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COUNTABLE INDUCTIVE DEFINITIONS IN AST

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Abstract: We transfer the notions of inductive definition, fixed point and inductive class in Alternative Set Theory and show that every Σ-semiset is a fixed point and every Σ-class is inductive.

Key words: Inductive definition, fixed point, inductive class, Alternative Set Theory, Σ-class.

Classification: 02K10, 02B99

The main source of reference is [MO]. In Section 1 we adapt key definitions and cite basic facts from [MO]. In Section 2 we prove the results which seem to be specific for the context of AST. In Section 3 we show that all Σ-classes are inductive.

We assume the reader's familiarity with all basic concepts of the Alternative Set Theory as exposed e.g. in [V].

§ 1. Adapted definitions and facts. Let $\varphi(Z)$ be a normal formula of FL_V, Z being among the class variables of φ. We say that φ is positive in Z, or simply φ is positive (if Z is the only class variable of φ), if φ belongs to the collection $\Phi_p(Z)$ of formulas defined as follows:

$\Phi_p(Z)$ is the smallest class of formulas such that:

(i) If Z does not occur in φ, then $\varphi \in \Phi_p(Z)$.

(ii) If t is a constant or variable, then $t \in Z$ is in $\Phi_p(Z)$.

(iii) If φ, ψ are in $\Phi_p(Z)$ and x is any (set) variable, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $(\exists x)\varphi$, $(\forall x)\varphi$ are in $\Phi_p(Z)$.

The main property of positive formulas is monotonicity.

Lemma 1.1. If $\varphi(Z)$ is positive in Z, then for any classes X, Y, $X \subseteq Y$ and $\varphi(X)$ imply $\varphi(Y)$.

Proof. By induction through the steps of the previous definition.

If the only class variable of $\varphi(x,Z)$ is Z, the only set variable is x and φ is positive in Z, then we can adjoin to φ an operator Γ_φ sending the class X to the class

$$\Gamma_\varphi(X) = \{x; \varphi(x,X)\}.$$

Γ_φ is monotone, i.e.

$$X \subseteq Y \longrightarrow \Gamma_\varphi(X) \subseteq \Gamma_\varphi(Y).$$

(Γ_φ is an informal object and we use it just to simplify the notation in some cases.) If φ contains only set-definable class parameters, then Γ_φ sends every set-definable class to a set-definable class.

Lemma 1.2. If $\varphi(x,Z)$ and $\psi(x,Z)$ are positive, then $\varphi(x, \Gamma_\varphi(Z))$ is positive in Z.

Proof. By induction on the length of φ.

Given a positive $\varphi(x,Z)$ we define an increasing sequence of classes $(I_\varphi^n)_{n \in FN}$ as follows:

$$I_\varphi^0 = \emptyset$$

$$I_\varphi^{n+1} = \Gamma_\varphi(I_\varphi^n) = \{x; \varphi(x, I_\varphi^n)\}.$$

This is a typical inductive definition which could probably be continued beyond the finite ordinals.

Here, however, we are interested in countable inductions, that is inductions which terminate in ω steps. These are defined by positive formulas φ such that $\Gamma_\varphi(\bigcup_n I^n_\varphi) = \bigcup_n I^n_\varphi$.

Let us put

$$I_\varphi = \bigcup\{I^n_\varphi;\ n \in Fn\}$$

for every positive φ.

We say that I_φ is a <u>fixed point</u> if φ contains no class parameters and $\Gamma_\varphi(I_\varphi) = I_\varphi$.

If $\varphi(Z)$ is positive in Z and X is a class, we say that <u>the parameter</u> X <u>is positive in</u> $\varphi(X)$.

If all the parameters $X_1, X_2, \ldots, X_k$ of φ are positive and $\Gamma_\varphi(I_\varphi) = I_\varphi$, then we say that I_φ is a <u>fixed point in</u> $X_1, X_2, \ldots, X_k$.

A class X is <u>inductive</u> (<u>inductive in</u> $X_1, \ldots, X_k$) if for some fixed point (fixed point in $X_1, \ldots, X_k$) I_φ and some set parameter a,

$$x \in X \leftrightarrow \langle x, a\rangle \in I_\varphi \leftrightarrow x \in I_\varphi'' \{a\}.$$

<u>Lemma 1.3</u>. (i) Every fixed point is inductive. (ii) Every inductive class is a Σ-class. (iii) Every set-definable class is a fixed point.

<u>Proof</u>. (i) Let I_φ be a fixed point in $X_1, \ldots, X_k$ and let a be a parameter. Put

$$\psi(x,y,Z) \equiv \varphi(x, Z''\{a\}) \wedge y = a.$$

If $\sigma(x,Z) \equiv \langle x,a\rangle \in Z$, then clearly σ is positive and $Z''\{a\} = \Gamma_\sigma(Z)$. By 1.2 ψ is positive and we can see inductively that $I^n_\psi = I^n_\varphi \times \{a\}$, whence $I_\psi = I_\varphi \times \{a\}$, I_ψ is a fixed point and $I_\varphi = I_\psi''\{a\}$.

