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STABILITY AND SADDLE-POINT PROPERTY FOR A LINEAR AUTONOMOUS FUNCTIONAL PARABOLIC EQUATION

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Abstract: A linear parabolic functional differential equation $\dot{u}(t) + Au(t) = Lu_t$ with infinite delay is investigated under assumptions that A is a sectorial operator in a Banach space X and L is a continuous linear operator from a space Y of continuous functions with fading memory norm into X. Values of functions from Y are in the domain of fractional power A^{α}, $0 \leq \alpha < 1$. The theorem on stability and the saddle-point property are proved.

Key words: Functional differential equations, parabolic equations with delay, infinite delay, solution operator and its generator, stability, saddle-point property.

Classification: 35R10, 34K30

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§ 1. Introduction and results. Two main difficulties occur in the investigation of linear functional differential equations with infinite delays, namely:

(i) The choice of a phase space on which the solution operator T(t) is considered. For example, it is necessary for asymptotic stability to endow a phase space with a property of fading memory (compare e.g. the results of [4] with [7]). Spaces with fading memories were introduced by several authors (see e.g. [3]) and their properties were generalized in an axiomatic way in [7]and later on in [10].

(ii) The solution operator T(t) forms a C_0-semigroup but it is difficult to obtain some information about its infinitesimal generator B. T. Naito has shown in [13] that asymptotic properties of $T(t)$ can be deduced from a localization of the essential spectrum of $T(t)$ and properties of the point spectrum of B.

In this paper we follow the main idea of T. Naito for a partial functional differential equation

$$(E) \qquad \dot{u}(t) + Au(t) = Lu_t.$$

We suppose that A is a sectorial operator in a Banach space X with a compact resolvent. The shift of u is denoted by u_t, i.e. $u_t(s) = u(t+s)$ for $s \in (-\infty, 0]$. In applications a linear operator L can depend on lower space derivatives but not on the highest ones. In other words, L is defined on a space Y^{α} of functions which map the interval $(-\infty, 0]$ into X^{α} for $0 \leq \alpha < 1$, where X^{α} is the domain of the fractional power A^{α} endowed with the graph norm. The spaces Y^{α} have the properties of an abstract phase space from [7] and [10]. Some estimates for the operators A are given in Section 2.

In Section 3 we shall prove that the question (E) determines a dynamical system $T(t)$ on the space Y^{α} and this system forms a C_0-semigroup. We remark that this problem for finite delays is generally investigated in the recent paper [12]. If a resolvent of A is compact then the system $T(t)$ differs by a compact operator from the solution operator of the homogeneous equation

$$(E_0) \qquad \dot{v}(t) + Av(t) = 0.$$

On the base of the R. Nussbaum formula for the radius of an essential spectrum ([14]) we obtain an estimate for the essential spectrum of $T(t)$ (Proposition 2). The main part of Section 4 is devoted to the investigation of the point spectrum of the generator B what leads to Theorem 2. As a corollary of this main result the sufficient conditions for asymptotic stability of the equation (E) are given (Corollary 1). Conclusions of Theorem 2 also allow to decompose the space Y^{α} into the direct sum $Y_1 \oplus Y_2$ of T(t)-invariant subspaces (Corollary 2). The space Y_1 has a finite dimension and $T(t)\varphi$ behaves like a solution of a totally unstable ordinary differential equation for $\varphi \in Y_1$. These results correspond to those ones for ordinary functional differential equations with finite delays as in [6].

We note that in [15] K. Schumacher has recently proved the existence of a resolvent operator for the equation (E) in which A can be time dependent. The stability for the equation (E) in which L is defined on Y_1 (i.e. L can depend on the highest derivatives) has been also recently investigated in [1], but only for finite delays and Hilbert spaces.

