F \subset A$, $x \in \overline{F}$ and $x\ r\ F$.

Definition 3.2'. X is gradually generated by $\mathcal{F}$ and r or tight with respect to $\mathcal{F}$ and r if for any non-closed subset A there are $F \in \mathcal{F}$ and x such that $F \subset A$, $x \in \overline{F} \setminus A$ and $x\ r\ F$.

Given two families $\mathcal{F}$ and $\mathcal{E}$ of subsets of X we denote by $\mathcal{F} \Diamond \mathcal{E}$ the family:

$\mathcal{F} \Diamond \mathcal{E} = \{F \in \mathcal{F}$:there exists $E \in \mathcal{E}$ such that $F \subset E\}$.

Finally if $\mathcal{F}$ is a family of subsets of X and r is a relation between points and subsets of X, we put respectively

$cl_{\mathcal{F}}(A) = \{x \in X$:there exists $F \in \mathcal{F}$ such that $F \subset A$ and $x \in \overline{F}\}$

$cl_{\mathcal{F},r}(A) = \{x \in X$:there exists $F \in \mathcal{F}$ such that $F \subset A, x \in \overline{F}$ and $x\ r\ F\}$.

Lemma 3.1. Let $\mathcal{F}$ and $\mathcal{E}$ be families of subsets of the topological space X; if X is Fréchet-tight with respect to $\mathcal{F}$ and tight with respect to $\mathcal{E}$, then it is tight with respect to $\mathcal{F} \Diamond \mathcal{E}$.

Proof. Let A be a non-closed subset of X; then there is $x \in \overline{E} \setminus A$, for some $E \in \mathcal{E}$ and $E \subset A$. As X is Fréchet-tight with respect to $\mathcal{F}$, we can find $F \in \mathcal{F}$ such that $F \subset E$ and $x \in \overline{F}$. Then $F \in \mathcal{F} \Diamond \mathcal{E}$ and, by definition, X is tight with respect to $\mathcal{F} \Diamond \mathcal{E}$.

The following proposition is obviously true.

Proposition 3.2. If X is Fréchet-tight with respect to both families $\mathcal{F}$ and $\mathcal{E}$ then it is Fréchet-tight with respect to $\mathcal{F} \Diamond \mathcal{E}$

and $\mathcal{E} \lozenge \mathcal{F}$.

Similarly to Lemma 3.1 one can prove

Lemma 3.1'. Let r be a relation between points and subsets of X and let $\mathcal{F}$ and $\mathcal{E}$ be two families of subsets of X such that X is Fréchet-tight relative to $\mathcal{F}$ and r and X is tight with respect to $\mathcal{E}$. Then X is tight with respect to $\mathcal{F} \lozenge \mathcal{E}$ and r.

The following result should be considered as well known; at least it has been proved in many particular cases (see [6],[7] or [19].

Lemma 3.3. Let r be a relation between points and subsets of X, and let $\mathcal{P}$ be a family of subsets of X, such that X is tight with respect to $\mathcal{P}$ and r; put $\tau = \sup\{|P| : P \in \mathcal{P}\}$. Define $\hat{A}^{\alpha} = \mathrm{cl}^{\alpha}_{\mathcal{P},r}(A)$ for $\alpha \leq \tau^{+}$ by transfinite induction:

$\hat{A}^{0} = A$;

$\hat{A}^{\alpha} = \bigcup\{\hat{A}^{\beta} : \beta < \alpha\} = \bigcup\{\mathrm{cl}^{\beta}_{\mathcal{P},r}(A) : \beta < \alpha\}$, if α is a limit ordinal;

$\hat{A}^{\alpha+1} = \mathrm{cl}^{\alpha+1}_{\mathcal{P},r}(A) = \mathrm{cl}_{\mathcal{P},r}(\mathrm{cl}^{\alpha}_{\mathcal{P},r}(A)) = \mathrm{cl}_{\mathcal{P},r}(\hat{A}^{\alpha})$.

Then $\mathrm{cl}_{\mathcal{P},r}(\hat{A}^{\tau+}) = \hat{A}^{\tau+}$.

Proof. Assume the contrary and put $A^{*} = \hat{A}^{\tau+}$. Then there exist $B \in \mathcal{P}$ and $x \in \overline{A^{*}} \setminus A^{*}$ such that $x \in \overline{B}$, $B \subset A^{*}$ and $x \; r \; B$. By the construction $A^{*} = \bigcup\{\hat{A}^{\alpha} : \alpha < \tau^{+}\}$ and $\hat{A}^{\alpha'} \subset \hat{A}^{\alpha''}$ if $\alpha' < \alpha'' < \tau^{+}$ (we identified τ^{+} with the initial ordinal number having the same cardinality).