(ii) Let I_φ be a fixed point. If I^n_φ is set-definable then obviously $I_\varphi^{n+1} = \{x; \varphi(x,I^n)\}$ is set-definable. Similarly, for any a $I_\varphi''\{a\} = \bigcup_n I_\varphi^{n}{}''\{a\}$ and $I_\varphi^{n}{}''\{a\}$ are set-definable. (This is no longer true, however, for classes inductive in other classes.)

(iii) If X is set-definable, put

$$\varphi(x,Z) \equiv x \in X.$$

Then $I_\varphi^n = I_\varphi = X$ for $n \geq 1$.

Since we are interested in countable inductions, we have to deal exclusively with positive formulas leading to such inductions.

Let $\varphi(Z)$ be positive and let $(Y_n)_{n\in FN}$ be an increasing sequence of set-definable classes. We say that φ <u>is stationary in</u> Z <u>w.r.t.</u> $(Y_n)_{n\in FN}$, if

$$\varphi(\bigcup_n Y_n) \longleftrightarrow (\exists n \in FN)\,\varphi(Y_n).$$

We say that φ <u>is stationary in</u> Z if it is stationary w.r.t. any such sequence.

<u>Lemma 1.4</u>. Let $\varphi(Z)$ be positive and stationary. Then for every increasing sequence of inductive (or, more generally, Σ-) classes $(D_n)_n$,

$$\varphi(\bigcup_n D_n) \longleftrightarrow (\exists n \in FN)\,\varphi(D_n).$$

<u>Proof</u>. Suppose D_n, $n \in FN$, are Σ-classes and $D_n = \bigcup_m D_n^m$. Let $(E_k)_{k\in FN}$ be an enumeration of all D_n^m, $m,n \in FN$. Define two functions H_1, H_2 from FN to FN by recursion as follows:

$H_1(0) = H_2(0) = 0$, and $H_1(k+1)$ = least m such that there is an n such that

$$E_{k+1} \cup D_{H_1(k)}^{H_2(k)} \subseteq D_m^n,$$

$H_2(k+1)$ = least n such that

$$E_{k+1} \cup D_{H_1(k)}^{H_2(k)} \subseteq D_{H_1(k+1)}^n$$

The definition makes sense for all $k \in FN$ because $(D_n)_n$ is increasing. Clearly $D^{H_2(n)}_{H_1(n)} \subseteq D^{H_2(n+1)}_{H_1(n+1)}$ and $\bigcup_n D_n = \bigcup_n D^{H_2(n)}_{H_1(n)}$.

Since φ is positive and stationary, we get

$$\varphi(\bigcup_n D_n) \leftrightarrow \varphi(\bigcup_n D^{H_2(n)}_{H_1(n)}) \leftrightarrow (\exists n)\, \varphi(D^{H_2(n)}_{H_1(n)}) \rightarrow (\exists n)\, \varphi(D_{H_1(n)}) \rightarrow$$
$$\rightarrow (\exists k)\, \varphi(D_k).$$

The other direction follows from positivity.

It is evident that if $\varphi(x,Z)$ is positive and stationary in Z and does not contain class parameters, then I_φ is a fixed point. A fixed point I_φ (or an inductive class $I''_\varphi\{a\}$) is called <u>stationary</u> if the defining formula is stationary.

We shall see later that stationary formulas form a sufficiently large part of all positive formulas.

The following is a version of the Transitivity Theorem (cf. [MO],1C.3).

<u>Theorem 1.5</u>. Let $\varphi(x,y,Z_1,\dots,Z_k,Z)$ be a formula positive and stationary in all its class variables. If $X_1,\dots,X_k$ are stationary inductive classes, $\varphi_0(x,y,Z) \equiv \varphi(x,y,X_1,\dots,X_k,Z)$, I_{φ_0} is a fixed point in $X_1,\dots,X_k$ and $X = I''_{\varphi_0}\{a\}$ for some a, then X is stationary inductive.

<u>Proof</u>. To simplify the argument suppose $k = 1$. The treatment for $k > 1$ is quite the same. Since X_1 is stationary inductive. there is a positive stationary formula $\varphi_1(x_1,y_1,Z)$ and a constant b_1 such that $X_1 = I''_{\varphi_1}\{b_1\}$. We shall combine the two inductions defined by φ_1 and φ into a single induction defined by a positive and stationary formula $\wp(x,y,x_1,y_1,t,Z)$. Consider arbitrary constants x^*, y^*, x_1^*, y_1^*, and put:

$\wp(x,y_1,y_1,t,Z) \equiv$
$[t = 1 \wedge \varphi_1(x_1,y_1,\{\langle x'_1,y'_1\rangle;\ \langle x^*,y^*,x'_1,y'_1,1\rangle \in Z\})] \vee$
$[t=2 \wedge \varphi(x,y,\{x'_1;\langle x,y,x'_1,b_1,1\rangle \in Z,\{\langle x',y'\rangle;\langle x',y',x^*_1,y^*_1 2\rangle \in Z\})]$.