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§ 2. Preliminaries. Let X be a Banach space and let A be a sectorial operator in X, i.e. (see [5],[8]) A is a closed operator with a dense domain $\mathfrak{D}(A)$ and the spectrum of A lies outside of a sector $S_{a,\omega} := \{\lambda \in C; \omega \le |\arg(\lambda - a)| \le \pi\}$ for some $a > 0, \omega < \pi/2$, and there is a constant M such that the inequality

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda - a|} \tag{2.1}$$

holds for the resolvent of A and $\lambda \in S_{a,\omega}$. Under these properties, -A generates a C_0-semigroups e^{-At} which has an analytic extension into a domain $\mathcal{K} := \{z \in C; |\operatorname{Arg} z| < \pi/2 - \omega\}$. All fractional powers A^{α} are defined, and, moreover, there is a constant c (in the sequel we shall denote by c an arbitrary constant) such that

$$\|A^{\alpha} e^{-At}\| \le c\, e^{-a\operatorname{Re} t}(\operatorname{Re} t)^{-\alpha} \tag{2.2}$$

for any $t \in \operatorname{Int} \mathcal{H}$. We denote by X^{α} the domain of A^{α} endowed with the graph norm.

We need the following generalization of the estimate (2,1).

<u>Proposition 1</u>. Let A be a sectorial operator for which (2.1) holds. Then for arbitrary $0 \leq \alpha < 1$, $\Delta < \pi - \omega$, there is a constant c such that the inequality

$$\| A^{\alpha} (\lambda I + A)^{-1} \| \leq \frac{c}{|\lambda + a|^{1-\alpha}} \tag{2.3}$$

is true for $|\arg(\lambda + a)| \leq \Delta$

<u>Proof</u>. As $(\lambda I + A)^{-1} = \int_0^{+\infty} e^{-\lambda s} e^{-As} ds$ for $\operatorname{Re} \lambda > -a$, we have

$$A^{\alpha} (\lambda I + A)^{-1} = \int_0^{+\infty} e^{-\lambda s} A^{\alpha} e^{-As} ds.$$

Let $\lambda = \tau + i\sigma$ with $\tau > -s$, $\sigma \leq 0$. Choose $\vartheta \in (0, \pi/2 - \omega)$. The Cauchy theorem yields the following expression

$$A^{\alpha} (\lambda I + A)^{-1} = e^{i\vartheta} \int_0^{+\infty} e^{-\lambda r e^{i\vartheta}} A^{\alpha} e^{-A r e^{i\vartheta}} dr.$$

Define $F_{\vartheta}(\lambda)$ by the integral on the right hand side. According to the estimate (2.2), F_{ϑ} is an analytic function in the domain $M_{\vartheta} := \{\lambda \in C; |\arg(\lambda + a) + \vartheta| < \pi/2\}$, and there is c such that $\| F_{\vartheta}(\lambda) \| \leq c |\lambda + a|^{\alpha - 1}$ for all $\lambda \in M_{\vartheta}$. But $M_{\vartheta} \subset \wp(-A)$ and $A^{\alpha} (\lambda I + A)^{-1} = e^{i\vartheta} F_{\vartheta}(\lambda)$ for $\lambda \in M_{\vartheta} \cap \{\lambda \in C; \operatorname{Re}(\lambda + a) > 0\}$. By the uniqueness theorem, this equality is valid on the whole set M_{ϑ}. Since the same idea can be used also for $\sigma \geq 0$, $\vartheta \in (-\pi/2 + \omega, 0)$, the estimate (2.3) follows.

§ 3. <u>A dynamical system</u>. As a space of solutions of the equation (E) we choose $Y_{\gamma,\alpha}(T) := \{ u : (-\infty, T] \to X^{\alpha}$; u is continuous on $(-\infty, T]$,

$$\| u \|_{Y_{\gamma,\alpha}(T)} := \sup_{t \in (-\infty, T]} \| e^{\gamma t} u(t) \|_{\alpha} < \infty \}$$

for $0 \leq \alpha < 1$ and a certain positive number γ . For the sake of simplicity we denote $Y_{\gamma,\alpha}(0)$ by Y and this space will be the basic phase space for the equation (E). We consider this equation together with an initial condition

(3.1) $$u_0 = \varphi \in Y.$$

A solution (in the space $Y_{\gamma,\alpha}(T)$, $T > 0$) of an integral equation

(IE) $$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} Lu_s \, ds, \quad u_0 = \varphi ,$$

is said to be a mild solution to the equation (E). We define a strong solution to (E) as a function $u \in Y_{\gamma,\alpha}(T)$ for some $T > 0$ such that $\dot{u}(t)$ exists, $u(t) \in \mathcal{D}(A)$, and (E) is satisfied for any $t \in (0,T)$. A strong solution is a mild one as well.

