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THE MONOTONE LIMIT CONVERGENCE THEOREM FOR ELEMENTARY FUNCTIONS WITH VALUES IN A VECTOR LATTICE

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Abstract: A necessary and sufficient condition for the monotone limit convergence theorem for elementary functions with values in a vector lattice is found.

Key words: Vector lattice, inner regular measure space, elementary function.

Classification: 28B15

All papers on the integration theory of functions with values in a vector lattice are based on the assumption that a measure space $(X, \mathcal{S}, \mu)$ and a vector lattice are such that the following statement holds for every sequence $\{f_n\}_{n=1}^{\infty}$ of elementary L-function s defined on X:

$$(\forall x \in X: f_n(x) \searrow 0) \Longrightarrow (\int_X f_n(x)\, d\mu(x)) \searrow 0.$$

This is the monotone limit convergence theorem.

This paper gives a necessary and sufficient condition for a vector lattice L so that the monotone limit convergence theorem holds for all "reasonable" measure spaces and any sequence of elementary L-functions.

Definition 1: A real vector space L is called a vector lattice if it has a partial ordering $\leq$ such that:

(i) $\forall a_1, a_2, b_1, b_2 \in L: a_1 \leq a_2,\ b_1 \leq b_2 \Longrightarrow a_1+b_1 \leq a_2+b_2$

(ii) $\forall a,b \in L \quad \forall \lambda \in \mathbb{R} : a \leq b,\ 0 \leq \lambda \Rightarrow \lambda a \leq \lambda b$

(iii) $\forall a,b \in L \quad \exists c,d \in L : c \leq a,\ c \leq b,\ \forall c' \in L : c' \leq a,\ c' \leq b \Rightarrow c' \leq c$

$a \leq d,\ b \leq d,\ \forall d' \in L : a \leq d',\ b \leq d' \Rightarrow d \leq d'.$

The elements c, d are called infimum and supremum of a and b respectively and they are denoted by $a \wedge b$ and $a \vee b$.

Definition 2: Let L be a vector lattice and $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of L. We say that $\{a_n\}_{n=1}^{\infty}$ decreases to $a \in L$ and write $a_n \searrow a (n \to \infty)$ if:

$\forall n : a_{n+1} \leq a_n,\ a \leq a_n$

$\forall a' \in L : (\forall n : a' \leq a_n) \Rightarrow a' \leq a.$

The symbol $a_n \nearrow a$ is defined dually and we say that $\{a_n\}_{n=1}^{\infty}$ increases to a.

Definition 3: A vector lattice L will be called Archimedean if $\forall a \in L : a \geq 0 \Rightarrow (n^{-1}a) \searrow 0\ (n \to \infty)$.

For a deeper theory of vector lattices see [1] and [4].

Definition 4: Let $(X, \mathcal{S}, \mu)$ be a measure space, i.e., X be a set, $\mathcal{S}$ be a σ-ring and μ be a σ-additive nonnegative set function, and L be a vector lattice.

A function $f : X \to L$ is called elementary, if

$\exists \{E_j\}_{j=1}^{m}\ \exists \{c_i\}_{i=1}^{m} : \forall j : E_j \in \mathcal{S},\ \mu(E_j) < \infty,\ c_j \in L$

$\forall x \in X : f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x).$

The element $\sum_{j=1}^{m} c_j \mu(E_j)$ is called an integral of f and is denoted by $\int_X f(x) d\mu(x)$.

Proposition 5: The integral $\int_X f(x)\, d\mu(x)$ of an elementary function $f : X \to L$ does not depend on the representation f(x) =

$= \sum_{j=1}^{m} c_j \chi_{E_j}(x)$. For any elementary function $f: X \to L$ there exist $\{E_j'\}_{j=1}^{k}$ and $\{c_j'\}_{j=1}^{k}$ such that $\forall i,j: i \neq j \Rightarrow E_i' \cap E_j' = \emptyset$ and

$$\forall x \in X: f(x) = \sum_{j=1}^{k} c_j' \chi_{E_j'}(x).$$

The proof does not differ from the case when L is the real line $\mathbb{R}$

Now we are going to find a condition for the monotone limit convergence theorem. Suppose that a vector lattice L is such that for any sequence $\{f_n\}_{n=1}^{\infty}$ of elementary functions defined on $[0,1)$ with the Lebesgue measure we have:

(1) $(\forall x \in [0,1): f_n(x) \searrow 0 \ (n \to \infty)) \Longrightarrow \int_0^1 f_n(x)dx \searrow 0 \ (n \to \infty)$.

Consider the sequence of decompositions $\{\mathcal{D}_n\}_{n=0}^{\infty}$ of the interval $[0,1)$ into the intervals $[(k-1)2^{-n}, k\,2^{-n})$ $k=1,\dots,2^n$. Suppose, we have a sequence $\{f_n\}_{n=0}^{\infty}$ of elementary functions $f_n: [0,1) \to L$ which are consistent with the decomposition $\mathcal{D}_n$, i.e.:

$\exists \{a(n,k)\}_{n=0,k=1}^{\infty\ \ 2^n}: \forall n \forall k \in \{1,\dots,2^n\}: a(n,k) \in L$ and

(2) $f_n(x) = a(n,k)$ whenever $x \in [(k-1)2^{-n}, k\,2^{-n})$.