By 1.2 $\wp$ is positive and we claim that:

i) $\wp$ is stationary,

ii) $\langle x,y\rangle \in I_{\varphi_0} \leftrightarrow \langle x,y,x^*_1,y^*_1,2\rangle \in I_{\wp}$.

To see i) put

$\{\langle x'_1,y'_1\rangle;\langle x^*,y^*,x'_1,y'_1,1\rangle \in Z\} = Z^1$,
$\{x'_1;\langle x,y,x'_1,b_1,1\rangle \in Z\} = Z^2$,
$\{\langle x',y'\rangle;\langle x',y',y^*_1,y^*_1,2\rangle \in Z\} = Z^3$.

Then,

$\wp(t,Z) \equiv [t=1 \wedge \varphi_1(Z^1)] \vee [t=2 \wedge \varphi(Z^2,Z^3)]$.

Let $(Y_n)_{n\in FN}$ be an increasing sequence of set-definable classes. Then so are the sequences $(Y^i_n)_n$, $i = 1,2,3$. Then,

$\wp(t,\bigcup_n Y_n) \equiv [t=1 \wedge \varphi_1(\bigcup_n Y^1_n)] \vee [t=2 \wedge \varphi(\bigcup_n Y^2_n,\bigcup_n Y^3_n)]$,

and since φ_1, φ are stationary in all variables, we get

$\wp(t,\bigcup_n Y_n) \leftrightarrow (\exists k,m,n \in FN)\ [t=1 \wedge \varphi_1(Y^1_k)] \vee [t=2 \wedge \varphi(Y^2_m,Y^3_n)]$,

whence, by monotonicity,

$\wp(t,\bigcup_n Y_n) \leftrightarrow (\exists\, n \in FN)\wp(t,Y_n)$.

The essential part of the theorem is, of course, claim ii) and this is just the content of the Combination Lemma 1C.2 of [MO]. From ii) we have

$$x \in X \leftrightarrow \langle x,a\rangle \in I_{\varphi_0} \leftrightarrow \langle x,a,x^*_1,y^*_1,2\rangle \in I_{\wp}\ ,$$

hence X is stationary inductive.

<u>Remarks</u>. 1) If φ_0 in the preceding theorem does not contain class variables, then the condition that I_{φ_0} be a fixed point is obviously satisfied, hence for any formula $\varphi(x,X_1,\dots,X_k)$ with $X_1,\dots,X_k$ as in the theorem, the class $\{x;\varphi(x,X_1,\dots,X_k)\}$ is stationary inductive.

2) It follows from the previous remark that, given $\varphi(x,X_1,\dots,X_k,Z)$ with $X_1,\dots,X_k$ as in the theorem, I^1_φ is stationary inductive and we see inductively that all I^n_φ are inductive. Thus, by 1.4 $\varphi(x,X_1,\dots,X_k,I_\varphi) \leftrightarrow x \in I_\varphi$, that is, I_φ is a fixed point in $X_1,\dots,X_k$. This shows that the requirement for I_{φ_0} in the preceding theorem to be a fixed point in $X_1,\dots,X_k$ is superfluous.

The reason that we use positive formulas instead of merely monotone ("up hereditary" in the terminology of Mlček,[M]) is that the former admit a canonical form. Namely the following holds:

<u>Theorem 1.6</u>. Let $\varphi(Z)$ be a positive formula. Then there is a quantifier-free set-formula $\Theta(\bar{z},u) \equiv \Theta(z_1,\dots,z_n,u)$ and a string $\bar{Q} \equiv Q_1 \dots Q_n$ of quantifiers such that

$$\varphi(Z) \equiv (Q_1 z_1)\dots(Q_n z_n)(\forall u)[\Theta(z_1,\dots,z_n,u) \vee u \in Z],$$

for all $Z \neq V$, or briefly,

$$(*) \quad \varphi(Z) \equiv (\bar{Q}\bar{z})(\forall u)\ (\Theta(\bar{z},u) \vee u \in Z),$$

for all $Z \neq V$.

<u>Proof</u>. [MO], 4B.1.

<u>Remark</u>. The restriction $Z \neq V$ is for the case that $\varphi(V)$ is false, since the right hand side of $(*)$ is always true for $Z = V$. However, given φ , we can put $\psi(x,Z) \equiv \varphi(x,Z) \vee (\forall y)(y \in Z)$. If φ is positive and stationary then so is ψ , $I_\varphi = I_\psi$ and, in addition, $(\forall x)\psi(x,V)$ is true. Thus studying fixed points we may always assume that $(*)$ holds for all Z.