Theorem 1. Let operators satisfy the following conditions:

(H1) $$\begin{cases} A \text{ is a sectorial operator in } X \text{ with the property (2.1) for } a > 0; \\ 0 \leq \alpha < 1, \ \gamma > 0; \\ L \text{ is a continuous linear operator from } Y \text{ into } X. \end{cases}$$

Then for any $\varphi \in Y$ there exists a unique mild solution to the equation (E) which satisfies the initial condition (3.1). This solution is defined on the interval $(-\infty, +\infty)$. Moreover, if $\varphi(0) \in X^{\alpha+\varepsilon}$ for some $\varepsilon > 0$, and $e^{\gamma \cdot}\varphi(\cdot)$ is a Hölder continuous function on the interval $(-\infty, 0]$ into X^{α}, then this solution is also a strong solution to (E).

Proof. (i) To prove the local existence to (IE) we choose $T > 0$, $r > 0$ and set $Z(r) := \{ u \in Y_{\gamma,\alpha}(T);\ u_0 = \varphi,\ \|u(t) - \varphi(0)\|_{\alpha} \leq r \text{ for } t \in [0,T] \}$. A map $t \to u_t$ is a continuous map of $[0,T]$ into Y for any $u \in Y_{\gamma,\alpha}(T)$, and the right hand side of (IE) determines (for sufficiently small $T > 0$) a contraction of Z(r) into itself.

(ii) We shall prove the global existence of a solution using a Gronwall type estimate. Suppose that for some φ, $\psi \in Y$ the corresponding solutions $u(\cdot,\varphi)$, $u(\cdot,\psi)$ exist on the interval $(-\infty,T)$ and let $v(t) := \|u_t(\varphi)-u_t(\psi)\|_Y$, $w(t) := \sup_{0\le s\le t} v(s)$. With help of (2.2) we have

$$v(t) = e^{-\gamma t}\sup_{s\le t}\|e^{\gamma s}[u(s,\varphi)-u(s,\psi)]\|_\alpha \le s^{-\gamma t}\|\varphi-\psi\|_Y +$$

$$+ e^{-\gamma t}\sup_{0\le s\le t}\|e^{\gamma s}[e^{-As}(\varphi(0)-\psi(0)) + \int_0^s e^{-A(s-\sigma)}L(u_\sigma(\varphi)-$$

$$-u_\sigma(\psi))d\sigma\| \le c\|\varphi-\psi\|_Y + c\|L\|t^{1-\alpha}w(t).$$

If Δ is such that $c\|L\|t^{1-\alpha} \le 2^{-1}$, then

(3.2) $\quad v(t) \le w(t) \le 2c\|\varphi-\psi\|_Y$

for $t \in [0,\Delta]$, $t \le T$. In the space Y the fundamental estimate of [7] holds, namely

(3.3) $\quad \|x_t\|_Y \le e^{-\gamma(t-\tau)}\|x_\tau\|_Y + \sup_{\tau\le s\le t}\|x(s)\|_\alpha$

for $\tau \le t \le T$ and $x \in Y_{\gamma,\alpha}(T)$. This means that the estimate (3.2) can be iterated and therefore the inequality

(3.4) $\quad v(t) \le 2ce^{bt}\|\varphi-\psi\|_Y$

holds on the whole interval $[0,T)$, where $b = \Delta^{-1}\log 2c$ is independent on t,T.