The sequence $\{f_n\}_{n=0}^{\infty}$ is uniquely determined by the double sequence $\{a(n,k)\}_{n=0,k=1}^{\infty\ \ 2^n}$.

Suppose that the double sequence $\{a(n,k)\}_{n=0,k=1}^{\infty\ \ 2^n}$ is such that $a(n,k_n) \searrow 0$ for every sequence $\{k_n\}_{n=0}^{\infty}$ such that $k_0 = 1$ and $\forall n: k_{n+1} = 2k_n \vee k_{n+1} = 2k_n - 1$. Then (2) implies $\forall x \in [0,1): f_n(x) \searrow 0 \ (n \to \infty)$.

From (1) we have $\int_0^1 f_n(x)dx \searrow 0 \ (n \to \infty)$. Looking at (2) we see that $(2^{-n} \cdot \sum_{k=1}^{2^n} a(n,k)) \searrow 0 \ (n \to \infty)$. The preceding consideration motivates us to formulate the following definition.

Definition 6. Let L be a vector lattice. A double sequence $\{a(n,k)\}_{n=0,k=1}^{\infty, \; 2^n}$ of elements of L is called a dyadic tree.
A sequence $\{b_n\}_{n=0}^{\infty}$ is called a chain of the dyadic tree

$\{a(n, k)\}_{n=0,k=1}^{\infty, \; 2^n}$ if there exists a sequence $\{k_n\}_{n=0}^{\infty}$ such that:

$k_0 = 1$

$\forall n: k_{n+1} = 2 k_n \vee k_{n+1} = 2 k_n - 1, \; b_n = a(n,k_n)$.

The dyadic tree $\{a(n,k)\}_{n=0,k=1}^{\infty, \; 2^n}$ is called chain-decreasing to zero if all its chains decrease to zero.

We say that L satisfies the dyadic tree condition (briefly DTC), if $(2^{-n} \sum_{k=1}^{2^n} a(n,k)) \searrow 0 \; (n \longrightarrow \infty)$ for every dyadic tree $\{a(n,k)\}_{n=0,k=1}^{\infty, \; 2^n}$ which is chain decreasing to zero.

Theorem 7: Let L be a vector lattice such that the implication

$(\forall x \in [0,1): f_n(x) \searrow 0) \implies \int_0^1 f_n(x)dx \searrow 0$

holds for every sequence $\{f_n\}_{n=1}^{\infty}$ of elementary functions defined on the interval $[0,1)$ with Lebesgue measure. Then L is Archimedean and satisfies DTC.

Proof: The fact that L satisfies DTC was proved before the Definition 6. Suppose that L is not Archimedean, i.e.

$\exists a \in L: a \geq 0, \; (n^{-1}a) \not\searrow 0 \; (n \longrightarrow \infty)$.

For every natural n define an elementary function

$f_n: [0,1) \longrightarrow L$

$f_n(x) = 0$ if $x \in [0,1 - \frac{1}{n})$

$f_n(x) = a$ if $x \in [1 - \frac{1}{n}, 1)$.

Then we have:

$\forall x \in [0,1): f_n(x) \searrow 0 \; (n \longrightarrow \infty)$ and

$\int_0^1 f_n(x)dx = n^{-1} a \not\searrow 0$, which is a contradiction.

Now we are going to precise the notion "reasonable" measure space.

Definition 8: Let $(X, \mathcal{S}, \mu)$ be a measure space. It is called inner regular if there exists a system $\mathcal{C} \subset \mathcal{S}$ such that:

$\forall \{K_n\}_{n=1}^{\infty} : (\forall n : K_{n+1} \subset K_n,\ K_n \in \mathcal{C},\ K_n \neq \emptyset) \Longrightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset.$

$\forall A \in \mathcal{S} : \mu(A) = \sup\{\mu(K) : K \subset A, K \in \mathcal{C}\}.$

Proposition 9: (i) If $(X, \mathcal{S}, \mu)$ is an inner regular measure space and $A \in \mathcal{S}$ then $(A, \mathcal{S}_A, \mu_A)$ is an inner regular space, where $\mathcal{S}_A = \{B : B \in \mathcal{S}, B \subset A\}$ and μ_A is the restriction of μ.

(ii) If X is a Hausdorff space and μ is a measure in Bourbaki´s sense then $(X, \mathcal{B}(X), \mu)$ is an inner regular space.

The part (i) is obvious. The part (ii) follows from the Bourbaki´s definition of measure, see [7] pp. 435-540. Our definition of the inner regular measure is less strict than Pfanzagl-Pierlo´s definition of a compact approximable measure, but the idea is the same, see [2].