§ 2. Some results on fixed points. In the proof of the next lemma we just use the fact that for every set-definable class X and for every Σ-class $\bigcup_n Y_n$,

$$X \subseteq \bigcup_n Y_n \leftrightarrow (\exists n)(X \subseteq Y_n).$$

Lemma 2.1. Let $\varphi(x,Z)$ be positive. If φ has a canonical form

$$(\bar{Q}\bar{z})(\forall u)(\Theta(x,\bar{z},u) \vee u \in Z),$$

where $\bar{Q} = \emptyset$, or $\forall^k$, or $\exists^k$, or $\exists^k\forall^k$, then φ is stationary.

Proof. Put $R = \{\langle x,\bar{z}, u\rangle;\ \neg\Theta(x,z,u)\}$, and let $R''(x,\bar{z}) = \{u;\ \langle x,\bar{z},u\rangle \in R\}$. Then

(1) $\varphi(x,Z) \equiv (\bar{Q}\bar{z})(R''(x,\bar{z}) \subseteq Z)$.

Let us show for example the case $Q = \forall^k$. The rest is shown similarly. Let $(Y_n)_n$ be an increasing sequence of set definable classes. Then:

$\varphi(x,\bigcup_n Y_n) \leftrightarrow (\forall z_1 \ldots z_k)(R''(x,z_1,\ldots,z_k) \subseteq \bigcup_n Y_n) \leftrightarrow$
$\bigcup\{R''(x,z_1,\ldots,z_k);\ z_1,\ldots,z_k \in V\} \subseteq \bigcup_n Y_n \leftrightarrow$
$(\exists n)\ [\bigcup\{R''(x,z_1,\ldots,z_k);\ z_1,\ldots,z_k \in V\} \subseteq Y_n] \leftrightarrow$
$(\exists n)(\forall z_1 \ldots z_k)(R''(x,z_1,\ldots,z_k) \subseteq Y_n) \leftrightarrow (\exists n)\varphi(x,Y_n)$.

Let us call an existential quantifier in the prefix $(\bar{Q}\bar{z})$ of the canonical form $(*)$ inessential if there is some set-definable Skolem-function for it. The following is obvious:

Lemma 2.2. If the prefix $(\bar{Q}\bar{z})$ of the canonical form of φ after the extraction of all inessential quantifiers is as in Lemma 2.1, then φ is stationary.

The last two lemmas imply that the simplest positive non-stationary formula cannot be less complicated than the formula $(\forall z_1)(\exists z_2)(R''(x,z_1,z_2) \subseteq Z)$, where $(\exists z_2)$ is not inessential.

However we do not know whether there exist non-stationary positive formulas.

The next result restricts further the possible φ for which there is not a fixed point.

Lemma 2.3. If $(u_n)_{n\in FN}$ is an increasing sequence of sets, then every positive formula is stationary w.r.t. $(u_n)_{n\in FN}$.

Proof. Let $\varphi(x,Z)$ be positive. Taking a canonical form of φ and defining R as in 2.1, we have

$$\varphi(x,Z) \equiv (\bar{Q}\bar{z})(R''(x,\bar{z}) \subseteq Z).$$

Then, $\varphi(x,\bigcup_m u_n \equiv (\bar{Q}\bar{z})(R''(x,\bar{z}) \subseteq \bigcup u_n) \leftrightarrow (\bar{Q}\bar{z})(\exists n)(R''(x,\bar{z}) \subseteq u_n)$.

Put $R''(x,\bar{z}) \subseteq u_n \equiv \psi(x,z,u_n)$.

ψ is positive in u_n and it suffices to show that for every set-formula $\psi(x,\bar{z},w)$, positive in w and every increasing sequence $(u_n)_{n\in FN}$,

$$(\bar{Q}\bar{z})(\exists n)\psi(x,\bar{z},u_n) \leftrightarrow (\exists n)\bar{Q}z)\psi(x,\bar{z},u_n).$$

Suppose ψ and $(u_n)_{n\in FN}$ are given. Prolong the sequence $(u_n)_{n\in FN}$ to a set $\{u_\beta;\ \beta \le \alpha\}$ such that $u_\beta \subseteq u_\gamma$ for $\beta \le \gamma$. If the string $\bar{Q}$ has length m, the above equivalence is shown in m steps by pulling at the step $\varkappa$ the quantifier $(\exists n)$ to the front of the quantifier Q_{m-k+1}. If $Q_{m-k+1} = \exists$ this is trivially possible. Thus it suffices to show that

(+) $(\forall z)(\exists n)\sigma(x,z,u_n) \leftrightarrow (\exists n)(\forall z)\sigma(x,z,u_n)$

for σ positive in u_n.

Let $F: V \longrightarrow N$ be a function such that

$F(z) = \beta \leftrightarrow \sigma(x,z,u_\beta) \wedge (\forall\gamma < \beta)\neg\sigma(x,z,u_\gamma)$.

The direction " $\leftarrow$ " of (+) is obvious.

