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ON BAIRE ISOMORPHISMS OF NON-METRIZABLE COMPACTA

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Abstract: Using a spectral theorem for Baire mappings between compacta it is shown that: (a) the first-level Baire isomorphisms preserve the dimension dim of compacta; (b) there is no Baire isomorphism between the Cantor cube of weight τ^+ and its hyperspace for $\tau > 2^{\omega}$.

Key words: τ-spectrum, Baire set, Baire mapping.

Classification: 54B25, 54C50

All topological spaces considered will be compact and Hausdorff.

The Baire sets of X of multiplicative class 0, denoted $Z_0(X)$, are the zero-sets of continuous real-valued functions. The sets of additive class 0, denoted $CZ_0(X)$, are the complements of the sets in $Z_0(X)$. Define inductively for each countable ordinal α the sets of multiplicative class $\alpha+1$, denoted $Z_{\alpha+1}(X)$, to be the countable intersections of the sets of additive class α and the sets of additive class $\alpha+1$, denoted $CZ_{\alpha+1}(X)$, to be their complements. The sets of multiplicative class λ (λ a limit ordinal), denoted $Z_\lambda(X)$, are defined to be the countable intersections of countable unions of sets in $\bigcup_{\alpha<\lambda} Z_\alpha(X)$, and the sets of additive class λ, denoted $CZ_\lambda(X)$, are defined to be their complements. The sets from the collection $\bigcup_{\alpha\in\omega_1} Z_\alpha(X)$ are called the Baire sets of X. It is well-known that the collection of all

Baire sets of X is the smallest collection of subsets of X which contains $Z_0(X)$ and is closed under complementation, countable union and countable intersection.

A mapping $f:X \longrightarrow Y$ is called a Baire mapping (of class γ) if an inverse-image of each cozero-set of Y is a Baire set (of additive class γ) of X. A bijection f is called a Baire isomorphism (of class (γ, δ)) if f is a Baire mapping (of class γ) and f^{-1} is a Baire mapping (of class δ).

Let us recall also that a hyperspace of X, denoted exp X, is a collection of all non-empty compact subsets of X in the Vietoris topology. For any mapping $f:X \longrightarrow Y$ there exists an associated mapping $\exp f: \exp X \longrightarrow \exp Y$. It is well-known [6] that $\exp: COMP \longrightarrow COMP$ is a covariant functor.

Unless noted, definitions and terminology concerning inverse spectra will be found in [6].

Spectral representations of Baire mappings. Results of this section were announced in [1].

Lemma 1. Let $S = \{X_\alpha, p_\alpha^\beta, A\}$ be a τ-spectrum and $f:X \longrightarrow Y$ be a Baire mapping (of class γ) where $X = \lim S$ and $wY \leq \tau$. Then there exist an index $\alpha \in A$ and a Baire mapping (of class γ) $f_\alpha : X_\alpha \longrightarrow Y$ such that $f = f_\alpha \cdot p_\alpha$.

Proof. By [6], our spectrum is factorizing and so, using transfinite induction, one can prove that each Baire set of X is cylindrical, i.e. if B is a Baire set (of additive class γ) of X then there exist an index $\alpha \in A$ and a Baire set (of additive class γ) B_α of X_α such that $B = p_\alpha^{-1}(B_\alpha)$.

Let $\{G_\lambda : \lambda \in \tau\}$ be any base of Y consisting of cozero-sets of Y. Since f is a Baire mapping (of class γ), the sets

$f^{-1}(G_\lambda)$, $\lambda \in \tau$, are Baire sets (of additive class γ) of X and consequently, by the above remark, for each $\lambda \in \tau$ there exists $\alpha_\lambda \in A$ such that $f^{-1}(G_\lambda) = p^{-1}_{\alpha_\lambda}(p_{\alpha_\lambda}(f^{-1}(G_\lambda)))$. By τ-completeness of A, $\alpha = \sup\{\alpha_\lambda : \lambda \in \tau\}$ is an element of A. Clearly, for each $\lambda \in \tau$, $f^{-1}(G_\lambda) = p^{-1}_\alpha(p_\alpha(f^{-1}(G_\lambda)))$.