Lemma 10: Let $(X, \mathcal{S}, \mu)$ be a probability measure space and $\{E_{m,j}\}_{m=1,\,j=1}^{\infty,\ \ell_m}$ be a system of $\mathcal{S}$-measurable sets such that

(3) $\forall m : X = \bigcup_{j=1}^{\ell_m} E_{m,j},\ j \neq i \Rightarrow E_{m,i} \cap E_{m,j} = \emptyset$

(4) $\forall m\ \forall s \in \{1, \dots, \ell_{m+1}\}\ \exists j \in \{1, \dots, \ell_m\} : E_{m+1,s} \subset E_{m,j}$

(5) $\forall m\ \forall i,j \in \{1, \dots, \ell_m\},\ \forall r,s \in \{1, \dots, \ell_{m+1}\}:$
$i < j,\ E_{m+1,r} \subset E_{m,i}, E_{m+1,s} \subset E_{m,j} \Rightarrow r < s$

(6) $\forall m\ \forall j \in \{1, \dots, \ell_m\} : \mu(E_{m,j}) > 0$

Then:

(i) For every $\varepsilon > 0$ there exists a system $\{\lambda_{m,j}\}_{m=1,\, j=1}^{\infty,\ \ell_m}$ of dyadic-rational numbers such that:

(7) $\forall m \ \sum_{j=1}^{\ell_m} \lambda_{m,j} = 1, \ \sum_{j=1}^{\ell_m} |\lambda_{m,j} - \mu(E_{m,j})| < \varepsilon(1 - 2^{-m})$

(8) $\forall m \ \forall j \in \{1, \dots, \ell_m\}: \ \lambda_{m,j} = \sum \lambda_{m+1,s}$, where the sum on the right hand side is taken over the set of all $s \in \{1, \dots, \ell_{m+1}\}$ such that $E_{m+1,s} \subset E_{m,j}$.

(9) $\forall m \ \forall j \in \{1, \dots, \ell_m\}: \ \lambda_{m,j} > 0$.

(ii) If moreover $(X, \mathcal{S}, \mu)$ is inner regular then $\forall \varepsilon > 0$ $\exists E \in \mathcal{S}: \mu(E) \leq \varepsilon$ and $\forall \{j_m\}_{m=1}^{\infty}: \forall m: j_m \in \{1, \dots, \ell_m\}$, $E_{m+1, j_{m+1}} \subset E_{m, j}$, $\bigcap_{m=1}^{\infty} E_{m, j_m} = \emptyset$ $\exists m_0: E_{m_0, j_{m_0}} \subset E$.

<u>Proof</u>: (i) The system $\{\lambda_{m,j}\}_{m=1,\, j=1}^{\infty,\ \ell_m}$ will be constructed by the induction with respect to m. Let $\varepsilon > 0$ be fixed. Take $m = 1$. If $\ell_1 = 1$ then put $\lambda_{1,1} = 1$. If $\ell_1 > 1$ then for every $j \in \{1, \dots, \ell_1 - 1\}$ let $\lambda_{1,j}$ be a dyadic rational number such that $0 < \lambda_{1,j} < \mu(E_{1,j}) < \lambda_{1,j} + \frac{\varepsilon}{4 \cdot \ell_1}$. Such $\lambda_{1,j}$ exists because the set of all dyadic rational numbers is a dense set in $[0, 1]$.

Put $\lambda_{1,\ell_1} = 1 - \sum_{j=1}^{\ell_1 - 1} \lambda_{1,j}$. Then $\lambda_{1,\ell_1} > \mu(E_{1,\ell_1}) > \lambda_{1,\ell_1} - \frac{\varepsilon}{4}$ and the system $\{\lambda_{1,j}\}_{j=1}^{\ell_1}$ has the required properties.

Suppose that $\{\lambda_{m,j}\}_{j=1}^{\ell_m}$ has already been constructed. We are going to construct the system $\{\lambda_{m+1,s}\}_{s=1}^{\ell_{m+1}}$. From the properties (3) - (5) it follows that there exists a sequence of integers $\{p_j\}_{j=1}^{\ell_m + 1}$ such that:

$1 = p_1 < p_2 < \dots < p_{\ell_m} < p_{\ell_m + 1} = \ell_{m+1} + 1$ and

$\bigcup_{s=p_j}^{p_{j+1}-1} E_{m+1,s} = E_{m,j}$ for every $j \in \{1, \dots, \ell_m\}$.