Now, if the left hand-side of (+) is true, then F is (set-) defined on V and $F''V \subseteq FN$. Therefore $F''V$ is finite. It follows that $(\forall z)(\exists n)\sigma(x,z,u_n) \leftrightarrow (\exists n)(\forall z)[\sigma(x,z,u_1) \vee \ldots \vee \sigma(x,z,u_n)]$.

Since σ is positive and $(u_n)_{n \in FN}$ is increasing, we get
$\sigma(x,z,u_1) \vee \ldots \vee \sigma(x,z,u_n) \longleftrightarrow \sigma(x,z,u_n)$
and the proof is complete.

A positive formula $\varphi(x,Z)$ will be called **reversible**, if there is some positive $\psi(x,Z)$ such that $\Gamma_\psi(\Gamma_\varphi(Z)) = Z$ for every class Z. ψ is called a reverse of φ .

For example the formula

$$\psi(x,Z) \equiv (\exists y)(x \in y \wedge y \in Z),$$

with operator $\Gamma_\psi = \bigcup$ is a reverse of the formula

$$\varphi(x,Z) \equiv x \subseteq Z,$$

with operator P (the power-set operator), since $\bigcup P(Z) = Z$.

<u>Lemma 2.4</u>. If I_φ is a fixed point and $\varphi(x,Z)$ is a stationary in $(I_\varphi^n)_{n \in FN}$ and reversible positive formula, then $\Gamma_\psi(I_\varphi)$ is a fixed point.

<u>Proof</u>. Let $\sigma(x,Z)$ be a reverse of ψ and put

$$\rho(x,Z) \equiv \psi(x, \Gamma_\varphi \Gamma_\sigma(Z)).$$

By 1.2 ρ is positive. We show that $I_\rho^n = \Gamma_\psi(I^n)$ for $n \geq 1$.

First notice that $\Gamma_\sigma(\emptyset) = \emptyset$ because $\Gamma_\sigma \Gamma_\psi(\emptyset) = \emptyset$ and $\emptyset \subseteq \Gamma_\psi(\emptyset)$ and $\Gamma_\sigma(\emptyset) \subseteq \Gamma_\sigma \Gamma_\psi(\emptyset)$. Hence $I_\rho^1 = \{x; \psi(x, \Gamma_\varphi \Gamma_\sigma(\emptyset) = \{x; \psi(x, \Gamma_\varphi(\emptyset) = \{x; \psi(x, I_\varphi^1) = \Gamma_\psi(I_\varphi^1)$.

Suppose $I_\rho^n = \Gamma_\psi(I_\varphi^n)$. Then

$$I_\rho^{n+1} = \Gamma_\rho(I_\rho^n) = \Gamma_\psi \Gamma_\varphi \Gamma_\sigma(I_\rho^n) = \Gamma_\psi \Gamma_\varphi \Gamma_\sigma \Gamma_\psi(I_\varphi^n) = \Gamma_\psi \Gamma_\varphi(I_\varphi^n) = \Gamma_\psi(I_\varphi^{n+1}).$$

Since ψ is stationary, we have

$I_\rho = \bigcup_n I_\rho^n = \bigcup_n \Gamma_\psi(I_\varphi^n) = \Gamma_\psi(\bigcup_n I_\varphi^n) = \Gamma_\psi(I_\varphi)$.

It remains to show that $\Gamma_\varphi(I_\rho) = I_\rho$.

$\Gamma_\rho(I_\rho) = \Gamma_\psi \Gamma_\varphi \Gamma_\sigma(I_\rho) = \Gamma_\psi \Gamma_\varphi \Gamma_\sigma \Gamma_\psi(I_\varphi) = \Gamma_\psi \Gamma_\varphi(I_\varphi) = \Gamma_\psi(I_\varphi) = I_\rho$

(If moreover φ , σ are stationary, then ρ is stationary.)

Corollary 2.5. If I_φ is a fixed point and F is a 1-1 set-definable function with $I_\varphi \subseteq dom(F)$, then $F''I_\varphi$ is a fixed point. If I_φ is stationary, $F''I_\varphi$ is stationary.

Proof. Consider the formula

$\psi(x,Z) \equiv (\exists y \in Z)(F(y) = x)$. Clearly $\Gamma'_\psi(Z) = F''Z$ for $Z \subseteq dom(F)$, ψ is stationary and the formula

$\sigma(x,Z) \equiv (\exists y \in Z)(F(x) = y)$ which is stationary, is a reverse for ψ. The conclusion follows from 2.4.

Corollary 2.6 Every countable class is a stationary fixed point.

Proof. Since for any countable class X there is a 1-1 set-function f such that $f''FN = X$, it suffices by 2.5 to prove that FN is a stationary fixed point.

Let $<$ be the ordering of natural numbers and put

$$\varphi(x,Z) \equiv (\forall u)(u < x \rightarrow u \in Z).$$

Clearly φ is positive, stationary and $I^n_\varphi = n$.

We shall now prove that every countable union of sets (Σ-semisets) or cosets is a fixed point. First a lemma.