Since $\mathrm{cf}(\tau^{+}) = \tau^{+}$ and $|B| \leq \tau$, one can find $\alpha^{*} < \tau^{+}$ such that $B \subset \hat{A}^{\alpha^{*}}$. Then $x \in \mathrm{cl}_{\mathcal{P},r}(\hat{A}^{\alpha^{*}}) = \hat{A}^{\alpha^{*}+1} \subset \hat{A}^{\tau+} = A^{*}$, which is in contradiction with the hypothesis $x \in \overline{A^{*}} \setminus A^{*}$.

Corollary. If X is tight with respect to $\mathcal{P}$ and r, then for any subset A of X,

$$\overline{A} = \hat{A}^{\tau+}.$$

Remarks. By considering particular families $\mathcal{P}$ and relations r between points and subsets of X, special interesting cases can be obtained, as shown by the following examples.

(1) If X is a sequential space, let $\mathcal{P}$ be the family of countable subsets of X and x r B be the relation: "There is a sequence (x_n) converging to x such that $B \subset \{x_n\}$". Since $\aleph_0 = \sup \{|B| : B \in \mathcal{P}\}$, we obtain that for any subset A of X, $\overline{A} = \hat{A}^{\omega_1}$, where $\hat{A}$ denotes the (non-idempotent) sequential closure (see [7],[19]).

(2) Let X be a k-space, $\mathcal{P}$ be the family of subsets of X of cardinality not greater than t(X) and r be the following relation: "B is a subset of a compact set K in X such that $x \in \overline{K \cap A}^K$". Then, if $\hat{A} = \{x \in X : x \in \overline{K \cap A}^K$ for some compact K in X$\}$, we have $\overline{A} = \hat{A}^{\tau^+}$, if $\tau = t(X)$ (see [6]).

(3) Let X be a pseudo-radial space, $\mathcal{P}$ be as in example (2) and r be the relation: "There is a λ-sequence converging to x, such that $B \in \mathcal{P}$ and $B \subset \{x_\alpha : \alpha < \lambda\}$". If $\hat{A}$ denotes the chain-net closure, then $\overline{A} = \hat{A}^{\tau^+}$. Here again $\tau = t(X)$.

Note that in sequential spaces the tightness is countable so that, in the three examples that we have examined , it is the first ordinal of cardinality greater than the tightness which gives an upper bound for the number of times the pseudo-closure has to be iterated in order to get the topological closure.

Proposition 3.4. Let X, $\mathcal{P}$, r and τ be as in the preceding Lemma 3.3. Assume, in addition, that for some cardinal $\mu \geq \tau^+$ the following condition is satisfied:

if $A \subset X$ and $|A| \leq \mu$, then $|cl_{\mathcal{P},r}(A)| \leq \mu$.

Let X be tight with respect to $\mathcal{P}$ and r. Then $|A| \leq \mu$ implies $|\overline{A}| \leq \mu$.

Proof. By transfinite induction it follows that if $|A| \leq \mu$ then $|\hat{A}^{\alpha}| \leq \mu$ for every $\alpha \leq \tau^+$. Hence $|\hat{A}^{\tau^+}| \leq \mu$: Since $\hat{A}^{\tau^+} = \overline{A}$, we have $|\overline{A}| \leq \mu$.

We recall here some definitions that are needed in the following theorem and remarks. The bitightness bt(X) of a topological space X is the least cardinal number τ such that if A is non-closed in X there are a point $x \in \overline{A} \setminus A$ and a family $\mathcal{P}$ of subsets of A for which $|P| \leq \tau$, for any P in $\mathcal{P}$, $|\mathcal{P}| \leq \tau$ and $\{x\} = \bigcap\{\overline{P} : P \in \mathcal{P}\}$. The bitightness can be defined for any Hausdorff space; the definition was given by Arhangel'skiĭ in Soviet Math. Dokl. 11(1970), 597-601. The closure character of the space X, denoted by kc(X), is the least cardinal number τ such that a set C is closed in X if and only if $C \cap K$ is closed for any closed set K of X, with $|K| \leq \tau$ (see [8]).

Theorem 3.5. For any topological space X we have $|X| \leq (d(X))^{t(X)} \cdot (kc(X))^{t(X)}$.