Let us consider now any two points x_1 and x_2 in X with $p_\alpha(x_1) = p_\alpha(x_2)$. Let us show that $f(x_1) = f(x_2)$. Suppose the converse. Then there exists $\lambda \in \tau$ such that $f(x_1) \in G_\lambda$ and $f(x_2) \notin G_\lambda$. Consequently, $x_1 \in f^{-1}(G_\lambda)$ and $x_2 \notin f^{-1}(G_\lambda)$. By the construction of α , $p_\alpha(x_1) \in p_\alpha(f^{-1}(G_\lambda))$ and $p_\alpha(x_2) \notin p_\alpha(f^{-1}(G_\lambda))$. Hence $p_\alpha(x_1) \neq p_\alpha(x_2)$. Contradiction.

Now we can define the mapping $f_\alpha : X_\alpha \longrightarrow Y$ by the following way: $f_\alpha = f\, p^{-1}_\alpha$. Clearly, $f = f_\alpha p_\alpha$ and so, it only remains to show that f_α is a Baire mapping (of class γ). For, let G be a cozero-set of Y. Then a Baire set (of additive class γ) $f^{-1}(G)$ is an inverse-image of a set $f^{-1}_\alpha(G)$ under a perfect mapping p_α. Consequently, $f^{-1}_\alpha(G)$ is a Baire set (of additive class γ) of X ([4], p. 153) and hence f_α is a Baire mapping (of class γ). The proof is complete.

Suppose we are given two inverse spectra $S = \{X_\alpha, p^\beta_\alpha, A\}$ and $S' = \{Y_{\alpha'}, q^{\beta'}_{\alpha'}, A'\}$. A Baire morphism (of class γ) of S to S' is a family $\{h, f_{\alpha'}\}$ consisting of a nondecreasing function h from A' to A such that the set h(A') is cofinal in A, and of Baire mappings (of class γ) $f_{\alpha'} : X_{h(\alpha')} \longrightarrow Y_{\alpha'}$ defined for all $\alpha' \in A'$ and such that $q^{\beta'}_{\alpha'} f_{\beta'} = f_{\alpha'} p^{h(\beta')}_{h(\alpha')}$ for any α', $\beta' \in A'$ satisfying $\alpha' \leq \beta'$. Any Baire morphism (of class γ) of S to S' induces a Baire mapping (of class γ) of lim S to lim S'. To show this let us consider a thread $x = \{x_\alpha\} \in X = \lim S$. Let us define a point $y = \{y_{\alpha'}\}$ of the product $\prod\{Y_{\alpha'} : \alpha' \in A'\}$ by

$y_{\alpha'} = f_{\alpha'}(x_{h(\alpha')})$. It is easy to see that in such a way we obtain a mapping $f: X \to Y = \lim S'$ with $q_{\alpha'} f = f_{\alpha'} p_{h(\alpha')}$ for any $\alpha' \in A'$. Let us show that f is a Baire mapping (of class γ). For, consider a cozero-set G in Y. By the compactness of Y there exist a countable collection of indexes $\alpha'_k \in A'$ and cozero-sets $G_{\alpha'_k}$ in $Y_{\alpha'_k}$ such that $G = \cup \{ q^{-1}_{\alpha'_k}(G_{\alpha'_k}) : k \in \omega \}$. Then

$$f^{-1}(G) = \cup \{ q^{-1}_{\alpha'_k}(G_{\alpha'_k}) : k \in \omega \} = \cup \{ p^{-1}_{h(\alpha'_k)}(f^{-1}_{\alpha'_k}(G_{\alpha'_k})) : k \in \omega \}.$$

Since $f_{\alpha'_k}$ are Baire mappings (of class γ) and the mappings p_α are continuous, the set $f^{-1}(G)$ is a countable union of Baire sets (of additive class γ). Thus f is a Baire mapping (of class γ). Thus we have

Lemma 2. The limit mapping of the Baire morphism (of class γ) is the Baire mapping (of class γ).