Lemma 2.7. Let $(u_n)_{n\in FN}$ be a sequence of sets. Then there is an increasing sequence $(v_n)_{n\in FN}$ such that $\bigcup_n u_n = \bigcup_n v_n$ and the sequence of natural numbers $|v_{n+1} - v_n|$ is either increasing or decreasing ($|u|$ is the unique $\alpha \in N$ such that $u \mathrel{\hat{\approx}} \alpha$).

Proof. Suppose, without loss of generality, that $(u_n)_{n\in FN}$ is increasing. Then either

(1) $(\exists n)(\forall m > n)(\exists k > m)(|u_m - u_n| \leq |u_k - u_m|)$,

or the negation of (1) is true:

(2) $(\forall n)(\exists m > n)(\forall k > m)(|u_k - u_m| < |u_m - u_n|)$.

In the first case we can find a subsequence $(u_{n_k})_k = (v_k)_k$, such that $|v_{k+1} - v_k|$ is increasing and in the second case we find a subsequence $(v_k)_k$ such that $|v_{k+1} - v_k|$ is decreasing.

<u>Theorem 2.8</u>. For every sequence $(u_n)_{n\in FN}$, $\bigcup_n u_n$ is a stationary fixed point.

<u>Proof</u>. <u>Case 1</u>. Suppose there is an increasing sequence $(v_n)_n$ with $|v_{n+1} - v_n|$ increasing such that $\bigcup_n u_n = \bigcup_n v_n$. Extend $(v_n)_n$ to a set $\{v_\beta ; \beta \leq \alpha\}$ with $|v_{\beta+1} - v_\beta|$ increasing.

For every β, $2 \leq \beta \leq \alpha$, let g_β be the surjection from $v_\beta - v_{\beta-1}$ onto $v_{\beta-1} - v_{\beta-2}$ which is least in the usual set-definable ordering of V. Let also g_1 be the least surjection from v_1 onto v_0. The correspondence $\beta \mapsto g_\beta$ is set-definable and put $f_\beta = \bigcup\{g_\gamma ; \gamma \leq \beta\}$.

Then f_β is a function from v_β onto $v_{\beta-1}$. Let $f = f_\alpha$. It is easily seen that $f \restriction v_\beta = f_\beta$ and $f^{-1}{}'' v_\beta = v_{\beta+1}$. (Some Venn-diagrams illustrate best the situation.)

If we put

$$\varphi(x,Z) \equiv x \in v_0 \vee (\exists y \in Z)(f(x) = y)$$

then φ is stationary and $I_\varphi^{n+1} = v_n$, hence $\bigcup_n u_n = \bigcup_n v_n = I_\varphi$.

<u>Case 2</u>. Let again $\bigcup_n u_n = \bigcup_n v_n$ where $|v_{n+1} - v_n|$ is decreasing. Extend as before $(v_n)_n$ to a set $\{v_\beta ; \beta \leq \alpha\}$. Let $g_\beta : v_{\beta-1} - v_{\beta-2} \to v_\beta - v_{\beta-1}$ be least surjections for $2 \leq \beta \leq \alpha$, while g_1 is the identity on v_0. Put $f_\beta = \bigcup\{g_\gamma ; \gamma \leq \beta\}$; then f_β maps $v_{\beta-1}$ onto $v_\beta - (v_1 - v_0)$. Put $f = f_\alpha$. Then $f \restriction v_\beta = f_{\beta+1}$ for $\beta < \alpha$. Consider the formula

$$\varphi(x,Z) \equiv x \in v_1 \vee (\exists y \in Z)(f(y) = x).$$

Again φ is stationary and $I_\varphi^n = v_n$ for $n \geq 1$.

A completely analogous result holds for sequences of cosets

$(V - u_n)_{n\in FN}$ or, more generally, $(X - u_n)_{n\in FN}$, where X is set-definable. It is evident that Lemma 2.7 is equally true if we substitute "decreasing" for "increasing" and $\cap$ for $\cup$. Then, given a set-definable X and a sequence $(u_n)_n$, such that

$$\ldots \subseteq u_1 \subseteq u_0 \subseteq X,$$

with $|u_n - u_{n+1}|$ either increasing or decreasing, it is easy to construct a set-definable function f with $dom(F) \subseteq X$ and such that either F" or F^{-1}" defines $X - u_{n+1}$ by means of $X - u_n$. Therefore:

Lemma 2.9. If X is set-definable and $(u_n)_n$ is any sequence of sets, then $\bigcup_{m} (X - u_n)$ is a stationary fixed point.

§ 3. All Σ-classes are inductive. Let Fix, Ind, Σ denote respectively the (codable) classes of fixed points, inductive classes and Σ-classes. By 1.3

$$\mathrm{Fix} \subseteq \mathrm{Ind} \subseteq \Sigma .$$

We shall prove in this section that

Theorem 3.1. Σ = Ind.

This will be done through a number of lemmas.

Lemma 3.2. If Sd_V has a code $\langle K,S\rangle$ such that the class S is stationary inductive, then Σ = Ind.