Let $s \in \{1,\ldots,\ell_{m+1}\}$, then $s \in \{p_j,\ldots,p_{j+1}-1\}$ for some $j \in \{1,\ldots$ $\ldots,\ell_m\}$. Let $\mu'_{m+1,s}$ be the number $\frac{\mu(E_{m+1,s})}{\mu(E_{m,j})} \cdot \lambda_{m,j}$

Then we have:

$$\sum_{s=p_j}^{p_{j+1}-1} \mu'_{m+1,s} = \frac{\lambda_{m,j}}{\mu(E_{m,j})} \sum_{s=p_j}^{p_{j+1}-1} \mu(E_{m+1,s}) =$$

$$= \frac{\lambda_{m,j}}{\mu(E_{m,j})} \cdot \mu(E_{m,j}) = \lambda_{m,j} \text{ and}$$

$$\sum_{s=1}^{\ell_{m+1}} | \mu'_{m+1,s} - \mu(E_{m+1,s})| = \sum_{j=1}^{\ell_m} \sum_{s=p_j}^{p_{j+1}-1} \mu(E_{m+1,s})|1 -$$

$$- \frac{\lambda_{m,j}}{\mu(E_{m,j})}| = \sum_{j=1}^{\ell_m} |1 - \frac{\lambda_{m,j}}{\mu(E_{m,j})}| \sum_{s=p_j}^{p_{j+1}-1} \mu(E_{m+1,s}) = \sum_{j=1}^{\ell_m} |1 -$$

$$- \frac{\lambda_{m,j}}{\mu(E_{m,j})}| \mu(E_{m,j}) = \sum_{j=1}^{\ell_m} |\mu(E_{m,j}) - \lambda_{m,j}| < \varepsilon (1 - 2^{-m}).$$

The last inequality is the inductive assumption.

Unfortunately $\mu'_{m+1,s}$ are not dyadic rational and they must be "repaired". We shall "repair" the numbers $\mu'_{m+1,s}$ in the following way. Let $j \in \{1,\ldots,\ell_m\}$ be fixed. If $p_{j+1} = p_j + 1$, then the set $\{p_j,\ldots,p_{j+1}-1\}$ has only one element p_j and $\mu'_{m+1,p_j} = \lambda_{m,j}$, which is dyadic rational by the induction hypothesis and we put $\lambda_{m+1,p_j} = \mu'_{m+1,p_j}$. In this case $|\lambda_{m+1,p_j} - \mu'_{m+1,p_j}| = 0$. If $p_{j+1} > p_j + 1$, then for all $s \in \{p_j,\ldots,p_{j+1}-2\}$ take dyadic rational $\lambda_{m+1,s}$ such that

$$0 < \lambda_{m+1,s} < \mu'_{m+1,s} < \lambda_{m+1,s} + \frac{\varepsilon}{2^{m+2}\,\ell_{m+1}}$$

Let $\lambda_{m+1,p_{j+1}-1}$ be the number $\lambda_{m,j} - \sum_{s=p_j}^{p_{j+1}-2} \lambda_{m+1,s}$. Then we have:

$\lambda_{m+1,p_{j+1}-1}$ is dyadic rational,

$$\lambda_{m+1,p_{j+1}-1} > \mu'_{m+1,p_{j+1}-1} > \frac{-\varepsilon(p_{j+1}-p_j-1)}{2^{m+2}\cdot\ell_{m+1}} + \lambda_{m+1,p_{j+1}-1}$$

$$\sum_{s=p_j}^{p_{j+1}-1} \lambda_{m+1,s} = \lambda_{m,j} \text{ and}$$

$$\sum_{s=p_j}^{p_{j+1}-1} |\lambda_{m+1,s} - \mu'_{m+1,s}| < \left(\sum_{s=p_j}^{p_{j+1}-2} \frac{\varepsilon}{2^{m+2}\ell_{m+1}} \right) + \frac{\varepsilon(p_{j+1}-p_j-1)}{2^{m+2}\ell_{m+1}} =$$

$$= \frac{\varepsilon(p_{j+1}-p_j-1)}{2^{m+1}\ell_{m+1}} < \frac{\varepsilon(p_{j+1}-p_j)}{2^{m+1}\ell_{m+1}}$$

This means

$$\sum_{s=1}^{\ell_{m+1}} |\lambda_{m+1,s} - \mu'_{m+1,s}| = \sum_{j=1}^{\ell_m} \sum_{s=p_j}^{p_{j+1}-1} |\lambda_{m+1,s} - \mu'_{m+1,s}| <$$

$$\frac{\varepsilon}{(\ell_{m+1})2^{m+1}} \sum_{j=1}^{\ell_m} p_{j+1}-p_j = \frac{\varepsilon}{2^{m+1}\ell_{m+1}}(p_{\ell_m+1}-p_1) =$$

$= \frac{\varepsilon}{2^{m+1}\ell_{m+1}}(\ell_{m+1}+1-1) = \frac{\varepsilon}{2^{m+1}}$. Therefore $\sum_{s=1}^{\ell_{m+1}} |\lambda_{m+1,s} -$

$- \mu(E_{m+1,s})| \le \sum_{s=1}^{\ell_{m+1}} |\lambda_{m+1,s} - \mu'_{m+1,s}| + \sum_{s=1}^{\ell_{m+1}} |\mu'_{m+1,s} -$

$- \mu(E_{m+1,s})| < \varepsilon(1 - \frac{1}{2^m}) + \frac{\varepsilon}{2^{m+1}} = \varepsilon(1 - \frac{1}{2^{m+1}})$.