The following theorem shows that for τ-spectra the converse also holds:

Theorem 1. Any Baire mapping (of class γ) between the limit spaces of two τ-spectra ($\tau \geq \omega$) with the same index sets is the limit mapping of some closed and cofinal Baire morphism (of class γ).

The validity of this theorem is an immediate consequence of the above lemmas an Proposition 1.3 from [6].

Corollary 1. Any Baire isomorphism (of class (γ, δ)) between the limit spaces of two τ-spectra ($\tau \geq \omega$) with the same index sets is the limit mapping of some closed and cofinal Baire morphism consisting of the Baire isomorphisms (of class (γ, δ)).

<u>Preservation of dimension</u>. Let us recall [4] that a bijection between X and Y is called the first-level Baire isomorphism

if an image of any countable union of zero-sets is a countable union of zero-sets in both directions. Clearly any first-level Baire isomorphism is a Baire isomorphism of class (1,1). An important theorem of Rogers and Jayne [4],[5] states that the first-level Baire isomorphisms preserve the dimension dim of metrizable compacta.

Theorem 2. If X and Y are first-level Baire isomorphic compacta, then dim X = dim Y.

Proof. Without loss of generality we can suppose that $w X = w Y > \omega$. Let $f:X \longrightarrow Y$ be the first-level Baire isomorphism and dim $X \leq n$. Of course, it is sufficient to prove that dim $Y \leq n$. Let us consider a sigma-spectrum (i.e. ω-spectrum) $S_1 = \{X_\alpha, p_\alpha^\beta, A\}$ such that lim $S_1 = X$ and dim $X_\alpha \leq n$ for each $\alpha \in A$. Since X and Y have the same weight we can suppose that Y is the limit space of some sigma-spectrum $S_2 = \{Y_\alpha, q_\alpha^\beta, A\}$ with the same index set as S_1. By the Corollary 1 there exists a Baire morphism $\{f_\alpha : \alpha \in A\}: S_1 \longrightarrow S_2$ such that $f = \lim f_\alpha$ and each $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ is a Baire isomorphism of class (1,1). It is easy to see, using closedness of the limit projections p_α and q_α, that each f_α is the first-level Baire isomorphism. (Indeed, let $Z = \cup Z_i$ be a countable union of zero-sets of X_α. Then, by continuity of p_α, $p_\alpha^{-1}(Z) = \cup p_\alpha^{-1}(Z_i)$ is a countable union of zero-sets of X. Since f is a first-level Baire isomorphism $f(p_\alpha^{-1}(Z)) = \cup T_i$, where each T_i is a zero-set of Y. Since q_α is a closed mapping we can conclude that $q_\alpha(T_i)$ is a closed subset and, consequently, by metrizability of Y_α, is a zero-set of Y_α. It only remains to note that $f_\alpha(Z) = q_\alpha(f(p_\alpha^{-1}(Z))) = q_\alpha(\cup T_i) = \cup q_\alpha(T_i)$.) Consequently, by the above mentioned theorem of Rogers and Jayne, dim $Y_\alpha \leq n$, $\alpha \in A$. Thus dim $Y \leq n$. The theorem is proved.

Let us recall that the transfinite dimensions ind and Ind are the ordinal valued functions obtained through the extension by transfinite induction of the classical notions of small or large inductive dimension respectively; the values of the transfinite dimensions considered in the class of separable metrizable spaces are always countable ordinals. The transfinite dimensions were first considered by W. Hurewicz who proved that for a Polish space X the transfinite dimension ind is defined iff X is countable-dimensional (i.e. X is a union of countably many zero-dimensional sets). It is known [5] that the last property is an invariant of first-level Baire isomorphisms in the class of metrizable compacta.