Proof. Let $\langle K,S\rangle$ be a code of Sd_V such that S is stationary inductive and let $X = \bigcup_{m} X_n$ be a Σ-class. Then evidently there s a countable $Y \subseteq K$ such that

$$\{X_n ; n \in FN\} = \{S"\{y\} ; y \in Y\},$$

hence we get

$$x \in X \longleftrightarrow (\exists y \in Y)(x \in S"\{y\}).$$

The formula $(\exists y \in Y)(x \in S''\{y\})$ contains the class-parameters Y, S in positive stationary positions and Y (by 2.6) as well as S (by assumption) are stationary inductive. It follows from the Transitivity Theorem 1.5 that X is stationary. Thus $\Sigma \subseteq$ ind.

Take a Gödelization of the language FSL_V, i.e. of all the finite set-formulas, as a mapping $G: FSL_V \to V$ defined as follows:

1) $G(x_n) = \langle 0,n \rangle$, for the set variables x_n, $n \in FN$.
2) $G(x) = \langle 1,x \rangle$, for the set-constants $x \in V$.
3) $G(t=s) = \langle 2,G(t),G(s) \rangle$, for constants or variables t, s.
4) $G(t \in s) = \langle 3,G(t),G(s) \rangle$, " " " " "
5) $G(\neg \varphi) = \langle 4,G(\varphi) \rangle$.
6) $G(\varphi \wedge \psi) = \langle 5,G(\varphi),G(\psi) \rangle$.
7) $G((\exists x_n)\varphi) = \langle 6,n,G(\varphi) \rangle$.

We say that $G(\varphi)$ is the Gödel-set of φ .

Lemma 3.3. The class $Fml = \{x;\ x$ is the Gödel-set of a set formula$\}$ is stationary inductive.

Proof. Let us denote by AFml the class of (Gödel-sets of) atomic formulas. Then , from the definition of G we have

$x \in AFml \leftrightarrow (\exists m,n \in FN)(\exists y,z)[x = \langle 2,\langle 0,m \rangle,\langle 0,n \rangle\rangle \vee$
$x = \langle 3,\langle 0,m \rangle,\langle 0,n \rangle\rangle \vee x = \langle 2,\langle 0,m \rangle,\langle 1,y \rangle\rangle \vee x = \langle 3,\langle 0,m \rangle,\langle 1,y \rangle\rangle \vee$
$x = \langle 2,\langle 1,y \rangle,\langle 1,x \rangle\rangle \vee x = \langle 3,\langle 1,y \rangle,\langle 1,x \rangle\rangle]$.

It is clear that the defining formula is positive and stationary in FN and FN is stationary inductive, thus AFml is stationary inductive by 1.5.

Next,

$x \in Fml \leftrightarrow (\exists f)(\exists n \in FN)(dom(f) = n+1 \wedge f(n) = x \wedge$
$(\forall k < n)[f(k) \in AFml \vee (\exists l,m < k)(f(k) = \langle 4,f(l) \rangle \vee f(k) =$
$= \langle 5,f(l),f(m) \rangle) \vee (\exists i \in FN)(\exists j < k)(f(k) = \langle 6,i,f(j) \rangle)]$.

Again the defining formula is positive and stationary in

FN, AFml while the latter are stationary inductive, **hence Fml is** stationary inductive.

Consider now the predicate of satisfaction Sat(x,g) expressing the fact: "x is the Gödel-set of a set-formula φ, g is a sequence of valuations for the variables x_n, $n \in FN$, and φ is true substituting g(i) for its free variable x_i".

We fix a number $\alpha_0 \in N-FN$ and let

$A = \{g; dom(g) = \alpha_0 \wedge (\exists n \in FN)(\forall k > n)(g(k) = 0)\}$.

Clearly A is stationary inductive. For $g \in A$ write $g \models \varphi$ for the fact that φ is true w.r.t. the valuation g.

The following is a version of 5.3.2 of [MO].

Lemma 3.4. The class $Sat = \{\langle x,g\rangle; Sat(x,g)\}$ is stationary inductive.

Proof. Define the predicate Val(x,g,t) by:

$Val(x,g,t) \equiv$ (x is the Gödel-set of a formula φ) $\wedge\, g \in A\, \wedge$

$\wedge [(t = 0 \wedge g \models \varphi) \vee (t = 1 \wedge g \models \neg\varphi)]$.

Then obviously

$$Sat(x,g) \longleftrightarrow Val(x,g,0),$$

and it suffices to prove that there is a formula $\wp$ positive and stationary in all its class parameters, the latter being stationary inductive, such that $I_\wp$ is a fixed point and

$$Val(x,g,t) \longleftrightarrow \langle x,g,t\rangle \in I_\wp .$$

Let $AVal(x,g,t) \equiv x \in AFml \wedge Val(x,g,t)$.