Suppose now that a solution $u(\cdot,\varphi)$ to the solution (IE) exists on the interval $(-\infty,T)$ and T is finite. By (3.4) for $\psi = 0$, this solution is bounded on the interval $[0,T)$. Choose $\beta \in (\alpha,1)$ and $\delta > 0$. For $t \in [\delta,T)$ we have

$$\|u(t,\varphi)\|_\beta \le \|A^{\beta-\alpha}e^{-At}A^\alpha\varphi(0)\| + \|\int_0^t e^{-A(t-s)}Lu_s\,ds\|_\beta \le$$

$$\le c\delta^{\alpha-\beta}\|\varphi\|_Y + cT^{1-\beta} \le c.$$

Therefore for $\delta \le \tau \le t < T$ we obtain

$$\| u(t,\varphi)-u(\tau,\varphi)\|_{\alpha} \leq \|(e^{-A(t-\tau)}-I)u(\tau)\|_{\alpha} +$$

(3.5)

$$+ \|\int_{\tau}^{t} e^{-A(t-s)} Lu_s \, ds\|_{\alpha} \leq c(t-\tau)^{\beta-\alpha} + c(t-\tau)^{1-\alpha},$$

since $\|(e^{-At}-I)x\|_{\alpha} \leq ct^{1-\alpha}\|x\|_{\beta}$ for $x \in X^{\beta}$, $0 \leq \alpha \leq \beta$ (see [8]). The estimate (3.5) shows that $\lim_{t \uparrow T} u(t)$ exists in the space X^{α} and therefore the solution u can be continued behind the point T.

(iii) With respect to the general theorem on the regularity of a mild solution to a nonhomogeneous equation $\dot{v}(t) + Av(t) = f(t)$ (see e.g. [8], Lemma 3.2.1) it is sufficient to prove that the map $t \to Lu_t$ is Hölder continuous from $[0,T)$ into X, i.e., by the additional assumptions on φ, a solution $u(\cdot,\varphi)$ is Hölder continuous from $[0,T)$ into X^{α}. With help of (2.2) and a local boundedness of u_t we get

$$\| u(t)-u(s)\|_{\alpha} \leq \|(e^{-A(t-s)}-I)\, A^{-\varepsilon}e^{-As}A^{\alpha+\varepsilon}\varphi(0)\| +$$

$$+ \|\int_0^s (e^{-A(t-s)}-I)e^{-A(s-\sigma)}\, Lu_{\sigma}\, d\sigma\|_{\alpha} +$$

$$(3.6) \quad + \|\int_s^t e^{-A(t-\sigma)} Lu_{\sigma}\, d\sigma\|_{\alpha} \leq c(t-s)^{\varepsilon} \|\varphi(0)\|_{\alpha+\varepsilon} +$$

$$+ c(t-s)^{\varepsilon}\int_0^s \frac{\|u_{\sigma}\|_Y}{(s-\sigma)^{\alpha+\varepsilon}}\, d\sigma + c\int_s^t \frac{\|u_{\sigma}\|_Y}{(t-\sigma)^{\alpha}}\, d\sigma \leq c(t-s)^{\varepsilon}$$

for $0 \leq s \leq t < T$.

<u>Corollary</u>. Let the hypotheses (H1) be satisfied and let $u(\cdot,\varphi)$ be a mild solution to (IE) on the interval $(-\infty,+\infty)$. If $T(t)\varphi$ denotes $u_t(\varphi)$ then $T(t)$ is a C_0-semigroup on the space Y.

We denote by $S(t)$ the solution operator to the equation (E_0) in the space Y, i.e. $S(t)\varphi := v_t(\varphi)$, where $v(\cdot,\varphi)$ is a solution to (E_0) with (3.1).

Lemma 1. Let the hypotheses (H1) be satisfied together with

(H2) A has a compact resolvent in X.

Then for any $t \in [0,+\infty)$ the operator $T(t)-S(t)$ is a compact map from Y into Y.