Proof. Put $\tau = t(X)$ and $\lambda = kc(X)$. Let $\mathcal{F} = \{B \subset X : |B| \leq \tau\}$ and $\mathcal{E} = \{B \subset X : \overline{B} = B$ and $|B| \leq \lambda\}$.

Define $\mathcal{P} = \mathcal{F} \lozenge \mathcal{E} = \{F \in \mathcal{F}$: there is $E \in \mathcal{E}$ with $F \subset E\}$. Clearly $\mathcal{P} = \{B \subset X : |B| \leq \tau$ and $|\overline{B}| \leq \lambda\}$. Let A be any subset of X and put $\mu = |A|^{\tau} \cdot \lambda^{\tau}$. Then $\mu = \mu^{\tau} \geq \tau^+$. Let $M \subset X$ be such that $|M| \leq \mu$. Then $cl_{\mathcal{P}}(M) = \bigcup\{\overline{B} : B \in \mathcal{P}$ and $B \subset M\}$. Since $|B| \leq \tau$ for any $B \in \mathcal{P}$, we have $|\{B \in \mathcal{P} : B \subset M\}| \leq |M|^{\tau} \leq \mu^{\tau} = \mu$. Besides $|\overline{B}| \leq \mu$ so that $|cl_{\mathcal{P}}(M)| \leq \mu \cdot \mu = \mu$. Then by Proposition 3.4 it follows that $|\overline{M}| \leq \mu$ since $|M| \leq \mu$ implies $|cl_{\mathcal{P}}(M)| \leq \mu$ and X is obviously tight with respect to the family $\mathcal{P}$.
In particular, since $|A| \leq \mu$, we also have $|\overline{A}| \leq \mu$: We have then shown that for any subset A of X the following holds:

$$|\bar{A}| \leq |A|^{t(X)} \cdot kc(X)^{t(X)}.$$

Taking, in particular, for A a dense subset of X, we get the thesis.

The case of the real line with the usual topology shows that the equality can hold in Theorem 3.5. In fact, for the real line density, tightness and closure character are countable.

We observe that for T_2 spaces the above upper bound for the cardinality of A is a little sharper than the one given by using the bitightness: $|\bar{A}| \leq |A|^{bt(X)}$, since in this case $kc(X) \leq 2^{bt(X)}$.

Theorem 3.6. For any topological space X the following conditions are equivalent:

(a) $kc(X) \leq 2^{t(X)}$

(b) $|\bar{A}| \leq |A|^{t(X)}$, for any subset A containing more than one point.

Proof. ((a) implies (b).) As $t(A) \leq t(X)$ and $kc(A) \leq kc(X)$ by Theorem 3.5 we have

$$|\bar{A}| \leq |A|^{t(X)} \cdot kc(X)^{t(X)} \leq |A|^{t(X)} \cdot (2^{t(X)})^{t(X)} = |A|^{t(X)}$$

if $|A| > 1$.

((b) implies (a).) Put $\mathcal{P} = \{\bar{B} : |B| \leq t(X)\}$. From the definition of tightness if A is non-closed in X there is a closed set F in $\mathcal{P}$ such that $A \cap F$ is not closed in X. By condition (b) $|F| \leq (t(X))^{t(X)} = 2^{t(X)}$ for every F in $\mathcal{P}$, and all elements of $\mathcal{P}$ are closed in X. Now the topology of X is generated, in the usual sense, by $\mathcal{P}$. So $kc(X) \leq 2^{t(X)}$ must hold in X.

4. Some open problems on pseudo-radial spaces. The following is a list of some of the more interesting problems and questions which were raised during this investigation and are still unsolved, as far as we know.

(1)$^{x)}$ Can "real" (i.e. without additional assumptions on ZFC set theory) examples be obtained of a T_2 (or T_3, T_4 or T_2 compact) pseudo-radial space with countable tightness which is not sequential?

(2)$^{x)}$ Find a T_1 pseudo-radial space such that the quasi-character is strictly less than the tightness.

(3) Find necessary or sufficient (or both) conditions for a space Y to be a subspace of a pseudo-radial space X.

(4) In particular: is $N \cup \{p\}$ a pseudo-radial space for any $p \in \beta N \setminus N$?

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x) The problems 1 (for T_2-spaces) and 2 are solved in the next paper of this issue (P. Simon, G. Tironi: Two examples of pseudo-radial spaces).

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