The proof of (i) is complete.

(ii) Now let $(X, \mathcal{S}, \mu)$ be an inner regular probability measure space and $\mathcal{C}$ be a system such that: $\mathcal{C} \subset \mathcal{S}$, $(\forall n: K_n \in \mathcal{C}\ .K_{n+1} \subset K_n,\ K_n \neq \emptyset) \Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset$ and $\forall A \in \mathcal{S} : \mu(A) > 0 \Rightarrow \forall \varepsilon > 0\ \exists K \in \mathcal{C} : \mu(A - K) < \varepsilon$. Let $\varepsilon > 0$ be fixed. We shall construct a system $\{K_{m,j}\}_{m=1,j=1}^{\infty,\ \ell_m}$ such that:

(10) $\forall m, \forall j \in \{1, \dots, \ell_m\} : K_{m,j} \subset E_{m,j},\ K_{m,j} \in \mathcal{C} \vee K_{m,j} = \emptyset$

(11) $\forall m, \forall j \in \{1, \dots, \ell_m\}\ \forall s \in \{1, \dots, \ell_{m+1}\} : E_{m+1,s} \subset E_{m,j} \Rightarrow$

$\Rightarrow K_{m+1,s} \subset K_{m,j}$

(12) $\forall m \sum_{j=1}^{\ell_m} \mu(K_{m,j}) > 1 - \varepsilon(1 - 2^{-m})$.

The system $\{K_{m,j}\}_{m=1,j=1}^{\infty,\ \ell_m}$ will be constructed by the induction with respect to m. Take m = 1. For all $j \in \{1, \dots, \ell_1\}$ let $K_{1,j}$ be a set such that $K_{1,j} \in \mathcal{C}$, $K_{1,j} \subset E_{1,j}$ and $\mu(E_{1,j} - K_{1,j}) < \frac{\varepsilon}{2 \cdot \ell_1}$. The system $\{K_{1,j}\}_{j=1}^{\ell_1}$ has the required properties. Let m be a fixed integer. Suppose that for all $m' \leq m$ we have constructed the systems $\{K_{m',j}\}_{j=1}^{\ell_{m'}}$ such that (10) - (12) are satisfied for all $m' \leq m$. We are going to construct the system $\{K_{m+1,s}\}_{s=1}^{\ell_{m+1}}$.

Let $\{p_j\}_{j=1}^{\ell_m+1}$ be a sequence of integers such that:

$$1 = p_1 < p_2 < \ldots < p_{\ell_m} < p_{\ell_m+1} = \ell_{m+1} + 1 \text{ and}$$

$$\forall j \in \{1, \ldots, \ell_m\}: \bigcup_{s=p_j}^{p_{j+1}-1} E_{m+1,s} = E_{m,j} \text{ (see the proof of (i))}$$

Let $s \in \{1, \ldots, \ell_{m+1}\}$ then $s \in \{p_j, \ldots, p_{j+1} - 1\}$ for some $j \in \{1, \ldots, \ell_m\}$. If $\mu(E_{m+1,s} \cap K_{m,j}) = 0$, let $K_{m+1,s}$ be $\emptyset$. If $\mu(E_{m+1,s} \cap K_{m,j}) > 0$, let $K_{m+1,s}$ be a set such that $K_{m+1,s} \in \mathcal{C}$, $K_{m+1,s} \subset E_{m+1,s} \cap K_{m,j}$ and $\mu((E_{m+1,s} \cap K_{m,j}) - K_{m+1,s}) < \frac{\varepsilon}{2^{m+1} \ell_{m+1}}$. Then (10) and (11) are satisfied and

$$\sum_{s=1}^{\ell_{m+1}} \mu(K_{m+1,s}) = \sum_{j=1}^{\ell_m} \mu(K_{m,j}) - \sum_{j=1}^{\ell_m} \sum_{s=p_j}^{p_{j+1}-1} \mu((E_{m+1,s} \cap K_{m,j}) - K_{m+1,s}) > 1 - \varepsilon\left(1 - \frac{1}{2^m}\right) - \varepsilon \frac{1}{2^{m+1}} = 1 - \varepsilon\left(1 - 2^{-(m+1)}\right).$$

The system $\{K_{m,j}\}_{m=1, j=1}^{\infty, \ell_m}$ is constructed

Let $E = X - \left(\bigcap_{m=1}^{\infty} \bigcup_{j=1}^{\ell_m} K_{m,j}\right)$. Then $\mu(E) \leq \varepsilon$ and E has the required property.

Let $\{j_m\}_{m=1}^{\infty}$ be a sequence such that for all m:

$j_m \in \{1, \dots, \ell_m\}$, $E_{m+1,j_{m+1}} \subset E_{m,j}$ and $\bigcap_{m=1}^{\infty} E_{m,j_m} = \emptyset$.