Proof. Since $[T(t)-S(t)]\varphi(\vartheta) = 0$ for $\vartheta \in (-\infty,-t)$ it is sufficient to prove that the map

$$\Phi\varphi : \tau \longrightarrow \int_0^\tau e^{-A(\tau-\sigma)} LT(\sigma)\varphi \, d\sigma , \quad \tau \in [0,t],$$

is compact as a map of Y into $C([0,t];X^\alpha)$. This can be shown by the Arzelá-Ascoli theorem. If $\mathcal{B}$ is a bounded set in Y then functions from $\Phi(\mathcal{B})$ are equicontinuous because of (3.6). According to (2.2) and (3.4) a set $\Phi(\mathcal{B})(\tau)$ is bounded in $X^{\alpha+\varepsilon}$ for $\alpha < \alpha + \varepsilon < 1$. Since the hypothesis (H2) implies that the imbedding of $X^{\alpha+\varepsilon}$ into X^α is compact (see e.g. [8]), the result follows.

§ 4. Spectrum of T(t) and of its generator. For a closed operator B with a dense domain in a Banach space X we denote $N_k(\lambda,B):=$ $:= \mathrm{Ker}(\lambda I-B)^k$ and $N(\lambda,B):=\bigcup_{k=1}^{\infty} N_k(\lambda,B)$. We shall use the notion of an essential spectrum in the sense of F. Browder ([2]), i.e. λ is said to belong to the essential spectrum of B ($\lambda \in \mathrm{ess}(B)$) whenever at least one of the following conditions is satisfied:

(i) $(\lambda I-B)$ is not closed;

(ii) the dimension of $N(\lambda,B)$ is infinite;

(iii) λ is a limit point of the spectrum of B.

The radius of ess(B) will be denoted by $r_e(B)$. R. Nussbaum proved in [14] that

(4.1) $r_e(B) = \inf \{ k \in R;\ \alpha(B(M)) \leq k\, \alpha(M)$ for every bounded set $M \}$,

where $\alpha(M)$ is the Kuratowski measure of noncompactness of M.

If λ_0 belongs to the spectrum of B but not to the essential spectrum then λ_0 is an eigenvalue of B (denotation $\lambda_0 \in P_\sigma(B)$) and the dimension of $N(\lambda_0, B)$ is said to be the multiplicity of λ_0. We note that λ_0 is a pole of the resolvent of B as well and the projector P which is given by

$$P = \frac{1}{2\pi i} \int_{\Gamma(\lambda_0)} (\lambda I - B)^{-1} \, d\lambda \tag{4.2}$$

($\Gamma(\lambda_0)$ is a sufficiently small circle with the center in λ_0) decomposes the space X into two B-invariant subspaces and $\mathcal{R}(P) = N(\lambda_0, B)$. Moreover, there exists n_0 such that $N_n(\lambda_0, B) = N_{n_0}(\lambda_0, B)$ for all $n \geq n_0$ and $\mathcal{R}(I-P) = \mathcal{R}(\lambda_0 I - B)^{n_0}$.

Proposition 2. Let the hypotheses (H1),(H2) be satisfied. Let T(t) be the solution operator to the equation (IE). Then for the radius of its essential spectrum the estimate

$$r_e(T(t)) \leq c e^{-\min(a,\gamma)t}$$

holds.

Proof. By the R. Nussbaum result (4.1) and Lemma 1, we have $r_e(T(t)) = r_e(S(t))$. Obviously, $r_e(S(t)) \leq \|S(t)\|$, and

$$\|S(t)\varphi\|_Y = e^{-\gamma t} \max\left[\sup_{\vartheta \leq -t} \| e^{\gamma(t+\vartheta)} \varphi(t+\vartheta) \|_\alpha , \right.$$

$$\left. \sup_{-t \leq \vartheta \leq 0} \| e^{\gamma(t+\vartheta)} e^{-A(t+\vartheta)} \varphi(0) \|_\alpha \right].$$

Thus the estimate (2.2) yields the result.

In the sequel B will stand for the infinitesimal generator of the C_0-semigroup T(t).

Lemma 2. Let the hypotheses (H1) be satisfied. Then:

(i) If $B\varphi = \lambda\varphi$ with $\varphi \neq 0$ then $\mathrm{Re}\,\lambda \geq -\gamma$ and $T(t)\varphi = e^{\lambda t}\varphi$. Moreover, $\varphi(\vartheta) = e^{\lambda\vartheta} d$, where $d \in \mathcal{D}(A)$ and it solves the characteristic equation

(4.3) $$D(\lambda)d := \lambda d + Ad - L(e^{\lambda\vartheta} d) = 0.$$

(ii) If Re $\lambda \geq -\gamma$ and (4.3) has a nontrivial solution, then $\lambda \in P_{\sigma}(B)$.