Let $\mathcal{K}$ denote the class of compacta each of which admits a zero-dimensional mapping onto some metrizable compactum.

Theorem 3. If X and Y are first-level Baire isomorphic compacta, $X \in \mathcal{K}$ and the transfinite dimension ind X is defined, then $Y \in \mathcal{K}$ and the transfinite dimension ind Y is defined.

Proof. By Theorem 5 from [2], there exists a zero-dimensional mapping g of X onto a countable-dimensional metrizable compactum K. By the well-known theorem of Tumarkin, K is a union of countable collection of zero-dimensional G_δ-sets K_i. Clearly, $X = \bigcup \{g^{-1}(K_i): i \in \omega\}$ and for each $i \in \omega$, $g^{-1}(K_i)$, denoted X_i, is a zero-dimensional, Lindelöf, Čech-complete space. Let $S_1 = \{X_\alpha, p_\alpha^\beta, A\}$ be any sigma-spectrum, the limit space of which coincides with X. Using the above representation of X, the spectral theorem of Ščepin [6] for sigma-spectra and the possibility of representation of any Lindelöf and Čech-complete space with $\dim \leq n$ as the limit space of sigma-spectrum consisting of Polish spaces with $\dim \leq n$ we can suppose without loss of generality that for each $\alpha \in A$, X_α is a countable-dimensional compactum. Let us note also that without loss of generality we can suppose the zero-dimensionality of all limit projections p_α in S_1. Indeed, X is the limit space of the sigma-spectrum $S_1 = \{X_\alpha, p_\alpha^\beta, A\}$ and $X \in \mathcal{K}$, i.e. there exists a zero-dimensional mapping $g: X \to K$, where K is a metrizable compactum. Clearly there exist an index $\alpha_0 \in A$ and a mapping $g_{\alpha_0}: X_{\alpha_0} \to K$ such that $g = g_{\alpha_0} \cdot p_{\alpha_0}$. Now, if $x \in X_{\alpha_0}$ then $p_{\alpha_0}^{-1}(x) \subseteq g^{-1}(k)$ where $k = g_{\alpha_0}(x) \in K$. By the zero-dimensionality of g, $\dim p_{\alpha_0}^{-1}(x) = 0$ and, consequently, p_{α_0} is zero-dimensional. Finally, let us consider the sigma-spectrum $S_1' = \{X_\alpha, p_\alpha^\beta, \alpha \in A, \alpha \geq \alpha_0\}$. Clearly, $\lim S_1' = X$ and all limit projections of S_1' are zero-dimensional. Let $f: X \to Y$ be a *first-level* Baire isomorphism. Let $S_2 = \{Y_\alpha, q_\alpha^\beta, A\}$ be any sigma-spectrum with $\lim S_2 = Y$. Since f is a *first-level Baire isomorphism* between X and Y, by Corollary 1, there exists a Baire morphism $\{f_\alpha : \alpha \in A\}: S_1 \to S_2$ such that $f = \lim f_\alpha$ and each f_α is a Baire isomorphism of class (1,1). Moreover, each f_α, as it is easy to se, is the first-level Baire isomorphism. (Precisely the same arguments are used in the proof of Theorem 2.) By [5], each Y_α is countable-dimensional. Let us note now that each limit projection q_α in S_2 is zero-dimensional. For, let us consider any fibre of the projection q_α. It is easy to see that it is first-level Baire isomorphic to the corresponding fibre of the projection p_α. Since these fibres are compact, we can conclude, by Theorem 2, that q_α is zero-dimensional. Consequently, $Y \in \mathcal{K}$. By [2], the transfinite dimension ind Y is defined. The theorem is proved.

I do not know if the last theorem holds without the additional assumption that $X \in \mathcal{K}$?

Hyperspaces and Baire isomorphisms

Theorem 4. There is no Baire isomorphism between D^{τ^+} and $\exp D^{\tau^+}$ for $\tau > 2^{\omega}$.