Then for some m_0 we must have $K_{m_0,j_{m_0}} = \emptyset$. In the opposite case we would have a sequence $\{K_{m,j_m}\}_{m=1}^{\infty}$ such that $\forall m: K_{m,j_m} \neq \emptyset$, $K_{m,j_m} \subset \mathscr{C}$, $K_{m+1,j_{m+1}} \subset K_{m,j_m}$ and $\emptyset \neq \bigcap_{m=1}^{\infty} K_{m,j} \subset \bigcap_{m=1}^{\infty} E_{m,j_m}$, which is a contradiction. Since $K_{m_0,j_{m_0}} = \emptyset$, then $E_{m_0,j_{m_0}} \cap K_{m_0,j_{m_0}} = \emptyset$. If $j \in \{1, \dots$ $\dots, \ell_{m_0}\}$, $j \neq j_{m_0}$, then $E_{m_0,j_{m_0}} \cap K_{m_0,j} = \emptyset$, because $K_{m_0,j} \subset$ $\subset E_{m_0,j}$ and $E_{m_0,j_{m_0}} \cap E_{m_0,j} = \emptyset$ by (3). Therefore $E_{m_0,j_{m_0}} \subset X -$ $- (\bigcup_{j=1}^{\ell_{m_0}} K_{m_0,j}) \subset X - (\bigcap_{m=1}^{\infty} \bigcup_{j=1}^{\ell_m} K_{m,j}) = E$.

The proof of (ii) is complete.

<u>Theorem 11</u>: For every vector lattice L the following properties are equivalent:

(i) L is Archimedean and satisfies DTC.

(ii) For every inner regular measure space $(X, \mathscr{S}, \mu)$ and every sequence $\{f_m\}_{m=1}^{\infty}$ of elementary functions $f_m: X \longrightarrow L$ the following implication holds:

$(\forall x \in X: f_m(x) \searrow 0 (m \longrightarrow \infty)) \Longrightarrow \int_X f_m(x)\, d\mu(x) \searrow 0 \; (m \longrightarrow \infty)$.

<u>Proof</u>: The implication (ii) $\Longrightarrow$ (i) follows from the Theorem 7. Let L be an Archimedean vector lattice with DTC property, $(X, \mathscr{S}, \mu)$ be an inner regular measure space and $\{f_m\}_{m=1}^{\infty}$ be a sequence of elementary L-functions decreasing to zero. There exist systems $\{E_{m,j}\}_{m=1,j=1}^{\infty, \ell_m}$, $\{c_{m,j}\}_{m=1,j=1}^{\infty, \ell_m}$ such that

$$\forall m \, \forall j \in \{1, \dots, \ell_m\}: E_{m,j} \in \mathscr{C}, \; \mu(E_{m,j}) < \infty, \; c_{m,j} \in L$$

$$\forall m \, \forall x \in X: f_m(x) = \sum_{j=1}^{\ell_m} c_{m,j} \cdot \chi_{E_{m,j}}(x).$$

Since $\mu(\bigcup_{j=1}^{\ell_1} E_{1,j}) < \infty$ and $f_m(x) \searrow 0$ for every $x \in X$, without loss of generality we may assume that $(X, \mathscr{S}, \mu)$ is an inner regular probability measure space and the system $\{E_{m,j}\}_{m=1,j=1}^{\infty,\ \ell_m}$ has the properties (3) - (6) of the Lemma 10.

Let $\{j_m\}_{m=1}^{\infty}$ be a sequence of integers such that

(13) $\forall m: j_m \in \{1,\dots,\ell_m\},\ E_{m+1,j_{m+1}} \subset E_{m,j_m}$.

Since $f_m(x) \searrow 0$, we have $c_{m+1,j_{m+1}} \leq c_{m,j_m}$, but we are not able to prove that $c_{m,j_m} \searrow 0$. But when $\bigcap_{m=1}^{\infty} E_{m,j_m} \neq \emptyset$, we have $c_{m,j_m} \searrow 0$, because $c_{m,j_m} = f_m(x)$ for some $x \in \bigcap_{m=1}^{\infty} E_{m,j_m}$. We are going to modify the system $\{c_{m,j}\}_{m=1,j=1}^{\infty,\ \ell_m}$. Let $\varepsilon > 0$ be a real number. By Lemma 10 there exists a system $\{\lambda_{m,j}\}$ with the properties (7) - (9) and a set $E \in \mathcal{S}$ such that:

(14) $\mu(E) < \varepsilon$

(15) $\forall \{j_m\}_{m=1}^{\infty}: (\forall m: j_m \in \{1,\dots,\ell_m\},\ E_{m+1,j_{m+1}} \subset E_{m,j_m}) \Rightarrow$

$\Rightarrow \exists m_0: E_{m_0,j_{m_0}} \subset E$

Put

(16) $$d_{m,j} = \begin{cases} c_{m,j} & \text{if } E_{m,j} \not\subset E \\ 0 & \text{if } E_{m,j} \subset E \end{cases}$$

Then we have:

(17) $d_{m,j_m} \searrow 0$ $(m \longrightarrow \infty)$ for every sequence $\{j_m\}_{m=1}^{\infty}$ with the property (13).