(iii) If $\mu \in P_{\sigma}(T(t))$ and $\mu \neq 0$, then there exists a finite number of $\lambda \in P_{\sigma}(B)$ such that $e^{\lambda t} = \mu$.

Proof. (i) The function $z(t) := T(t)\varphi$ is a solution to $\dot z(t) = T(t)B\varphi = \lambda z(t)$, i.e. $z(t) = e^{\lambda t}\varphi$ By the definition of $T(t)$, we have $z(t)(\vartheta) = u_t(\vartheta, \varphi) = u_{t+\vartheta}(0, \varphi) = e^{\lambda(t+\vartheta)}\varphi(0)$ for $t + \vartheta \geq 0$. Thus $\varphi(\vartheta) = e^{\lambda\vartheta}\varphi(0)$ for any $\vartheta \leq 0$ and Re $\lambda \geq -$ $-\gamma$. The function $u(t,\varphi)$ solves (IE) and the function $s \to L(e^{\lambda s}\varphi) = e^{\lambda s}L(\varphi)$ has a bounded derivative on the interval $[0,T]$, $T < \infty$. This means (see the third part of the proof of Theorem 1) that $u(t,\varphi)$ is a strong solution to (E), i.e. $\varphi(0) \in \mathcal{D}(A)$ and

$$\frac{d}{dt} e^{\lambda t} \varphi(0) + A(e^{\lambda t} \varphi(0)) = L(e^{\lambda t} \varphi).$$

Hence $d = \varphi(0)$ solves the characteristic equation (4.3).

(ii) Under the assumption the function $\varphi(\vartheta) = e^{\lambda\vartheta} d \in Y$ and $T(t)\varphi = e^{\lambda t}\varphi$. By the definition of the generator, $B\varphi = \lambda\varphi$.

(iii) With the exception to the number of λ, the assertion can be found in [9], Th. 16.7.2. All solutions to the equation $e^{\lambda t} = \mu$ have the form $\lambda_n = t^{-1}\log\mu + i2\pi n t^{-1}$. As A is a sectorial operator, all λ_n, $|n| \geq n_0$, belong to the resolvent set of -A. This means that for this λ_n the equation (4.3) is equivalent to the equation

(4.4) $$d = (\lambda I + A)^{-1} L(e^{\lambda\vartheta} d).$$

If $\lambda_n \in P_{\sigma}(B)$ then there is a solution d of (4.4) such that $\|d\|_{\alpha} = \|e^{\lambda_n\vartheta} d\|_Y = 1$. But from the estimate (2.3) we get

$$1 = \|d\|_{\alpha} = \|A^{\alpha}(\lambda_n I+A)^{-1} L(e^{\lambda_n\vartheta} d)\| \leq c\|L\| |\lambda_n+a|^{1-\alpha}.$$

As the right hand side tends to zero for $|n| \to \infty$, the result follows.

More information about the structure of spaces $N_k(\lambda, B)$ is included in the following lemma. Notice that $D^{(j)}(\lambda)d :=$

$$:= \frac{d^j}{d\lambda^j} D(\lambda)d = - L(\vartheta^j e^{\lambda\vartheta} d) \text{ for } j \geq 1.$$

Lemma 3. Let the hypotheses (H1) be satisfied and let $\operatorname{Re} \lambda > -\gamma$. Then

(i) $x \in N_k(\lambda, B)$ if and only if

$$x(\vartheta) = e^{\lambda\vartheta} \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(k-j)!} \vartheta^{k-j} d_j, \tag{4.5}$$

where $d_1, \ldots, d_k \in \mathcal{D}(A)$ satisfy the relations

$$\sum_{\ell=0}^{j-1} \frac{(-1)^{\ell}}{\ell!} D^{(\ell)}(\lambda)\, d_{j-\ell} = 0, \quad j=1,\ldots,k. \tag{4.6}$$