Proof. Consider the Cantor cube D^{τ^+} of weight τ^+. Clearly D^{τ^+} is the limit space of the τ-spectrum $S' = \{D^A, p_A^B, \exp_{\tau} \tau^+\}$ where $\exp_{\tau} \tau^+$ denotes the collection of all subsets of τ^+ of cardinality $\leq \tau$ and p_A^B is the natural projection of D^B onto D^A. Fix some $A_0 \in \exp_{\tau} \tau^+$ of cardinality τ. Clearly, the limit space of the τ-spectrum $S = \{D^A, p_A^B, A, B \supseteq A_0; A, B \in \exp_{\tau} \tau^+\}$ coincides with D^{τ^+} and all the limit projections p_A of S are homeomorphic to the natural projection of D^{τ^+} onto D^{τ}. It is easy to see that $\exp S = \{\exp D^A, \exp p_A^B\}$ is a τ-spectrum and its limit space is homeomorphic to $\exp^{\tau^+}$. Let us suppose now that there exists a Baire isomorphism f between D^{τ^+} and $\exp D^{\tau^+}$. By the Corollary 1, we can conclude that there exists a Baire isomorphism $f_A : D^A \longrightarrow \exp D^A$ (where $A \subseteq \tau^+$ and $|A| = \tau$) such that $f_A \cdot p_A = \exp p_A \cdot f$. Consequently, for each point F of $\exp D^A$ the fibre $(\exp p_A)^{-1}(F)$ is Baire isomorphic to the corresponding fibre $p_A^{-1}(f_A^{-1}(F))$ which is homeomorphic with D^{τ^+}. Let T be a subspace of D^A such that T is discrete in the relative topology and $|T| = \tau$. Let F denote the closure of T in D^A. Obviously, F is a point of $\exp D^A$ and there exists a pair-wise disjoint collection of cardinality τ of open subsets of F. Indeed, by the construction, F is the closure in $D^A (|A| = \tau)$ of a subspace $T \subseteq D^A$ such that $|T| = \tau$ and T is discrete in the topology induced from D^A. Let $T = \{t_{\alpha} : \alpha \in \tau\}$. Let G_{α} be an open subset of F with $G_{\alpha} \cap T = \{t_{\alpha}\}$ $(\alpha \in \tau)$. The collection $\{G_{\alpha} : \alpha \in \tau\}$ of open subsets of F is desired. It only remains to note that $\{G_{\alpha} : \alpha \in \tau\}$ is a pair-wise disjoint collection. For, let us consider the intersection $G_\alpha \cap G_\beta = G$. Clearly, G is open in F and its intersection with T is empty. Consequently, by the density of T in F, $G = G_\alpha \cap G_\beta$ is also empty. One can easily check that the fibre $(\exp p_A)^{-1}(F)$ also contains a pair-wise disjoint collection of cardinality τ of open sets. Then we can conclude that there exists a pair-wise disjoint collection of cardinality τ of Baire sets in $p_A^{-1}(f_A^{-1}(F)) = D^{\tau^+}$. Since each Baire set is a union of zero-sets we can conclude that there exists a pair-wise disjoint collection of cardinality τ of zero-sets in D^{τ^+}. But this is impossible by the result of R. Engelking [3] and the inequality $\tau > 2^\omega$. The proof is complete.

Corollary 2. (CH) There is no Baire isomorphism between D^{ω_3} and $\exp D^{\omega_3}$.

As L.B. Shapiro informed me, the above corollary holds even without using CH,[7]. It should be observed also that the Cantor cubes of weights ω_n, $n = 0,1,2$, are Baire isomorphic with their hyperspaces. (For $n = 0,1$ these assertions follow from the well-known facts that the Cantor cubes D^{ω_0} and D^{ω_1} are even homeomorphic with their hyperspaces. For $n = 2$ the assertion follows from Shapiro's result ([7], Corollary 2).)

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