If $\bigcap_{m=1}^{\infty} E_{m,j_m} \neq \emptyset$, then $0 \leq d_{m,j_m} \leq c_{m,j_m} = f_m(x) \searrow 0$ for some $x \in \bigcap_{m=1}^{\infty} E_{m,j_m}$. If $\bigcap_{m=1}^{\infty} E_{m,j_m} = \emptyset$, then $d_{m,j_m} = 0$ for all $m \geq m_0$ by (15) and (16).

We are going to prove that $(\sum_{j=1}^{\ell_m} d_{m,j} \cdot \lambda_{m,j}) \searrow 0$ $(m \longrightarrow \infty)$. We shall construct a dyadic tree $\{a(n,k)\}_{n=0,k=1}^{\infty,\ 2^n}$ which is closely related to the systems $\{d_{m,j}\}_{m=1,j=1}^{\infty,\ \ell_m}$ and $\{\lambda_{m,j}\}_{m=1,j=1}^{\infty,\ \ell_m}$. Since all $\lambda_{m,j}$ are dyadic rational with the properties (7) and (8), there exist sequences of natural numbers $\{n_m\}_{m=1}^{\infty}$ and $\{t_{m,j}\}_{m=1,j=1}^{\infty,\ \ell_m}$ such that

(18) $\lambda_{m,j} = t_{m,j}\, 2^{-n_m},\ t_{m,j} \in \{1,\dots,2^{n_m}\}$.

We may assume that the sequence $\{n_m\}_{m=1}^{\infty}$ is increasing, i.e. $n_1 < n_2 < \dots < n_m < n_{m+1} < \dots$. If $0 < n < n_1$ we put:

(19) $a(n,k) = \bigvee_{j=1}^{\ell_1} d_{1,j}$ for all $k \in \{1,\dots,2^n\}$.

If $n_m < n < n_{m+1}$ and $k \in \{1,\dots,2^n\}$ we put:

(20) $a(n,k) = d_{m,j}$ where j is a natural number such that:

(21) $\left(\sum_{s=1}^{j-1} t_{m,s}\right) 2^{n-n_m} < k \leq \left(\sum_{s=1}^{j} t_{m,s}\right) 2^{n-n_m}$

From (18) and (7) we have $\left(\sum_{s=1}^{\ell_m} t_{m,s}\right) 2^{n-n_m} = 2^n$, which means that j is uniquely determined by k and $j \in \{1,\dots,\ell_m\}$. We are going to show that the dyadic tree $\{a(n,k)\}_{n=0,k=1}^{\infty,\ 2^n}$ is chain-decreasing to zero. Let $\{k_n\}_{n=0}^{\infty}$ be a sequence such that:

(22) $k_0 = 1,\ \forall n: k_{n+1} \in \{2k_n, 2k_n - 1\}$.

If $n < n_1$ then we have from (19):

(23) $a(n,k_n) = \bigvee_{j=1}^{\ell_1} d_{1,j}$.

For every natural m let $j_m \in \{1,\dots,\ell_m\}$ be such that

(24) $\sum_{s=1}^{-1+j_m} t_{m,s} < k_{n_m} \leq \sum_{s=1}^{j_m} t_{m,s}$.

Then we have:

(25) $a(n,k_n) = d_{m,j_m}$ whenever $n_m \leq n < n_{m+1}$ and

(26) $E_{m+1,j_{m+1}} \subset E_{m,j_m}$ for all m.

(25) follows from (24),(22),(21) and (20).

We are going to show (26).

Let m be a fixed natural number and $\{p_j\}_{j=1}^{1+\ell_m}$ be a sequence such that:

(27) $1 = p_1 < p_2 < \ldots < p_{\ell_m} < p_{\ell_m+1} = \ell_{m+1} + 1$ and

(28) $\bigcup_{s=p_j}^{-1+p_{j+1}} E_{m+1,s} = E_{m,j}$

(see the proof of Lemma 10).

Since the both sides in (24) are integers, it may be rewritten as $(\sum_{s=1}^{-1+j_m} t_{m,s}) + 1 \leq k_{n_m} \leq \sum_{s=1}^{j_m} t_{m,s}$. Using (28),(3) - (5), (8) and (18), we have

$2^{-(n_{m+1}-n_m)}((\sum_{i=1}^{-1+p_{j_m}} t_{m+1,i}) + 1) \leq k_{n_m} \leq (\sum_{i=1}^{-1+p_{1+j_m}} t_{m+1,i})2^{-(n_{m+1}-n_m)}$.