(ii) If x is of the form (4.5) then

$$T(t)x = e^{\lambda t} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} t^j x_{k-j}, \tag{4.7}$$

where

$$x_j(\vartheta) = e^{\lambda\vartheta} \sum_{\ell=1}^{j} \frac{(-1)^{j-\ell}}{(j-\ell)!} \vartheta^{j-\ell} d_\ell, \quad j=1,\ldots,k. \tag{4.8}$$

Proof. We proceed by induction. For k=1 the assertion is true according to Lemma 2. Suppose first that $x \in N_{k+1}(\lambda, B)$ and set $y = \lambda x - Bx \in N_k(\lambda, B)$. Therefore, by (4.7),

$$T(t)y = e^{\lambda t} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} t^j y_{k-j}.$$

Solving the differential equation $\frac{d}{dt}T(t)x = \lambda T(t)x - T(t)y$, we find

$$T(t)x = e^{\lambda t} \sum_{j=0}^{k} \frac{(-1)^j}{j!} t^j x_{k+1-j}, \tag{4.9}$$

where $x_{k+1} = x$, $x_j(\vartheta) = y_j(\vartheta) = e^{\lambda\vartheta} \sum_{\ell=1}^{j} \frac{(-1)^{j-\ell}}{(j-\ell)!} \vartheta^{j-\ell} d_\ell$,

$j=1,\dots,k$, and $d_1,\dots,d_k$ satisfy (4.6). Taking $t = -\vartheta > 0$ in (4.9) we obtain

$$x(0) = e^{-\lambda\vartheta}x(\vartheta) + \sum_{j=1}^{k}\sum_{\ell=1}^{k+1-j}\frac{(-1)^{k-j-\ell}}{j!(k+1-j-\ell)!}\vartheta^{k+1-\ell}d_\ell =$$

$$= e^{-\lambda\vartheta}x(\vartheta) + \sum_{\ell=1}^{k}\frac{(-1)^{k+1-\ell}}{(k+1-\ell)!}\vartheta^{k+1-\ell}d_\ell .$$

It remains to prove that $d_{k+1} := x(0)$ fulfils the relation

$$D(\lambda)d_{k+1} + \sum_{\ell=1}^{k}\frac{(-1)^\ell}{\ell\,!}D^{(\ell)}(\lambda)d_{k+1-\ell} = 0.$$

But this follows by substituting

$$x(t) = e^{\lambda t}\sum_{j=0}^{k}\frac{(-1)^j}{j!}t^j d_{k+1-j}$$

(set $\vartheta = 0$ in (4.9)) into the equation (E).

Conversely, let x be given by (4.5) with k+1. Put

$$\varphi(t) = e^{\lambda t}\sum_{j=0}^{k}\frac{(-1)^j}{j!}t^j x_{k+1-j},$$

where $x_1,\dots,x_{k+1}$ satisfy (4.8). Then it is easy to prove that φ is a solution to (E) which satisfies the initial condition $\varphi(0) = x_{k+1} = x$. As the initial problem for (E) has a unique solution, $\varphi(t) = T(t)x$ and thus $Bx = \lim_{t\to 0}\frac{\varphi(t)-\varphi(0)}{t} = \lambda x - x_k$. By the inductive assumption, $x_k \in N_k(\lambda,B)$ and $x \in N_{k+1}(\lambda,B)$ follows.

We remark that the explicit form of $N_2(\lambda,B)$ yields a condition on λ to be a simple eigenvalue of B. It follows from (4.7), (4.8) that T(t)x is a solution of a system of ordinary differential equations in the Jordan canonical form for $x \in N(\lambda,B)$.

<u>Corollary</u>. Under the assumptions of Lemma 3, the space $N_k(\lambda,B)$ is T(t)-invariant and $N_k(\lambda,B) \subset N_k(e^{\lambda t},T(t))$.