Then:

$AVal(x,g,t) \longleftrightarrow g \in A \wedge (\exists m,n \in FN)(\exists y,z)$

$\{x = \langle 2,\langle 0,m\rangle,\langle 0,n\rangle\rangle \wedge [(t = 0 \wedge g(m) = g(n)) \vee (t = 1 \wedge g(m) \neq g(n))]\} \vee$

$\{x = \langle 3,\langle 0,m\rangle,\langle 0,n\rangle\rangle \wedge [(t = 0 \wedge g(m) \in g(n)) \vee (t=1 \wedge g(m) \notin g(n))]\} \vee$

$\{x = \langle 2,\langle 0,m\rangle,\langle 1,y\rangle\rangle \wedge [(t=0 \wedge g(m) = y) \vee (t=1 \wedge g(m) \neq y)]\} \vee$

$\{x = \langle 3,\langle 0,m\rangle,\langle 1,y\rangle\rangle \wedge [(t=0 \wedge g(m) \in y) \vee (t=1 \wedge g(m) \notin y)]\} \vee$

$\{x = \langle 2,\langle 1,y\rangle,\langle 1,z\rangle\rangle \wedge [(t=0 \wedge y = z) \vee (t=1 \wedge y \neq z)]\} \vee$

$\{x = \langle 3,\langle 1,y\rangle,\langle 1,z\rangle\rangle \wedge [(t=0 \wedge y \in z) \vee (t=1 \wedge y \notin z)]\}$

Then the class $AVal = \{\langle x,g,t\rangle ; AVal(x,g,t)\}$ is stationary inductive since $Aval(x,g,t)$ is positive stationary in A, FN.

Define the function F on $V^{\alpha_0} \times \alpha_0 \times V$ as follows:

$F(g,\beta,u) = (g - \{\langle \beta, g(\beta)\rangle\}) \cup \{\langle \beta,u\rangle\}$

i.e., $F(g,\beta,u)$ is the function resulting from g if we replace its value at β by u. Now define the required formula $\wp$ as follows:

$\wp(x,g,t,Z) \equiv x \in Fml \wedge g \in A \wedge \{\langle x,g,t\rangle \in AVal \vee$

$(\exists y)[(x = \langle 4,y\rangle \wedge t = 0 \wedge \langle y,g,1\rangle \in Z) \vee$

$(x = \langle 4,y\rangle \wedge t = 1 \wedge \langle y,g,0\rangle \in Z)] \vee$

$(\exists y,z)[(x = \langle 5,y,z\rangle \wedge t = 0 \wedge \langle y,g,0\rangle \in Z \wedge \langle z,g,0\rangle \in Z) \vee$

$(x = \langle 5,y,z\rangle \wedge t = 1 \wedge (\langle y,g,1\rangle \in Z \vee \langle z,g,1\rangle \in Z)$

$(\exists k \in FN)(\exists y)[(x = \langle 6,k,y\rangle \wedge t=0 \wedge (\exists z)(\langle y,F(g,k,z),0\rangle \in Z)) \vee$

$(x = \langle 6,k,y\rangle \wedge t=1 \wedge (\forall z)(\langle y,F(g,k,z),1\rangle\ Z))]\}$.

We can summarize $\wp$ as follows:

$\wp(x,g,t,Z) \equiv x \in Fml \wedge g \in A \wedge [\langle x,g,t\rangle \in AVal \vee$

$(\exists k \in FN)(\exists \bar{y})(\forall \bar{z})\,\sigma(t,x,g,\bar{y},\bar{z},Z)]$,

where σ is positive in Z and contains only inessential existential quantifiers. It follows from 2.1 and 2.2 that $\wp$ is stationary in Z. That $\wp$ is positive stationary in Fml, A, AVal, FN is evident. Also all these class-parameters are stationary inductive as we proved earlier. We must also prove that induction by $\wp$ closes in ω steps but this is clear from the remarks following Th. 1.5.

It remains to see that

$$Val(x,g,t) \longleftrightarrow \langle x,g,t\rangle \in I_\wp .$$

Direction " $\longrightarrow$ " is shown by induction on the length of the formula x, while by induction on n we show that

$$\langle x,g,t\rangle \in I_\wp^n \longrightarrow Val(x,g,t).$$

<u>**Proof of Theorem 3.1**</u>. The pair $\langle Fml, Sat\rangle$ is a code for Sd_V since for every $X \in Sd_V$ such that $X = \{x; \varphi(x)\}$, we have $X = \{g; Sat(x,g)\} = Sat''\{x\}$, where x is the Gödel-set of φ. From 3.4 it follows that the code is inductive, and by 3.2 $\Sigma = Ind$.

R e f e r e n c e s

[M] J. MLČEK: Approximations of Σ-classes and Π-classes, Comment. Math. Univ. Carolinae 20(1979), 669-679.

[MO] Y. MOSCHOVAKIS: Elementary induction on abstract structures, North-Holland, 1974.

[V] P. VOPĚNKA: Mathematics in the Alternative Set Theory, Teubner Texte, Leipzig, 1979.

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