The inequality $2^{n_{m+1}-n_m}(k_{n_m} - 1) < k_{n_{m+1}} \leq 2^{n_{m+1}-n_m} k_{n_m}$ follows from (22) by the induction. Comparing the last two inequalities we have: $\sum_{i=1}^{-1+p_{j_m}} t_{m+1,i} < k_{n_{m+1}} \leq \sum_{i=1}^{-1+p_{1+j_m}} t_{m+1,i}$. Looking at (24) we see that j_{m+1} must be found in the set $\{p_{j_m}, \ldots, p_{j_m+1}-1\}$, which proves (26).

Finally $\{a(n,k)\}_{n=0,k=1}^{\infty,\ 2^n}$ is chain-decreasing to zero by (25), (26) and (17). Since L satisfies DTC, we have:

$(2^{-n} \sum_{k=1}^{2^n} a(n,k)) \searrow 0$ $(n \to \infty)$ and $(2^{-n_m} \sum_{k=1}^{2^{n_m}} a(n_m,k)) \searrow 0$ $(m \to \infty)$,

which means

(29) $(\sum_{j=1}^{\ell_m} \lambda_{m,j} d_{m,j}) \searrow 0$ $(m \to \infty)$ by (20) and (18).

Computing integrals of f_m and using (16),(14), (3) and (7) we obtain:

$$\int_X f_m(x)\, d\mu(x) = \sum_{j=1}^{\ell_m} c_{m,j}\, \mu(E_{m,j}) = \sum_{j=1}^{\ell_m} d_{m,j} \lambda_{m,j} +$$

$$+ \sum_{j=1}^{\ell_m} d_{m,j}(\mu(E_{m,j}) - \lambda_{m,j}) + \sum_{j=1}^{\ell_m} (c_{m,j} - d_{m,j})\mu(E_{m,j}) \leq$$

$$\leq \sum_{j=1}^{\ell_m} d_{m,j} \lambda_{m,j} + 2C\varepsilon, \text{ where } C = \bigvee_{j=1}^{\ell_1} c_{1,j}.$$

From (29) it follows

$$\bigwedge_{m=1}^{\infty} \left(\int_X f_m(x)\, d\mu(x) \right) \leq 2\, C\, \varepsilon \ .$$

Since L is Archimedean and ε is an arbitrary positive real number, we have: $(\int_X f_m(x)\, d\mu(x)) \searrow 0$ $(m \to \infty)$. The proof is complete.

Now, we shall give some examples of vector lattices satisfying DTC.

Proposition 12: The vector lattice $\mathbb{R}$ of all real numbers with natural operations and order satisfies DTC.

Proof: The monotone limit convergence theorem holds for real functions.

Proposition 12 may be proved also in a direct way using Dini´s theorem for compact spaces.

Definition 13: A vector lattice L is called separative if for every $x, y \in L$, $x \neq y$, there exists a linear form $f: L \to \mathbb{R}$ such that:

$\forall a \geq 0: f(a) \geq 0$

$\forall \{a_n\}_{n=1}^{\infty} . a_n \searrow 0$ $(n \to \infty) \Rightarrow f(a_n) \searrow 0$

$f(x) \neq f(y)$.

Theorem 14: Any separative vector lattice satisfies DTC. This fact follows from the Definitions 2, 6 and 13. It follows also from the results of Šipoš´ paper [5].

Theorem 15: Let $(Y, \mathcal{T}, \nu)$ be a σ-finite measure space (not necessarily inner regular). For all $p \in (0, \infty]$ the vector lattice $L^p(Y, \mathcal{T}, \nu)$ satisfies DTC.

Proof: If $p \in [1, \infty]$ then $L^p(Y, \mathcal{T}, \nu)$ is separative and satisfies DTC - Theorem 14. If $p \in (0,1)$ then $L^p(Y, \mathcal{T}, \nu)$ need not be separative (see [4] p. 318), but it also satisfies DTC. We shall use the fact that $L^{\infty}(Y, \mathcal{T}, \nu)$ satisfies DTC which was shown above. Let $\{f_{n,k}\}_{n=0,k=1}^{\infty, 2^n}$ be a dyadic tree of functions $f_{n,k} \in L^p(Y, \mathcal{T}, \nu)$ which is chain-decreasing to zero.

$$\text{Put } g_{n,k}(x) = \begin{cases} \dfrac{f_{n,k}(x)}{f_{o,1}(x)}, & \text{if } f_{o,1}(x) \neq 0 \\ 0, & \text{if } f_{o,1}(x) = 0. \end{cases}$$

Then $\forall n : \forall k \in \{1,\dots,2^n\} : 0 \leq g_{n,k} \leq 1$, i.e.: $g_{n,k} \in L^{\infty}(Y, \mathcal{T}, \nu)$. Moreover, the dyadic tree $\{g_{n,k}\}_{n=0,k=1}^{\infty, 2^n}$ is chain-decreasing to zero. Therefore, $(2^{-n} \sum_{k=1}^{2^n} g_{n,k}) \searrow 0$ which means $(2^{-n} \sum_{k=1}^{2^n} f_{n,k}) \searrow 0$.

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