<u>Theorem 2</u>. Let the hypotheses (H1),(H2) be satisfied. Then for any $\varepsilon > 0$ the set $G = \{\lambda \in C; \mathrm{Re}\,\lambda > -\min(a,\gamma) + \varepsilon\}$

contains only a finite number of points of $P_\sigma(B)$ and all of these points are of the finite multiplicity.

Proof. (i) The set G is a subset of the resolvent set of the operator -A and therefore the equation (4.3) is equivalent to (4.4). If we denote the right hand side of (4.4) as $F(\lambda)d$, we have $\lambda_0 \in P_\sigma(B) \cap G$ if and only if $1 \in P_\sigma(F(\lambda_0))$. But the operator $F(\lambda_0): X^\alpha \longrightarrow X^\alpha$ is compact what implies that 1 is an isolated point of the spectrum of $F(\lambda_0)$. It is easy to see that the function $\lambda \longrightarrow F(\lambda)$ is analytic in G. According to the Smulyan theorem ([16] or [11], Th. 7.1.9) there are two possibilities: $G \subset P_\sigma(B)$ or $P_\sigma(B)$ is isolated in G. By Lemma 2, the first case is impossible. Similar arguments as in the end of the proof of Lemma 2 show that $P_\sigma(B) \cap G$ is finite.

(ii) Now, we prove that $\lambda \in P_\sigma(B) \cap G$ is of the finite multiplicity. According to Corollary of Lemma 3, the multiplicity of λ cannot exceed the multiplicity of $e^{\lambda t} \in P_\sigma(T(t))$. For $t \geq t_0$ we have

$$|e^{\lambda t}| \geq e^{(-\min(a,\gamma)+\varepsilon)t} > ce^{-t\min(a,\gamma)} \geq r_e(T(t)).$$

This means that $e^{\lambda t} \notin \sigma_{ess} T(t)$ for sufficiently large t and the proof is complete.

The last theorem has two important corollaries:

Corollary 1 (asymptotic stability). Let the hypotheses (H1), (H2) be satisfied and let $\operatorname{Re} \lambda < 0$ for any solution to the characteristic equation (4.3). Then 0 is an asymptotically stable solution to (E). Moreover, there is $\delta > 0$ and a constant c such that

$$\| T(t) \| \leq ce^{-\delta t}. \tag{4.10}$$

Proof. By assumptions and Theorem 2, $\Delta := \sup \operatorname{Re} P_\sigma(B) < 0$. This means that $\sup \{ |\lambda| ; \lambda \in P_\sigma(T(t))\} = e^{\Delta t}$ (Lemma 2). With respect to an estimate of a radius of an essential spectrum

(Proposition 2) there is $\delta_1 > 0$ and a constant c such that $r(T(t)) \leq ce^{-\delta_1 t}$ This implies the result by standard arguments (see Lemma 7.4.2 in [6]).

Corollary 2 (saddle-point property). Let the hypotheses (H1), (H2) be satisfied. Then there exists a decomposition $Y = Y_1 \oplus Y_2$ such that

(i) Y_1 has a finite dimension;

(ii) Y_1. Y_2 are T(t)-invariant;

(iii) the zero solution is asymptotically stable for $T(t)/_{Y_2}$;

(iv) $Y_1 \subset \mathfrak{D}(B)$ and $B/_{Y_1}$ is a continuous linear operator generating a group which is an extension of $T(t))/_{Y_1}$.

Proof. According to Theorem 2, the set $\sigma_+ := \{\lambda \in P_\sigma(B);\ \mathrm{Re}\,\lambda \geq 0\}$ is finite and for any $\lambda_0 \in \sigma_+$ the projector $P(\lambda_0)$, which is given by (4.2), has a finite dimensional range $N(\lambda_0, B)$. The projector $P(\lambda_0)$ commutes with T(t) as well. If we set P = $= \sum_{\lambda \in \sigma_+} P(\lambda)$ then P is a continuous projector onto $Y_1 = \bigoplus_{\lambda \in \sigma_+} N(\lambda, B)$ with Ker P = Y_2 and the spaces Y_1, Y_2 satisfy (i) - (iv).

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